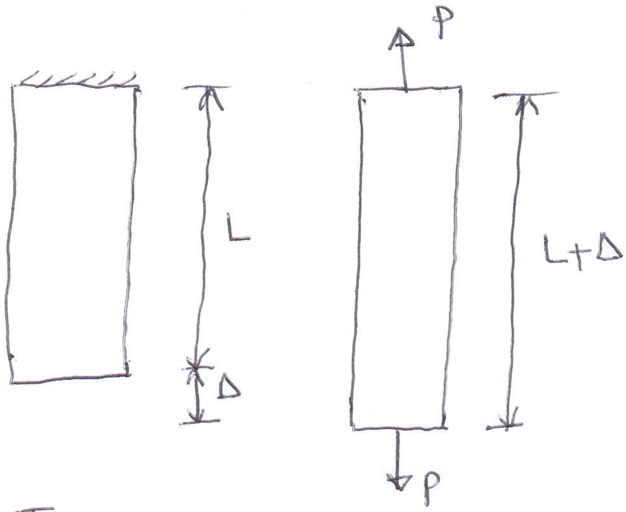
Solution TD N° 4

Ex. 1 :



Eq. 157

$$U = \frac{1}{2}\int_V \{\varepsilon\}^T [D]\{\varepsilon\}\, dV - \int_V \{\varepsilon\}^T [D]\{\varepsilon_0\}\, dV + \int_V \{\varepsilon\}^T\{\sigma_0\}\, dV$$
$$- \int_V \{u\}^T\{b\}\, dV - \int_V \{u\}^T\{\phi\}\, dS - \sum_{p=1}^{n} \{u\}^T\{f_p\}$$

on prend : $\varepsilon_0 = 0$, $\sigma_0 = 0$, $b = 0$, $\phi = 0$, $D = E$, $\varepsilon = \frac{\Delta}{L}$, $V = AL$, $f_p = P$, $u = \Delta$

Donc :

$$U = \frac{1}{2}E\varepsilon^2 AL - P\Delta = \frac{1}{2}E\left(\frac{\Delta}{L}\right)^2 AL - P\Delta$$

$$\frac{dU}{d\Delta} = 0 \Rightarrow \frac{E\Delta A}{L} - P = 0$$

$$\Rightarrow \Delta = \frac{PL}{AE}$$ (Résultat Connu)

Ex. 2 :

(Eq. 158)

$$\int_V \{\delta\varepsilon\}^T [D]\{\varepsilon\}\, dV - \int_V \{\delta\varepsilon\}^T \{\sigma_0\}\, dV$$

$$\int_V \{\delta\varepsilon\}^T\{\sigma\}\, dV = \int_V \{\delta u\}^T\{b\}\, dV + \int \{\delta u\}^T\{\phi\}\, dS + \sum_{p=1}^{n} \{\delta u\}^T\{f_p\}$$

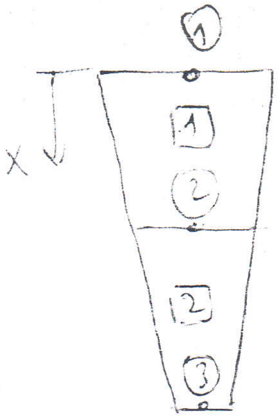
on prend : $b = 0$, $\phi = 0$, $\delta u = \Delta$, $\delta\varepsilon = \frac{\Delta}{L}$, $\sigma = E\varepsilon = \frac{E\Delta}{L}$, $f_p = P$, $V = AL$

Donc :

$$\frac{\Delta}{L} \cdot \frac{E\Delta}{L} AL = P\Delta$$

$$\Rightarrow \Delta = \frac{PL}{AE}$$ (Résultat Connu)

EX. 3



Données des nœuds et des éléments.

Nœud	X (cm)	r (cm)
1	0,0	1,00
2	12,0	0,75
3	24,0	0,5

$r \equiv$ rayon

$$r(x=\bar{x}) = \frac{1}{2}\left[\frac{D_1 L - (D_1 - D_3)\bar{x}}{L}\right]$$

Element	Nœud i	Nœud j
1	1	2
2	2	3

Element 1 : $X_i = 0$; $X_j = 12$ cm

$$X_j - X_i = 12 \text{ cm}$$

$$\bar{X} = \frac{X_i + X_j}{2} = 6 \text{ cm}$$

$$\bar{r} = \frac{1}{2}\left[2 - \frac{(2-1)\times 6}{24}\right] = 0{,}875 \text{ cm}$$

$$\bar{S} = \pi \bar{r}^2 = (3{,}14)(0{,}875)^2 = 2{,}404 \text{ cm}^2$$

$$[K^{(1)}] = \frac{(2{,}404)(30\times10^6)}{12}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}6010 & -6010\\ -6010 & 6010\end{bmatrix}\times10^3 \text{ N/cm}$$

$$[K^{(a)}] = \begin{bmatrix}6010 & -6010 & 0\\ -6010 & 6010 & 0\\ 0 & 0 & 0\end{bmatrix}\times10^3 \text{ N/cm}$$

Solution T.D. N° 4

$$\{f^{(1)}\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \; N$$

$$\{f^{(a)}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \; N$$

Element 2 : $X_i = 12\ cm \;;\; X_j = 24\ cm$

$$X_j - X_i = 12\ cm$$

$$\bar{X} = \frac{X_i + X_j}{2} = 18\ cm$$

$$\bar{r} = \frac{1}{2}\left[2 - \frac{(2-1)(18)}{24}\right] = 0{,}625\ cm$$

$$\bar{S} = (3.14)(0.625)^2 = 1.228\ cm^2$$

$$S_3 = (3.14)(0.5)^2 = 0.784\ cm^2$$

$$[K^{(2)}] = \frac{(1.228)(30\times10^6)}{12}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3070 & -3070 \\ -3070 & 3070 \end{bmatrix} \times 10^3 \; N/cm$$

$$[K^{(a)}] = \begin{bmatrix} 6010 & -6010 & 0 \\ -6010 & 9080 & -3070 \\ 0 & -3070 & 3070 \end{bmatrix} \times 10^3 \; N/cm$$

$$\{f^{(2)}\} = \{f_{cp}^{(2)}\} + \{f_s^{(2)}\} = \begin{Bmatrix} 0 \\ 12000 \end{Bmatrix} + \begin{Bmatrix} 0 \\ (0.784)(10200) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20000 \end{Bmatrix} \; N$$

$$\{f^{(a)}\} = \begin{Bmatrix} 0 \\ 0 \\ 20000 \end{Bmatrix} \; N$$

$$[K^{(a)}]\{a\} = \{f^{(a)}\}$$

ou

$$10^3\begin{bmatrix} 6010 & -6010 & 0 \\ -6010 & 9080 & -3070 \\ 0 & -3070 & 3070 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 20000 \end{Bmatrix}$$

$u_1 = 0$ (extrémité encastrée)

Solution TD, N° 4

$$10^3\begin{bmatrix} 9080 & -3070 \\ -3070 & 3070 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20000 \end{Bmatrix}$$

Solution $u_1 = 0$, $u_2 = 0{,}00332\ \text{cm}$; $u_3 = 0{,}00984\ \text{cm}$

Résultats des éléments.

Élément 1

$$[B] = \left[\frac{-1}{x_j - x_i} \quad \frac{1}{x_j - x_i}\right] = \left[\frac{-1}{12-0} \quad \frac{1}{12-0}\right] = \frac{1}{12}[-1 \quad 1]\ \text{1/cm}$$

$$\bar{\varepsilon} = [B]\{a^{(1)}\} = \frac{1}{12}[-1 \quad 1]\begin{Bmatrix} 0 \\ 0{,}00332 \end{Bmatrix} = 0{,}000277$$

$$\bar{\sigma} = E(\bar{\varepsilon} - \cancel{\varepsilon_0}^{\,0}) + \cancel{\sigma_0}^{\,0} = (30\times10^6)(0{,}000277) = 8310\ \text{N/cm}^2$$

$$\bar{F} = \bar{\sigma}\bar{S} = (8310)(2{,}404) = 20000\ \text{N}$$

Élément 2 :

$$[B] = \left[\frac{-1}{24-12} \quad \frac{1}{24-12}\right] = \frac{1}{12}[-1 \quad +1]\ \text{1/cm}$$

$$\bar{\varepsilon} = [B]\{a^{(2)}\} = \frac{1}{12}[-1 \quad +1]\begin{Bmatrix} 0{,}00332 \\ 0{,}00984 \end{Bmatrix} = 0{,}000543$$

$$\bar{\sigma} = E\bar{\varepsilon} = (30\times10^6)(0{,}000543) = 16300\ \text{N/cm}^2$$

$$\bar{F} = \bar{\sigma}\bar{S} = (16300)(1{,}228) = 20000\ \text{N}$$

Solution T.D. N° 4

Ex. 4.

1)
$$A = \frac{1}{2}\det\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 8 & 4 \\ 1 & 0 & 4 \\ 1 & 0 & 0 \end{bmatrix} = 16\ \text{cm}^2$$

$$m_{21} = \frac{y_j - y_k}{2A} = \frac{y_2 - y_3}{2A} = \frac{4-0}{2\times 16} = 0{,}125\ \text{cm}^{-1}$$

$$m_{31} = \frac{x_k - x_j}{2A} = \frac{x_3 - x_2}{2A} = \frac{0-0}{2\times 16} = 0$$

$$m_{22} = \frac{y_k - y_i}{2A} = \frac{y_3 - y_1}{2A} = \frac{0-4}{2\times 16} = -0{,}125\ \text{cm}^{-1}$$

$$m_{32} = \frac{x_i - x_k}{2A} = \frac{x_1 - x_3}{2A} = \frac{8-0}{2\times 16} = 0{,}25\ \text{cm}^{-1}$$

$$m_{23} = \frac{y_i - y_j}{2A} = \frac{y_1 - y_2}{2A} = \frac{4-4}{2\times 16} = 0$$

$$m_{33} = \frac{x_j - x_i}{2A} = \frac{x_2 - x_1}{2A} = \frac{0-8}{2\times 16} = -0{,}25\ \text{cm}^{-1}$$

$$[B] = \begin{bmatrix} m_{21} & 0 & m_{22} & 0 & m_{23} & 0 \\ 0 & m_{31} & 0 & m_{32} & 0 & m_{33} \\ m_{31} & m_{21} & m_{32} & m_{22} & m_{33} & m_{23} \end{bmatrix}$$

$$[B] = \begin{bmatrix} 0{,}125 & 0 & -0{,}125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0{,}25 & 0 & -0{,}25 \\ 0 & 0{,}125 & 0{,}25 & -0{,}125 & -0{,}25 & 0 \end{bmatrix}$$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$ (Cas de contrainte plane)

$$[D] = \frac{30\times 10^6}{1-0{,}3^2}\begin{bmatrix} 1 & 0{,}3 & 0 \\ 0{,}3 & 1 & 0 \\ 0 & 0 & \frac{1-0{,}3}{2} \end{bmatrix} = \begin{bmatrix} 33{,}0 & 9{,}89 & 0 \\ 9{,}89 & 33{,}0 & 0 \\ 0 & 0 & 11{,}5 \end{bmatrix}\times 10^6\ \text{N/cm}^2$$

$$[k^e] = [B]^T[D][B]\,h\,A \qquad (h = 0{,}5\ \text{cm})$$

Solution T.D. N° 4

$$[K^e] = \begin{bmatrix} 4.12 & 0 & -4.12 & 2.48 & 0 & -2.48 \\ & 1.44 & 2.88 & -1.44 & -2.88 & 0 \\ & & 9.88 & -5.36 & -5.76 & 2.48 \\ & & & 17.94 & 2.88 & -17.50 \\ & \text{SYM.} & & & 5.76 & 0 \\ & & & & & 17.50 \end{bmatrix} \times 10^6 \ \text{N/cm}$$

2) $$\{\varepsilon_0\} = \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix} = \begin{Bmatrix} 3\times 10^{-6} \times 75 \\ 3\times 10^{-6} \times 75 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 225 \\ 225 \\ 0 \end{Bmatrix} \times 10^{-6}$$

$$\{f^e_{\varepsilon_0}\} = [B]^T [D] \{\varepsilon_0\}\, t A$$

$$\{f^e_{\varepsilon_0}\} = \begin{bmatrix} 0.125 & 0 & 0 \\ 0 & 0 & 0.125 \\ -0.125 & 0 & 0.25 \\ 0 & 0.25 & -0.125 \\ 0 & 0 & -0.25 \\ 0 & -0.25 & 0 \end{bmatrix} \begin{bmatrix} 33.0 & 9.89 & 0 \\ 9.89 & 33.0 & 0 \\ 0 & 0 & 11.5 \end{bmatrix} \begin{Bmatrix} 225 \\ 225 \\ 0 \end{Bmatrix} \times 10^{-6} (0.5)(16)$$

$$\{f^e_{\varepsilon_0}\} = \begin{Bmatrix} 9.65 \\ 0 \\ -9.65 \\ 19.30 \\ 0 \\ -19.30 \end{Bmatrix} \ \text{N}$$

3) $S_y = 0$

S_x varie linéairement le long du côté jk

S_x effective est prise égale à la moyenne de 1600 et 2000 N/cm². Donc : $S_x = 1800$ N/cm²

Solution T.D. N° 4 (7)

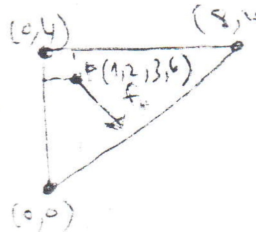
$$\{f_s^e\} = \frac{h\, l_{jk}}{2} \begin{Bmatrix} 0 \\ 0 \\ S_x \\ S_y \\ S_x \\ S_y \end{Bmatrix} = \frac{(0,5)(4)}{2} \begin{Bmatrix} 0 \\ 0 \\ 1800 \\ 0 \\ 1800 \\ 0 \end{Bmatrix}$$

$$\{f_s^e\} = \begin{Bmatrix} 0 \\ 0 \\ 1800 \\ 0 \\ 1800 \\ 0 \end{Bmatrix} \quad N$$

4)

$$N_i(x,y) = m_{11} + m_{21}x + m_{31}y$$
$$N_j(x,y) = m_{12} + m_{22}x + m_{32}y$$
$$N_k(x,y) = m_{13} + m_{23}x + m_{33}y$$

; $x_0 = 1,2$ cm ; $y_0 = 3,2$ cm



$$m_{11} = \frac{x_j y_k - x_k y_j}{2A^e} = \frac{x_1 y_3 - x_3 y_1}{2A^e} = \frac{(0)(0)-(0)(4)}{2(16)} = 0$$

$$m_{12} = \frac{x_k y_i - x_i y_k}{2A^e} = \frac{x_3 y_2 - x_2 y_3}{2A^e} = \frac{(0)(4)-(4)(0)}{2(16)} = 0$$

$$m_{13} = \frac{x_i y_j - x_j y_i}{2A^e} = \frac{x_2 y_1 - x_1 y_2}{2A^e} = \frac{(8)(4)-(0)(4)}{2(16)} = 1$$

$$N_i = 0 + 0,125 \times 1,2 + 0 \times 3,2 = 0,15$$

$$N_j = 0 + (-0,125) \times 1,2 + 0,25 \times 3,2 = 0,65$$

$$N_k = 1 + 0 \times 1,2 + (-0,25) \times 3,2 = 0,20$$

$$\{f_{cp}^e\} = [N]^T \{f_p\} = \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_k & 0 \\ 0 & N_k \end{bmatrix} \begin{Bmatrix} f_{px} \\ f_{py} \end{Bmatrix} = \begin{bmatrix} 0,15 & 0 \\ 0 & 0,15 \\ 0,65 & 0 \\ 0 & 0,65 \\ 0,2 & 0 \\ 0 & 0,2 \end{bmatrix} \begin{Bmatrix} 6000 \\ -9200 \end{Bmatrix} = \begin{Bmatrix} 900 \\ -1380 \\ 3900 \\ -5980 \\ 1200 \\ -1840 \end{Bmatrix} \quad N$$