

A photograph of a waterfall cascading over rocks, with water splashing and creating white foam at the base. The background is a natural, rocky environment.

Fluid Mechanics

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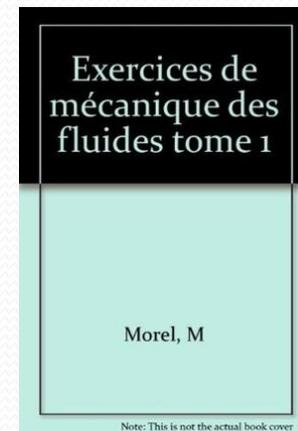
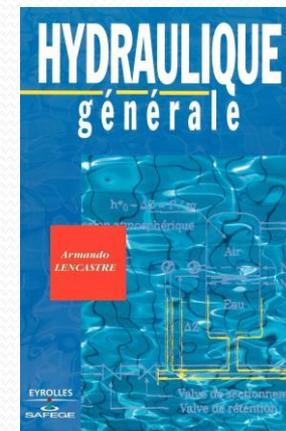
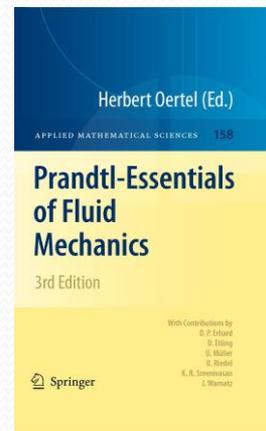
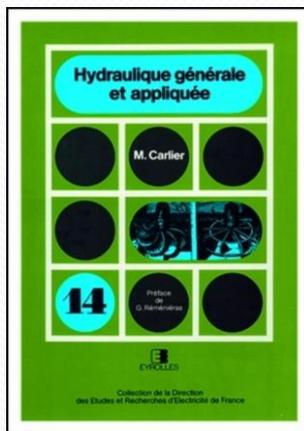
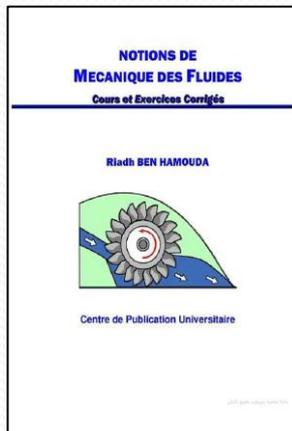
Program

- Chapter I: Properties of fluids
- Chapter II: Hydrostatics
- Chapter III: Fluid Kinematics
- Chapter IV: The Hydrodynamic of Perfect Fluids
- Chapitre V: Hydrodynamics of Real Fluids

Reference:

The main part of this course is based on the following books:

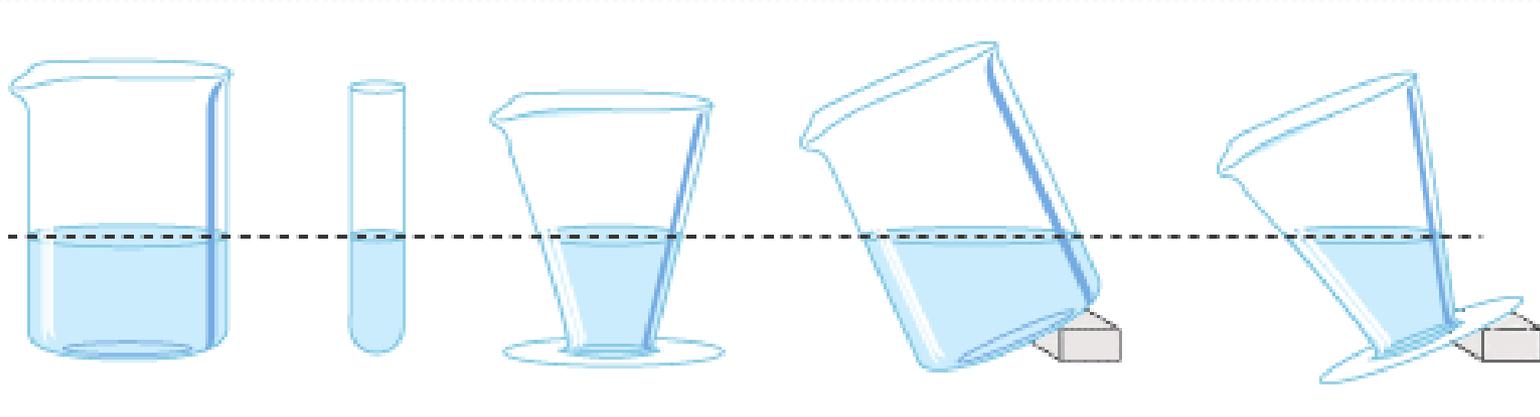
- **BEN HAMOUDA, R.** : *Notions de mécanique des fluides*, Centre de Publication Universitaire; (2008).
- **CARLIER, M.** : *Hydraulique générale et appliquée*, Editions Eyrolles, (1972).
- **COMOLET, R.** : *Mécanique expérimentale des fluides*, Masson, ed., (1982).
- **ERHARD, P., et al** : *Prandtl-essentials of fluid mechanics*. Vol. 158. Springer Science & Business Media, (2010).
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Chapter I: Properties of fluids

I.1 Definition of a fluid

A fluid is "a body with no form of its own that follows the shape of the vessel containing it and can flow".



Of the three main states of matter, solid, liquid and gaseous, the notion of fluid covers the last two.

The main difference between these three states is the intensity of the binding forces that ensure a certain cohesion between the molecules.

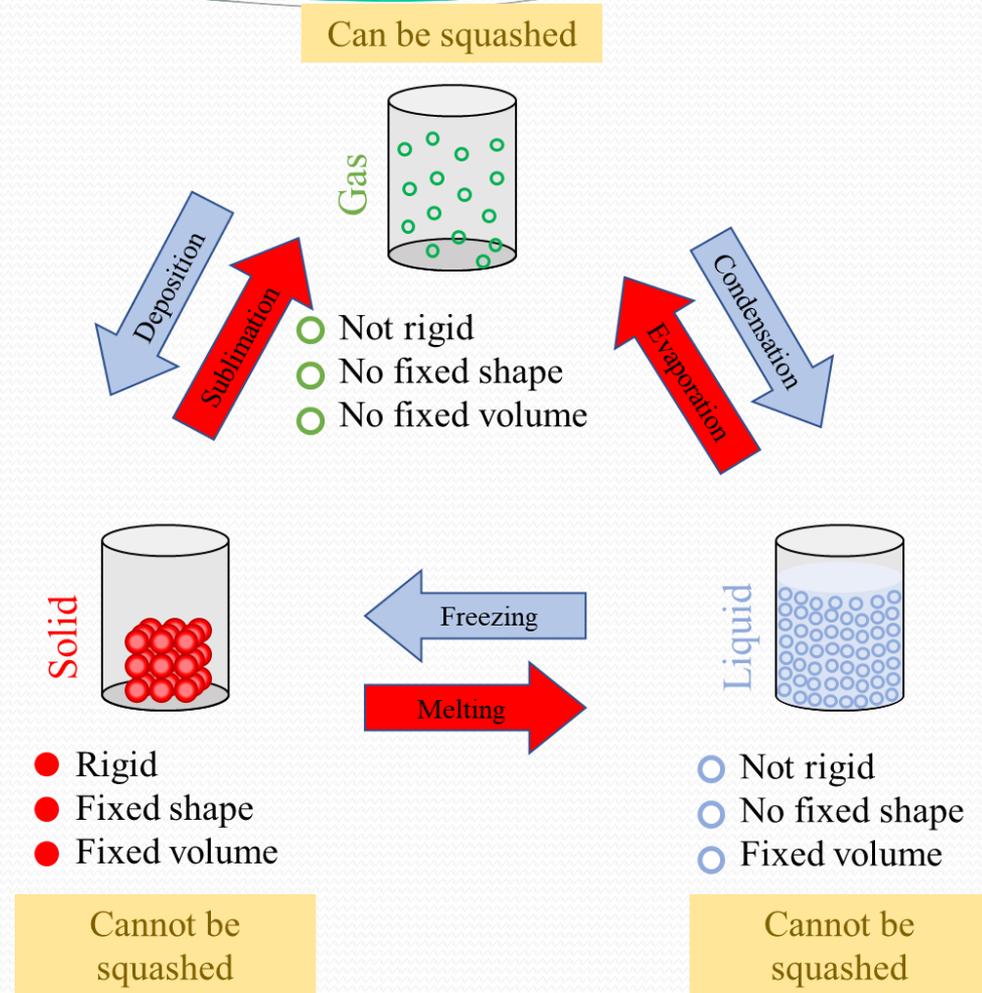


Fig. I.1: states of matter

For a body in the solid state, a relatively large force must be exerted to cause its deformation; moreover, if we remain within the elastic limit, the removal of this external force is accompanied by a return of the solid to its initial form.

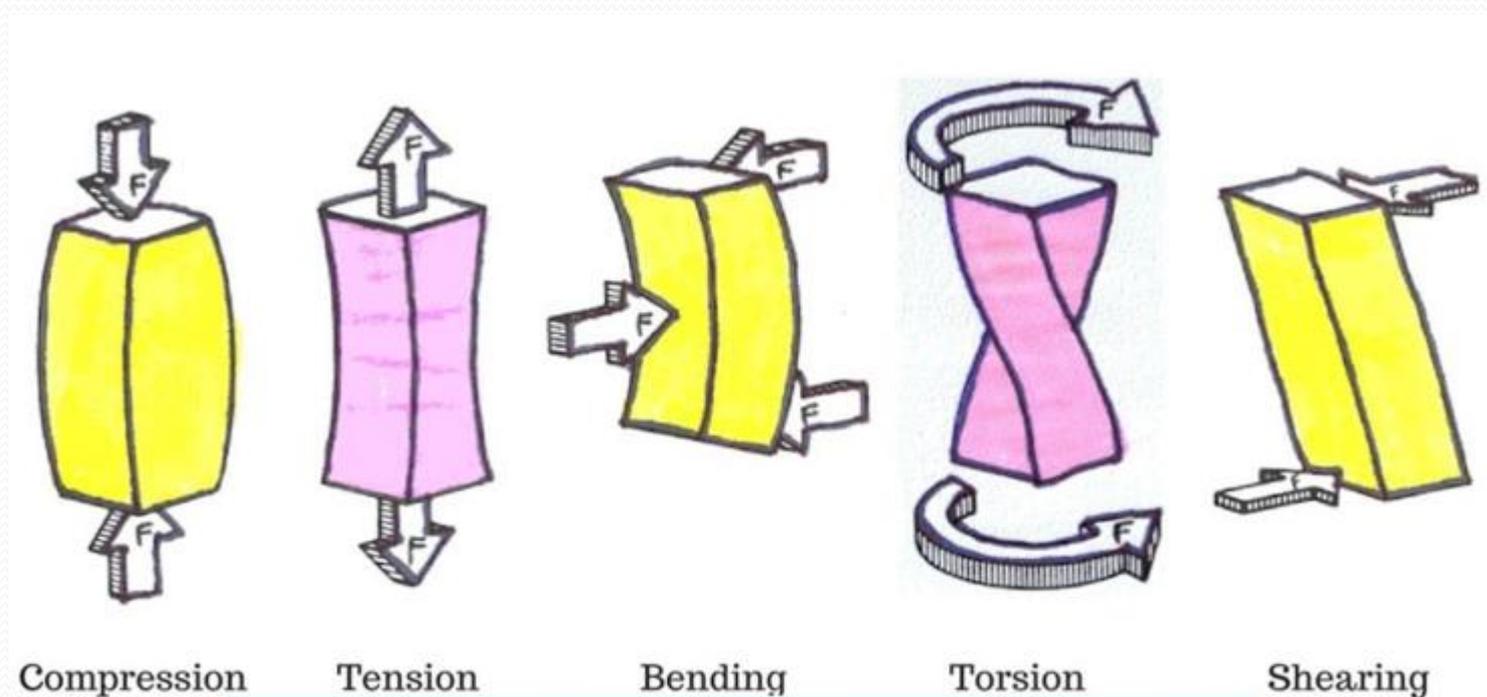


Fig. I.2: Types of stresses in engineering



In a body in the liquid state, the forces of attraction between molecules are much weaker. When an external force is applied, even a small one, the molecules can move relative to each other, this movement being definitively acquired.



Finally, in the case of a gas, the molecules are practically independent of each other but exhibit high-speed agitation which essentially depends on the temperature of the gas. This absence of internal cohesion within a fluid means that it cannot accept any traction force.

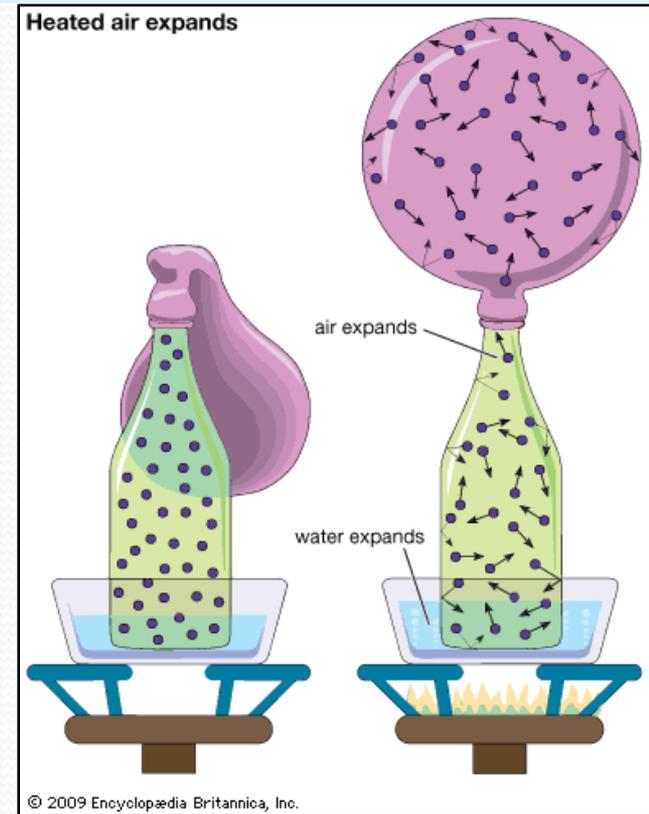
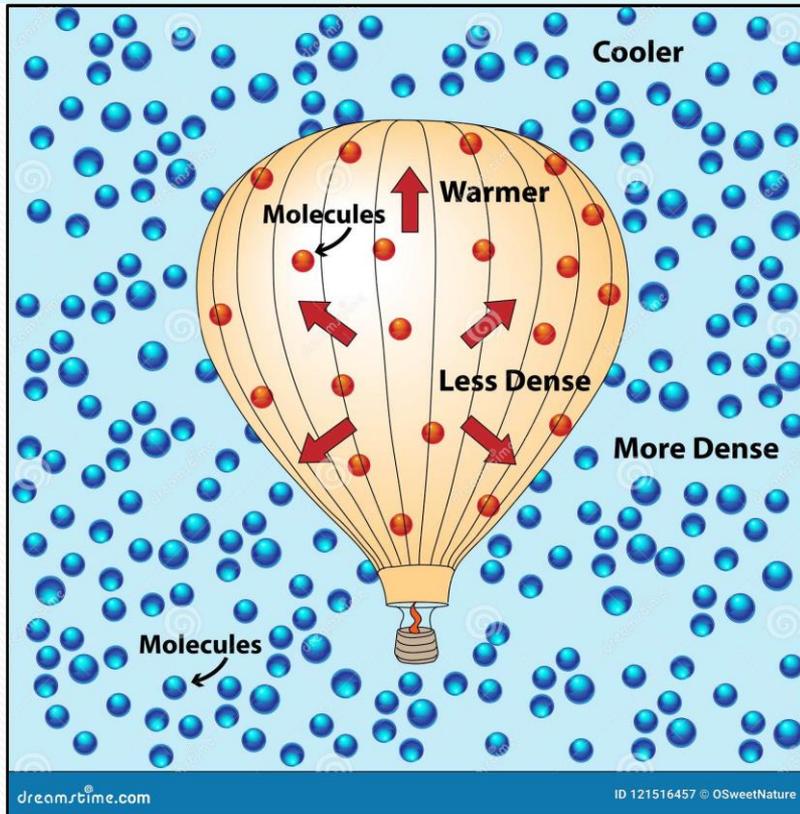


Fig. I.3: Expansion of gas volume with increasing temperature

Fluid mechanics studies the behavior of fluids:

- at rest (au repos): \longrightarrow hydrostatic
- moving fluid: \longrightarrow hydrodynamic
(fluides en mouvement)

There are two types of fluids:

- Liquid: « incompressibles »
- Gas: « compressibles »

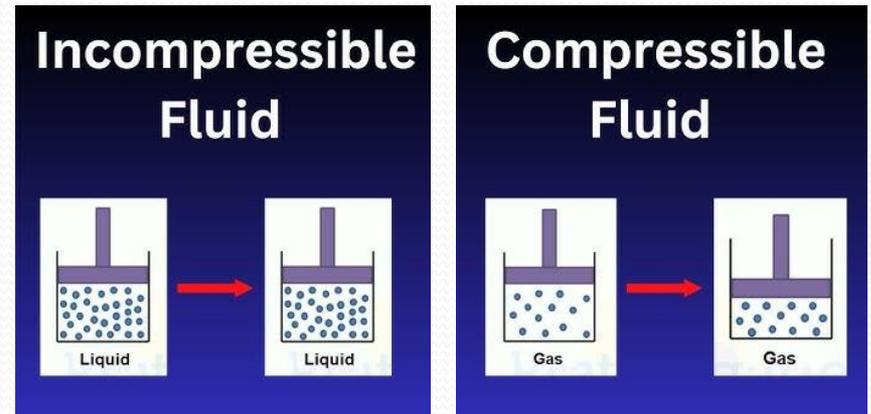
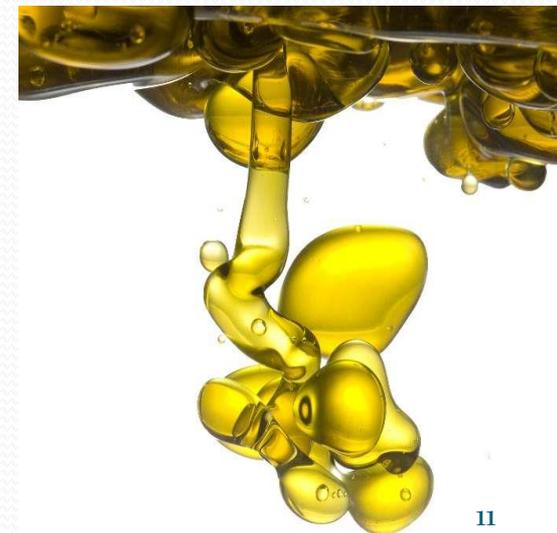


Fig. I.4: Incompressible fluid Vs compressible fluid

I.2 Principal physical properties of fluids

These general properties, defined at the particle scale, make it possible to evaluate the behavior of the fluid with respect to external stresses.

Since they are also a representation of the influence of the molecular structure, the pressure and temperature parameters play a particular role in the specific properties of the fluid.



I.2.1 Density or Mass density of the fluid

The density or specific mass is the mass of the unit volume of the body considered. If M is the mass of a volume V of fluid, the density is written:

$$\rho = \frac{M}{V} [\text{Kg/m}^3]$$

For liquids, the volume is practically insensitive to pressure variations, in the majority of cases it increases slightly when the temperature increases, water being an exception to this rule below 4°C.

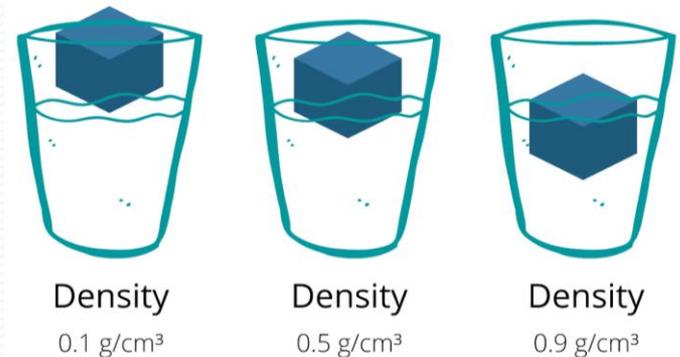


Fig. I.5: Fluid density

Table I.1 : Density

Fluid	Density Kg/m ³	Solid	Density Kg/m ³
Hydrogen	0.09	Cork	240
Helium	0.20	Wood	450 - 550
Dry air	1.205	Sand	1600
Oxygen	1.40	Salt	2160
Carbon dioxide	2.00	Aluminium	2700
Machine oil	900	Titanium	4500
Water	1000	Zinc	7150
Seawater	1030	Nickel	8900
Glycerol	1260	Lead	11340
Honey	1400	Uranium	18 700
Mercury	13546	Iridium	22640

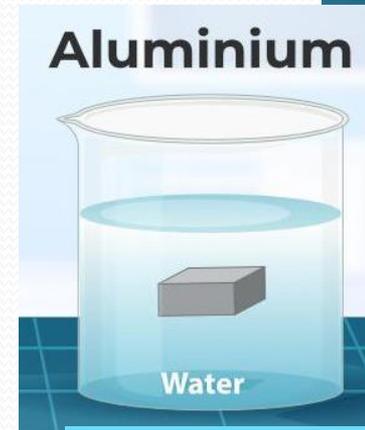
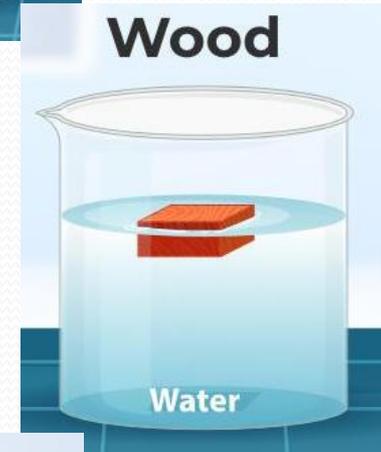
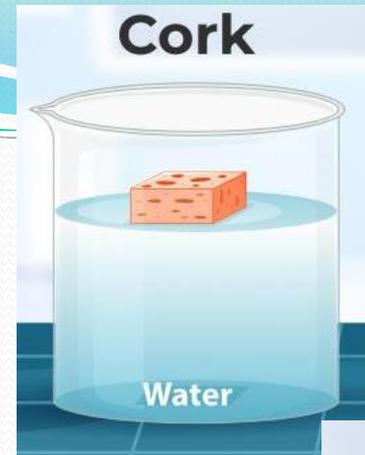


Fig. I.6: Matter density

Table I.2: Density of dry air as a function of temperature

ϑ in °C	ρ in kg/m ³	ϑ in °C	ρ in kg/m ³
-10	1.341	+40	1.127
-5	1.316	+45	1.109
0	1.293	+50	1.092
+5	1.269	+55	1.076
+10	1.247	+60	1.060
+15	1.225	+65	1.044
+20	1.204	+70	1.029
+25	1.184	+75	1.014
+30	1.164	+80	1.000
+35	1.146	+85	0.986

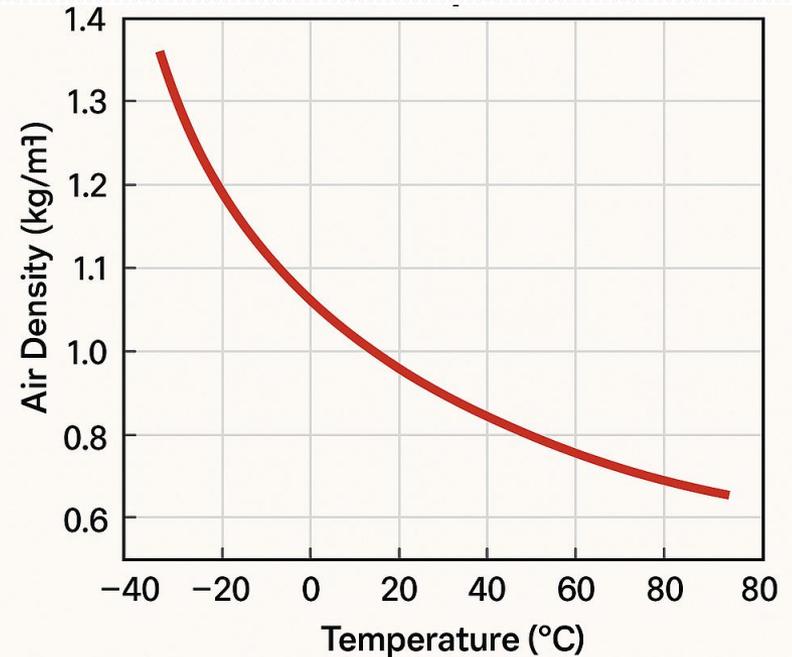


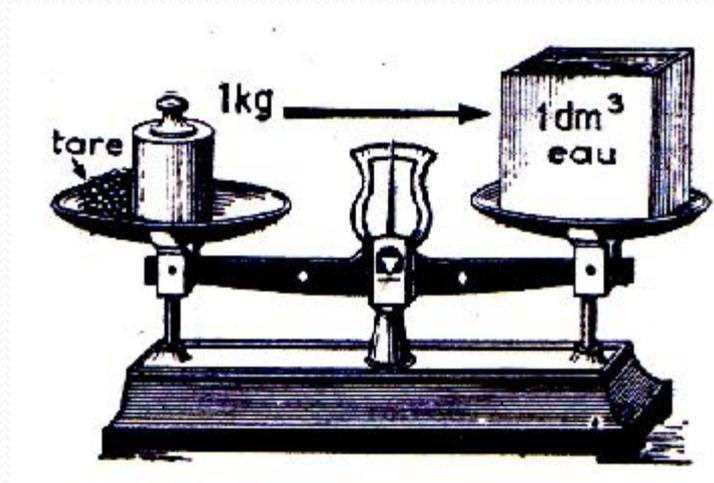
Fig. I.7: Variation of air density with temperature

I.2.2- Specific weight

Specific weight represents the force of gravity acting on mass per unit volume. It is written as:

$$\gamma = g \cdot \rho \text{ [N/m}^3\text{]}$$

$$\gamma_{\text{eau}} = 10^4 \text{ N/m}^3$$



I.2.3- Viscosity

Just like density, the viscosity of a fluid is a common observation: everyone has found that fluids flow more or less easily and that for some (especially oils), temperature has a great influence.

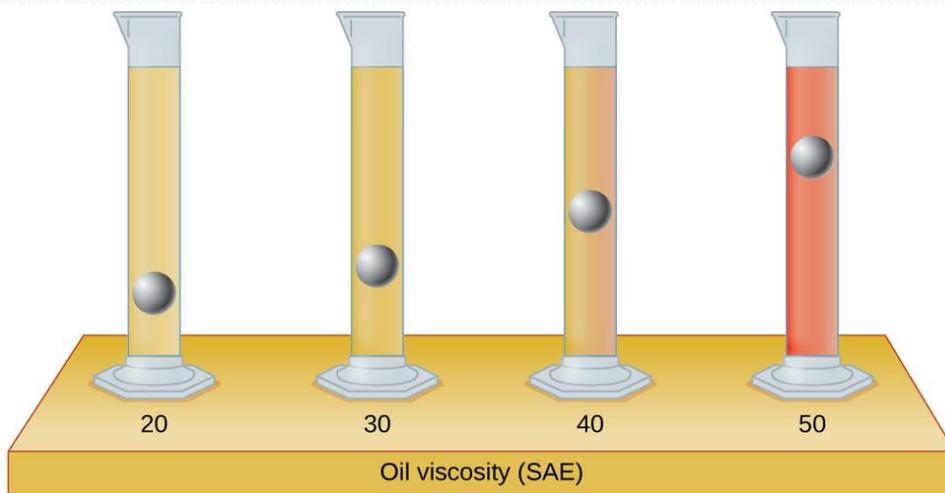


Fig. I.8: Viscosity of fluid

The viscosity is related to the existence of intermolecular forces which results on the one hand by the adhesion of the fluid to a wall and on the other hand by a resistance to the relative movement (slippage) of two adjacent particles of fluid.

This resistance corresponds to a loss of kinetic energy in the fluid, this lost mechanical energy being dissipated as heat within the fluid.

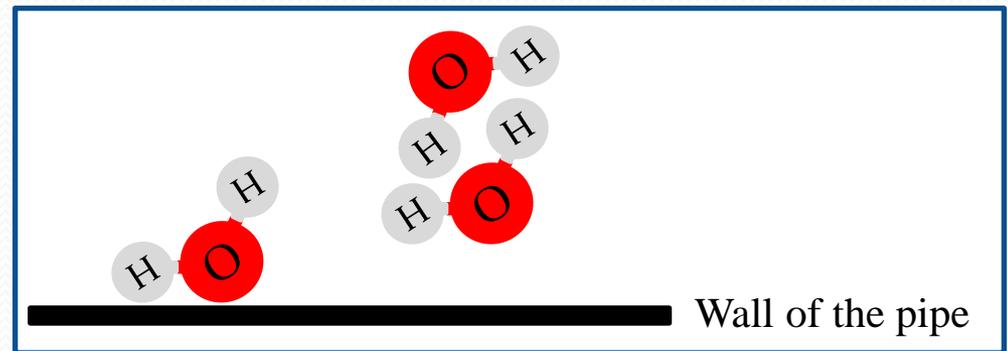


Fig. I.9: Intermolecular forces Vs adhesion of the fluid to a wall

The viscosity of a moving fluid is the property that expresses its resistance to a tangential force.

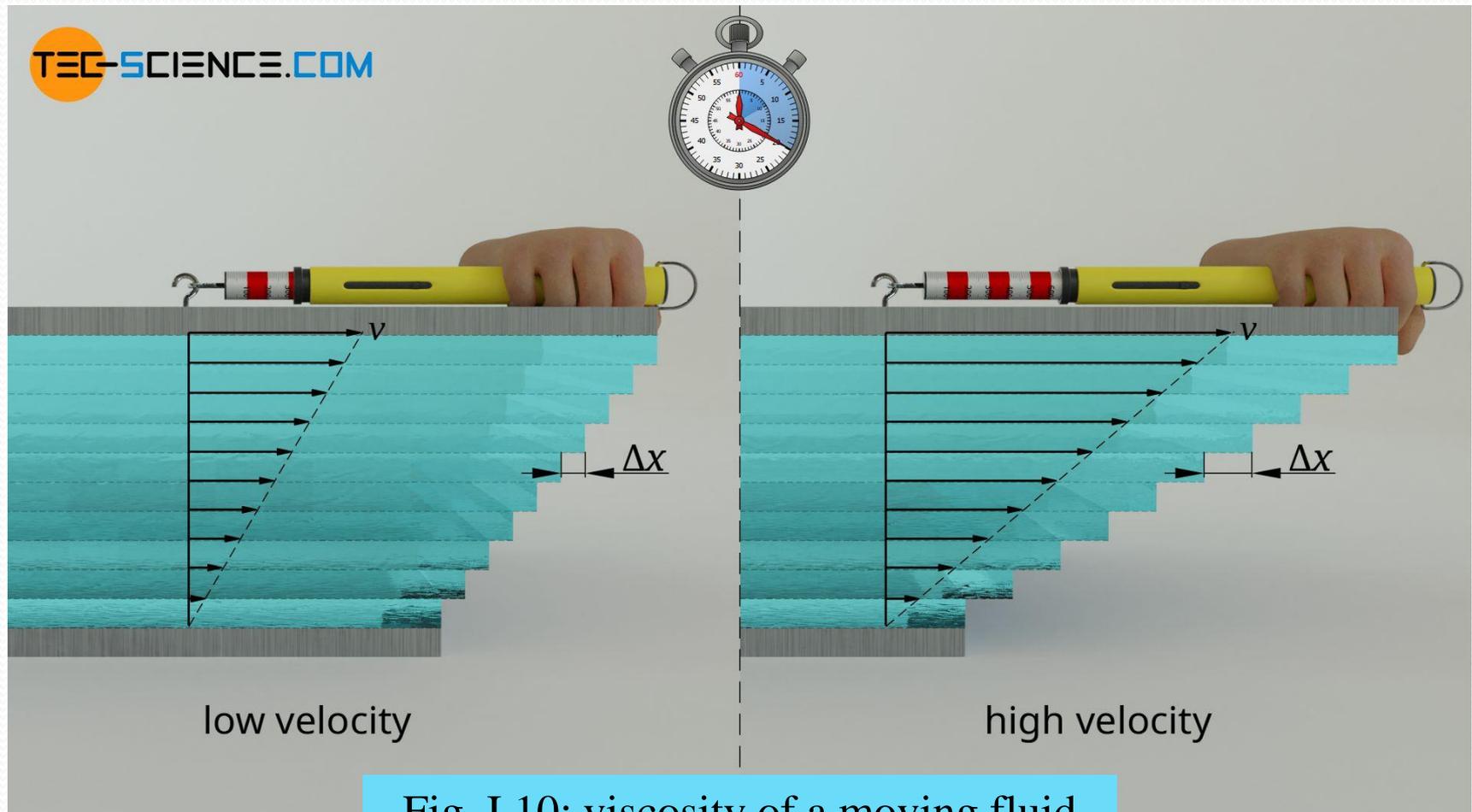


Fig. I.10: viscosity of a moving fluid

Imagine a fluid film placed between two flat plates, parallel and horizontal. One fixed and the other in uniform motion of speed V_{max} . For the speed to occur, a force F must be applied on the upper plate. This force is the result of viscous friction forces, which are:

$$F = \mu A \frac{dV}{dy}$$

A : plate area [m^2]

μ : viscosity [$N.s/m^2$]

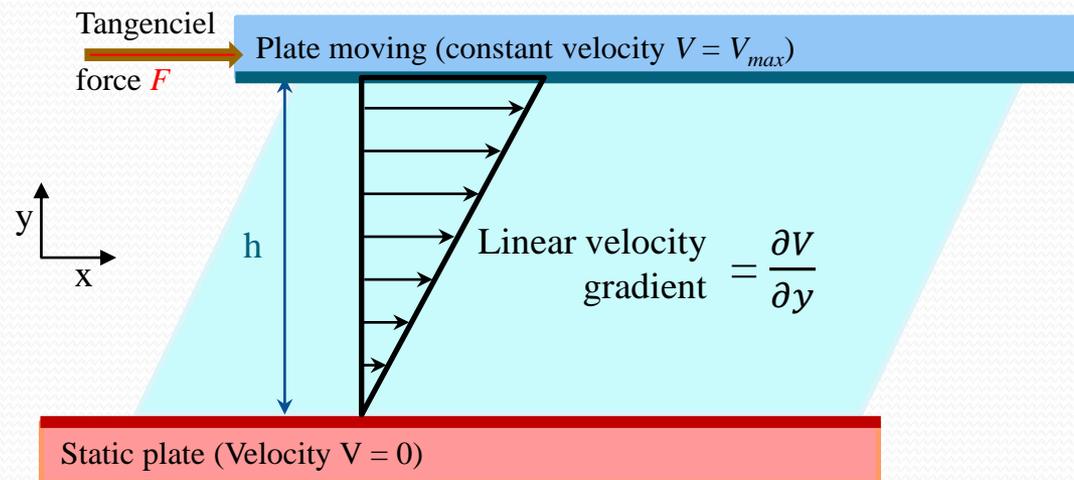


Figure I.11: Viscous drag between two plates

Two coaxial cylinders separated by a gap e , the space between them being filled with a liquid. The outer cylinder is rotated at a constant velocity (ω) and the inner cylinder is held in place. The fluid in contact with the outer cylinder will adhere to it and will therefore be animated by the velocity V ($V = \omega \times (r+e)$).

The fluid in contact with the fixed cylinder will have a zero velocity. It occurs in the fluid of shear forces. Viscosity is due to the interaction between the molecules of the fluid. This interaction creates a friction force that is measured by the torque M .

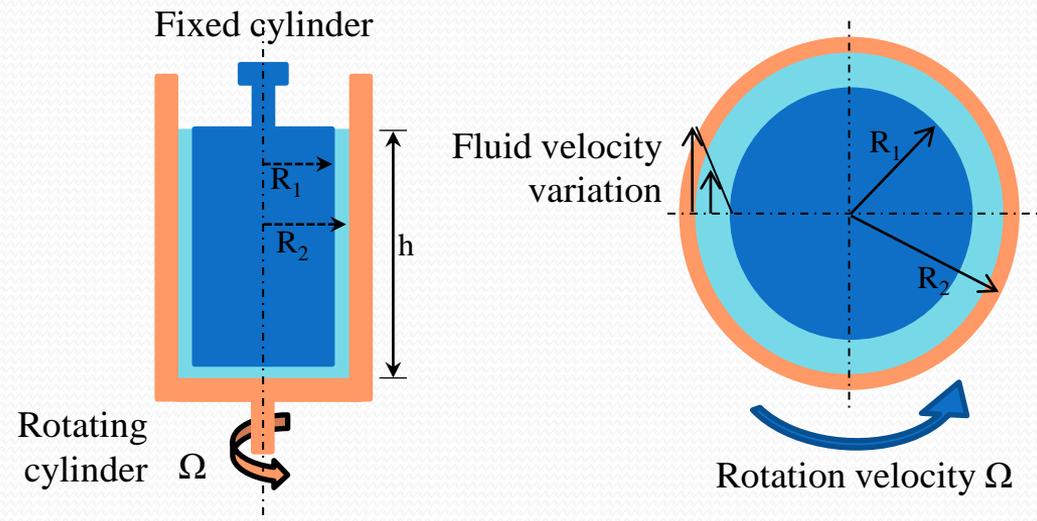


Figure I.12: Couette cell

If e is small compared to r , the curve representing the variation in velocity between r and $r+e$ is a straight line.

Experiments have shown that the torque M varies proportionally to the velocity and that we have:

$$M = \mu \cdot \frac{SR}{e} \cdot V$$

μ [N.s/m²] is called dynamic or absolute viscosity.

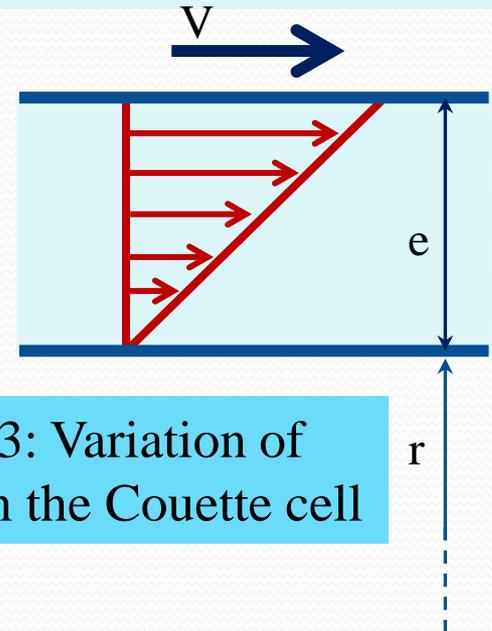


Figure I.13: Variation of velocity in the Couette cell

We define a second viscosity coefficient called kinematic viscosity ν .

With:

$$\nu = \frac{\mu}{\rho} \quad [\text{m}^2/\text{s}]$$



Figure I.13: Kinematic viscosity tester

$$\begin{aligned} \mu_{\text{water at } 20^\circ\text{c}} &= 10^{-3} \text{ N.s/m}^2 \\ \nu_{\text{water et } 20^\circ\text{c}} &= 10^{-6} \text{ m}^2/\text{s} \end{aligned}$$

$$\begin{aligned} \mu_{\text{mercury}} &= 1,554 \cdot 10^{-3} \text{ N.s/m}^2 \\ \nu_{\text{mercury}} &= 0,1147 \cdot 10^{-6} \text{ m}^2/\text{s} \end{aligned}$$

$$\begin{aligned} \mu_{\text{air}} &= 18,5 \cdot 10^{-6} \text{ N.s/m}^2 \\ \nu_{\text{air}} &= 15,6 \cdot 10^{-6} \text{ m}^2/\text{s} \end{aligned}$$

Table I.3 : the influence of temperature on the dynamic viscosity of some common fluids;

	Temperature (°C)	viscosity (Pa s)
Hydrogen	0	$8,4 \times 10^{-6}$
	50	$9,3 \times 10^{-6}$
	100	$10,3 \times 10^{-6}$
Air	0	$17,1 \times 10^{-6}$
	50	$19,4 \times 10^{-6}$
	100	$22,0 \times 10^{-6}$
Water	0	$1,79 \times 10^{-3}$
	20,2	0,001
	50	$0,55 \times 10^{-3}$
	100	$0,28 \times 10^{-3}$
Ice	-13	15×10^{12}
Mercury	20	$1,55 \times 10^{-3}$
Aceton	20	$0,326 \times 10^{-3}$
Ethanol	20	$1,20 \times 10^{-3}$
Methanol	20	$0,59 \times 10^{-3}$
Benzene	20	$0,64 \times 10^{-3}$

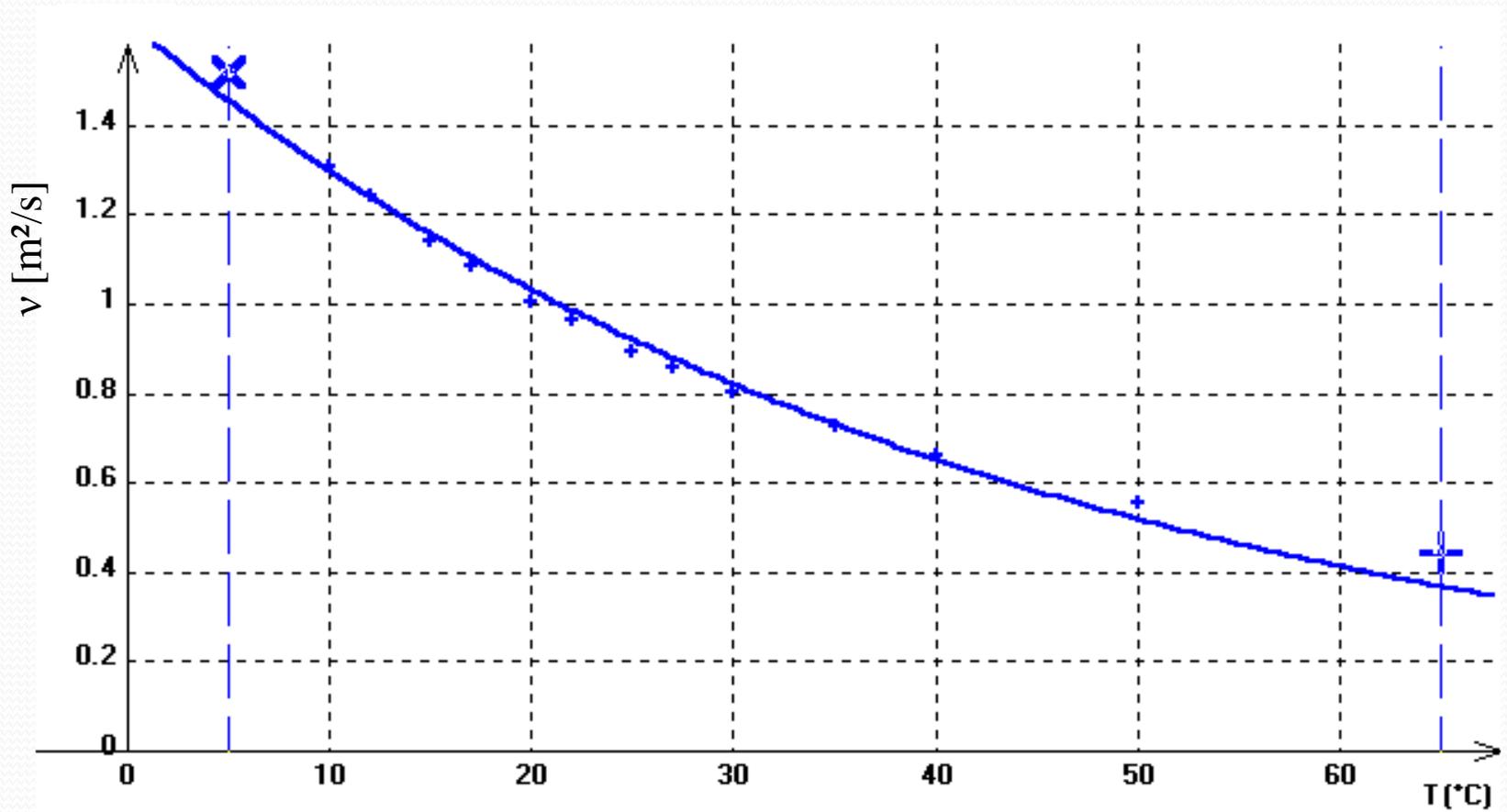
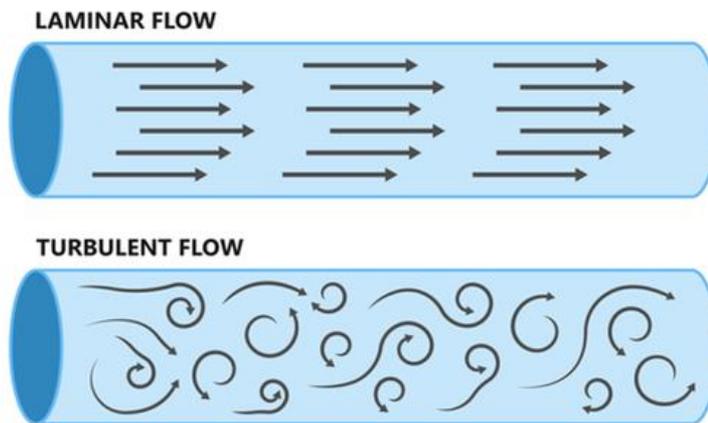


Figure I.14: Water viscosity versus temperature.

A fluid whose viscosity would be zero (non-existent in nature) is called a perfect fluid. Viscosity exists as soon as there is relative movement between particles whether in laminar or turbulent flow.



(a) Laminar flow



(b) Turbulent flow

Figure I.15: Laminar flow / turbulent flow.

I.2.4- Vapor pressure

All liquids tend to evaporate, that is, from liquid to gas. During this transformation, the vapor molecules exert pressure resulting from the separation of the molecules from each other, which is called vapor pressure.

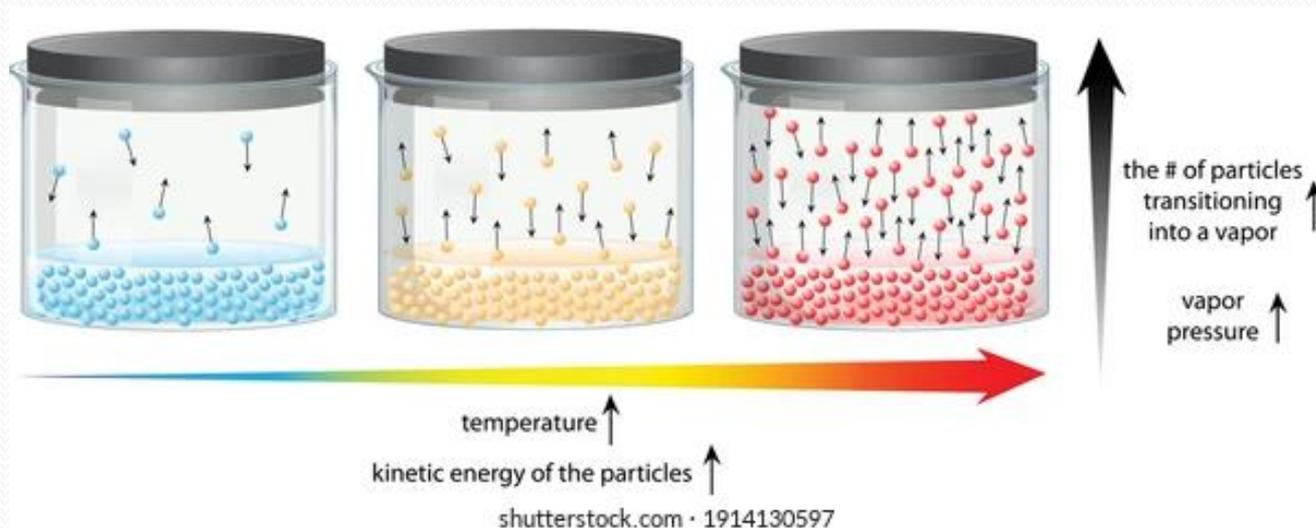


Figure I.16: Effect of temperature on vapor pressure of liquids.

In the case of water, vapor pressure increases with temperature increase.

Table I.4: Temperature Vs Vapor pressure

Temperature C°	Vapor pressure (Pa)
- 10	260
0	610
10	1230
20	2338
40	7376
60	19915
80	47342
100	101325 (1atm)
120	198536



Figure I.17: Vapor pressure of water below 100°C

Saturating vapor pressure is the pressure at which the gaseous phase of this substance is in equilibrium with its liquid or solid phase. It depends exclusively on the temperature.

Ordinary evaporation is a surface phenomenon - some molecules have enough kinetic energy to escape. If the container is closed, an equilibrium is reached where an equal number of molecules return to the surface. The pressure of this equilibrium is called the saturation vapor pressure

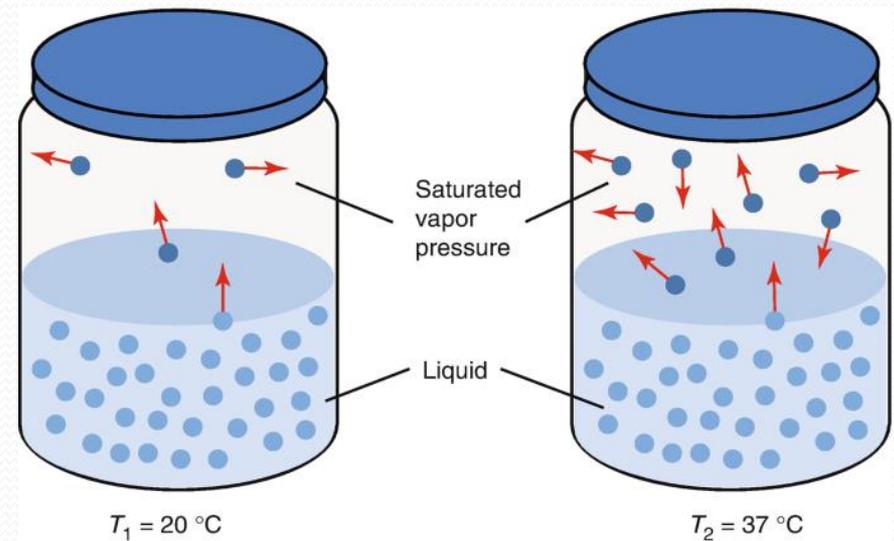


Figure I.18: Saturating vapor pressure

When the partial pressure of vapor is equal to the saturation vapor pressure of a substance, the gas, liquid, or solid phases are in equilibrium.

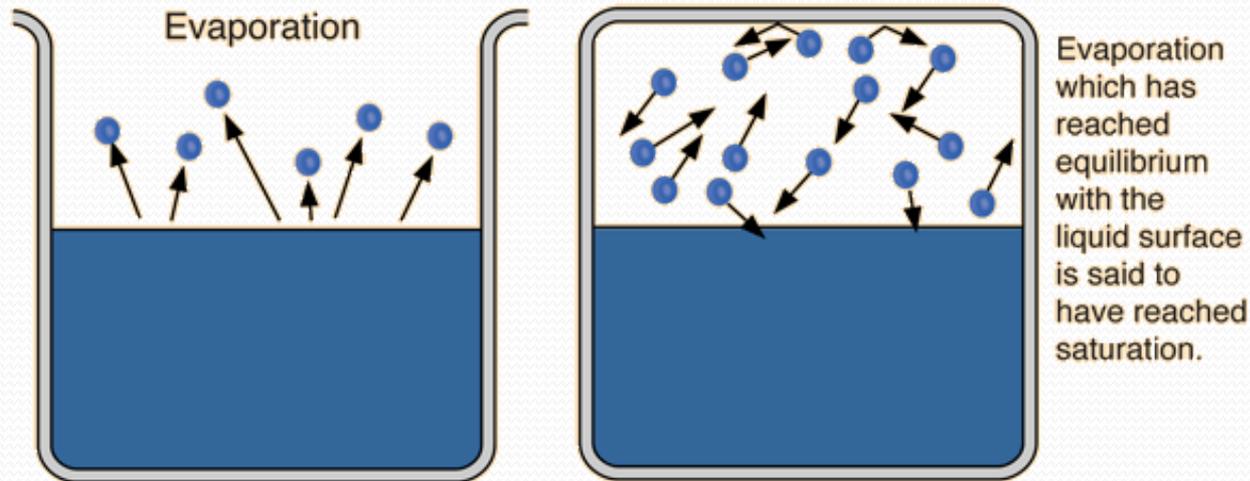


Figure I.19: Difference between evaporation and boiling

If the partial pressure of the vapor exceeds the saturation vapor pressure, then liquefaction or condensation occurs. From an equilibrium situation, this can be done by increasing the partial pressure of the vapor (for example by decreasing the volume), or by decreasing the saturation vapor pressure, i.e. by decreasing the temperature.



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Figure I.20: Pressure cooker:
operating principle

I.2.5- Surface tension

In a liquid, intermolecular forces are forces whose action occurs over distances of the order of magnitude of the dimensions of the molecules. If one of these is drowned in the middle of similar molecules, the actions of the latter oppose each other and the molecule is in equilibrium.

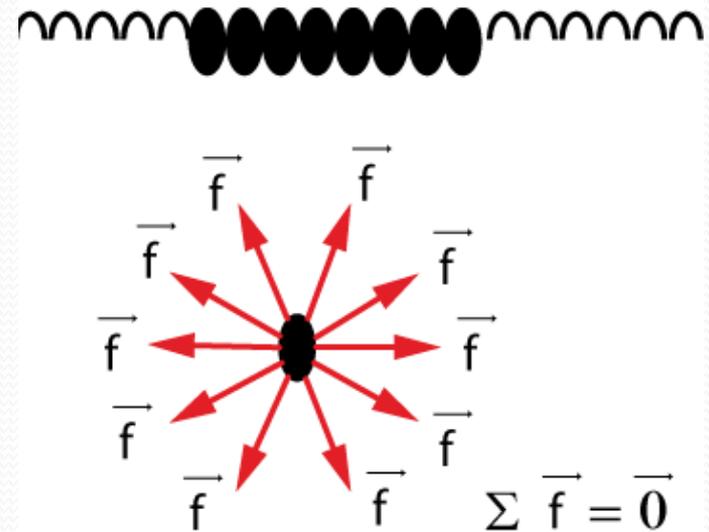


Figure I.21: Surface tension for a submerged molecule

If, on the other hand, we consider a molecule located at the boundary (or interface) of the fluid and another body, it is subjected to the action of different molecules: its energy balance leads us to consider a new type of force acting on the fluid but localized on the interface. Reduced to the unit area of the interface, this is the surface tension or interfacial tension.

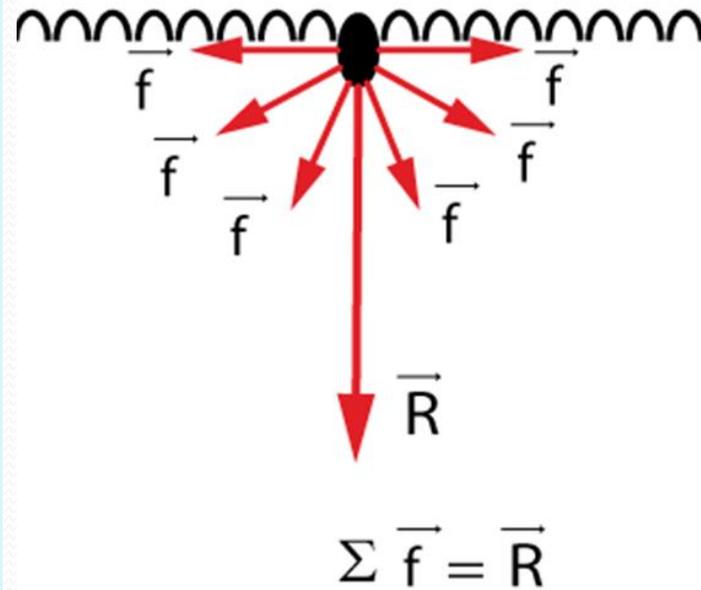


Figure I.22: Surface tension for an interface molecule

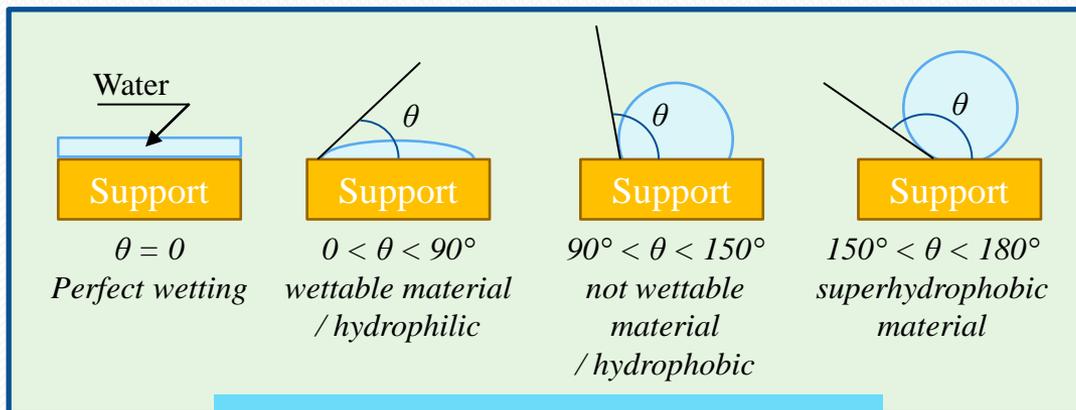


Figure I.23: Surface wettability

This tension always tends to reduce the area of the interface: it is for example the basis of the quasi-spherical shape of fine liquid droplets suspended in a gas.



The surface tension σ is expressed in units of force per unit of length and its value is relative to the interface between two given fluids. Thus, for an interface:

air-water at 20°C	$\sigma = 0,0728 \text{ N/m}$
air-mercury	$\sigma = 0.487 \text{ N/m}$
water-mercury	$\sigma = 0,375 \text{ N/m}$

I.3 Characterization of forces in a flow

I.3.1- The forces

The forces that act on a finite volume of fluid are of two types:

- Volume forces: exerted on each element of volume dV (weight, inertial force, magnetic, etc.)
- Surface forces: exerted on each element of surface dS (pressure, viscous friction, etc.).

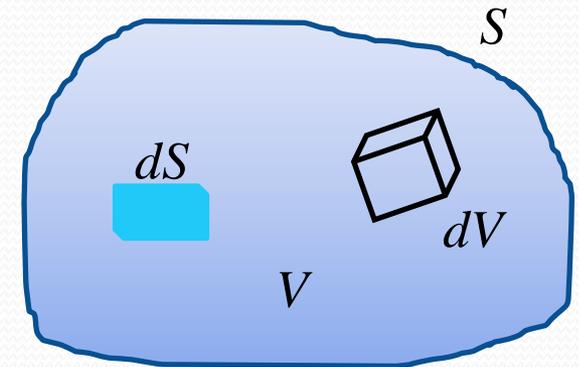


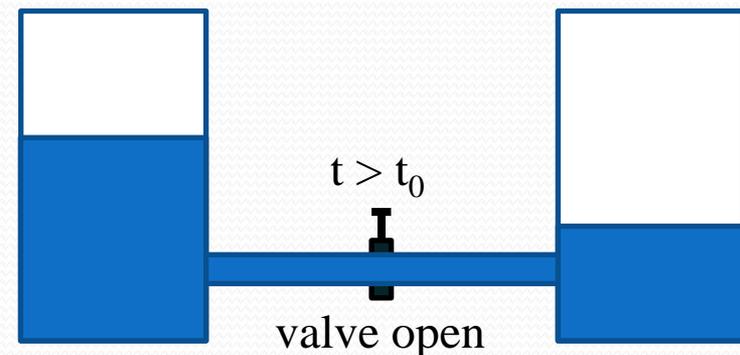
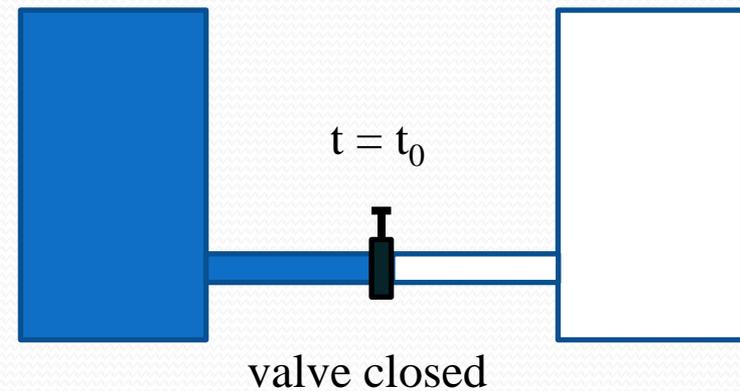
Figure I.23: Surface element ; volume elemen

I.3.1.1- Volume forces:

a) **The forces of gravity:** coming from gravity.

b) **Pure acceleration forces:** These forces come from the variation of velocity over time. If two tanks are at the same altitude, one full of water and the other empty, connected by a horizontal pipe of constant diameter and equipped with a valve. Once the valve is opened, the water will flow. The variation in velocity creates a pure acceleration force within the flow.

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c) Convective acceleration forces:

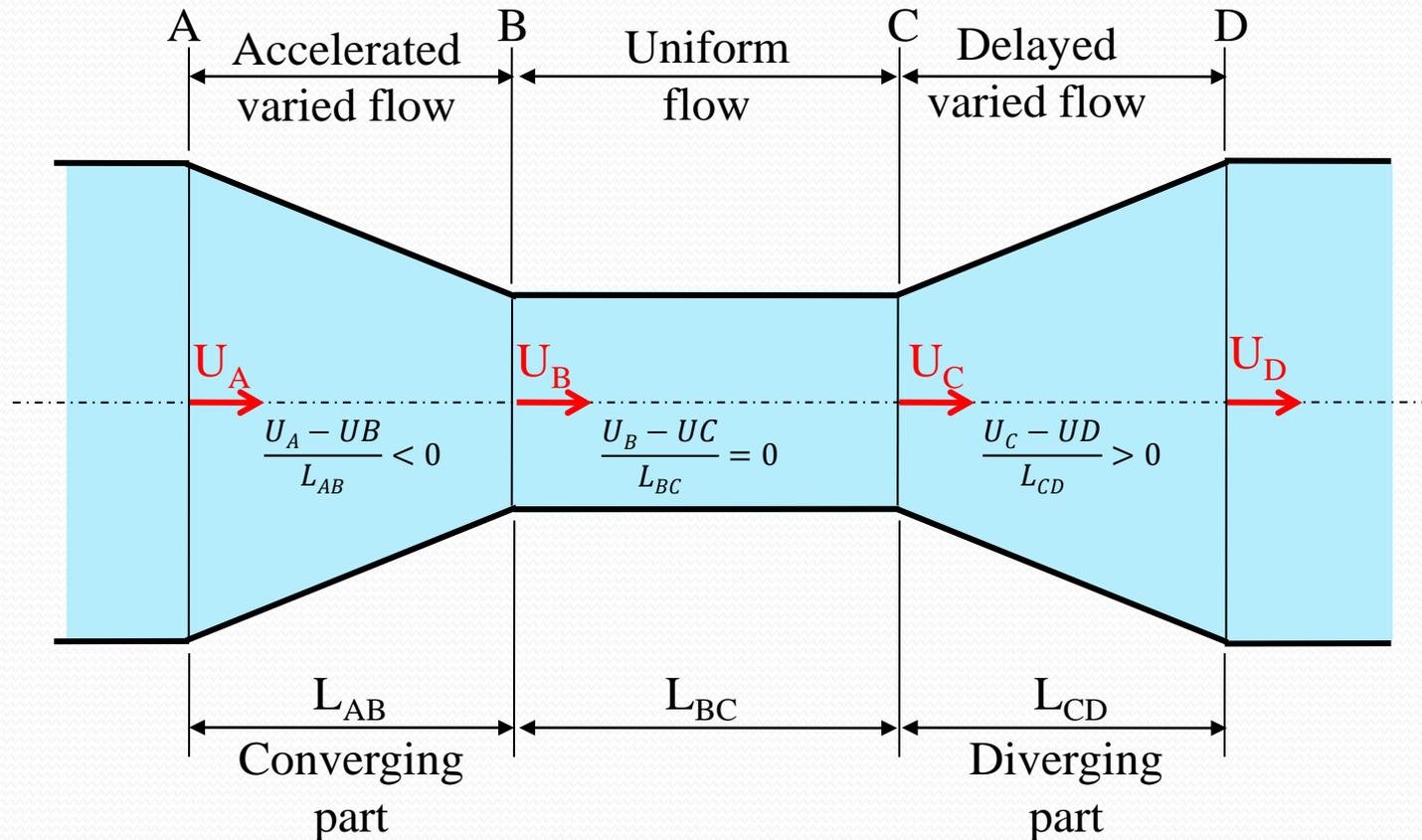
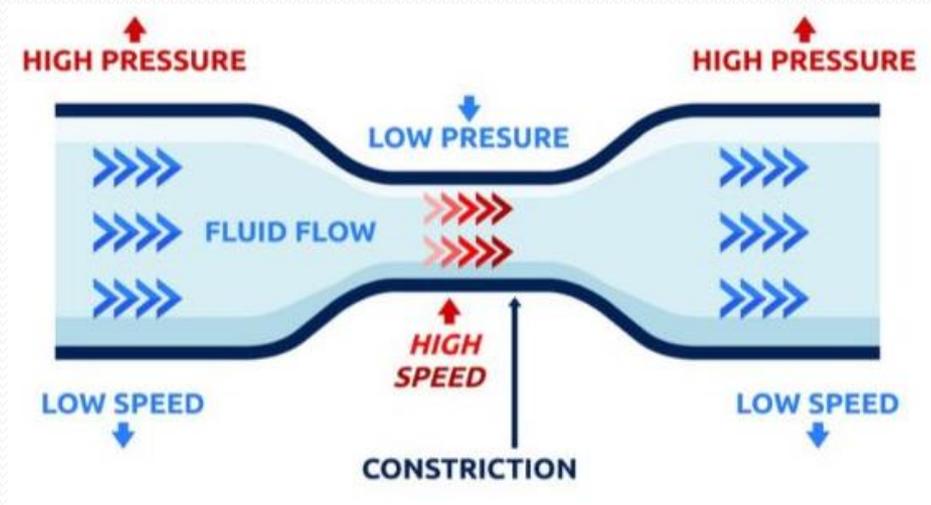


Figure I.24: Convective acceleration forces

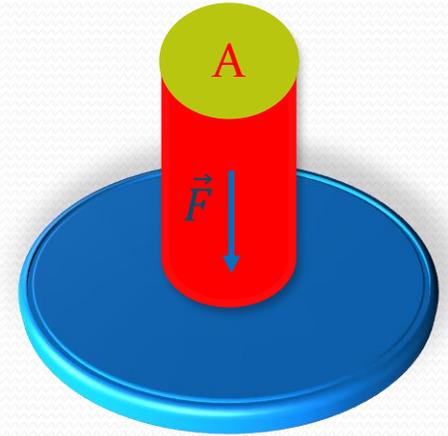
These forces come from the variation of velocity in space. In the inlet of a pipe (constant diameter before point A), the flow does not vary in time ($\frac{\partial U}{\partial t}=0$). As the flow is permanent (inlet flow = outlet flow), and the section S_A is larger than the section S_B , therefore $U_A < U_B$. This variation in speed will cause an acceleration, which in turn will generate a force that will be called: convection force.



I.3.1.2- surface forces

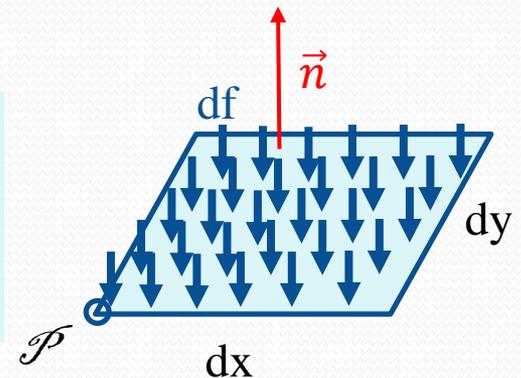
a) Pressure forces

The pressure P is the ratio of a force F acting perpendicular to the surface A of a fluid.



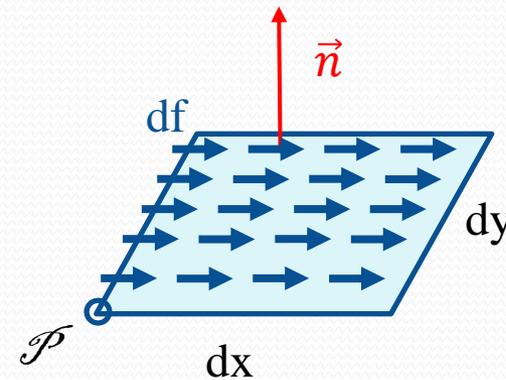
$$P = \frac{df}{dx.dy} = \frac{dF}{dS} = \frac{F}{A}$$

At a point \mathcal{P} of the fluid, the pressure is identical whatever the orientation of \vec{n} the normal to the surface.



b) Viscosity forces

In a fluid the molecules are in relative movement, which generates friction forces due to viscosity.

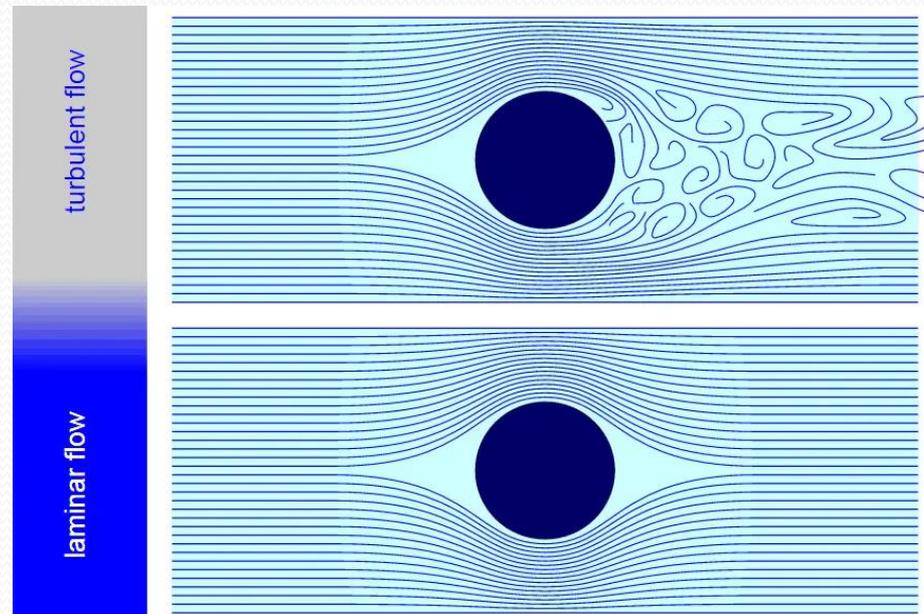


c) Friction forces due to turbulence

Turbulence has an important role in the flow of real fluids. In turbulent movement, the intensity and direction of the point velocity vary without any regularity. Turbulence tends to slow down the flow.



Figure I.25: Laminar flow Vs Turbulent flow



I.3.2- Importance of different forces

The relative importance of the different forces acting on a liquid is parameterized by non-dimensional numbers representing the ratios between these forces. Dimensional analysis makes it possible to simplify these ratios.

The different dimensions used are:

L: length,
T: time,
 ρ : density.



I.3.2.1- Euler's number

This number represents the ratio between the inertial force (**Ma**) and the pressure forces (**PA**).



$$\frac{Ma}{PA} = \frac{(\rho L^3)(L/T^2)}{P L^2} = \rho \frac{V^2}{P}$$

With:

M : the mass,

a : acceleration of the fluid,

P: pressure,

A: surface of application of the pressure,

V: characteristic velocity of the flow.

This number is used in calculations of the forces (static and dynamic) of water on a wall or structure..

I.3.2.2- Reynolds number



This number represents the ratio between the inertial force (**Ma**) and the viscosity forces (**τA**).

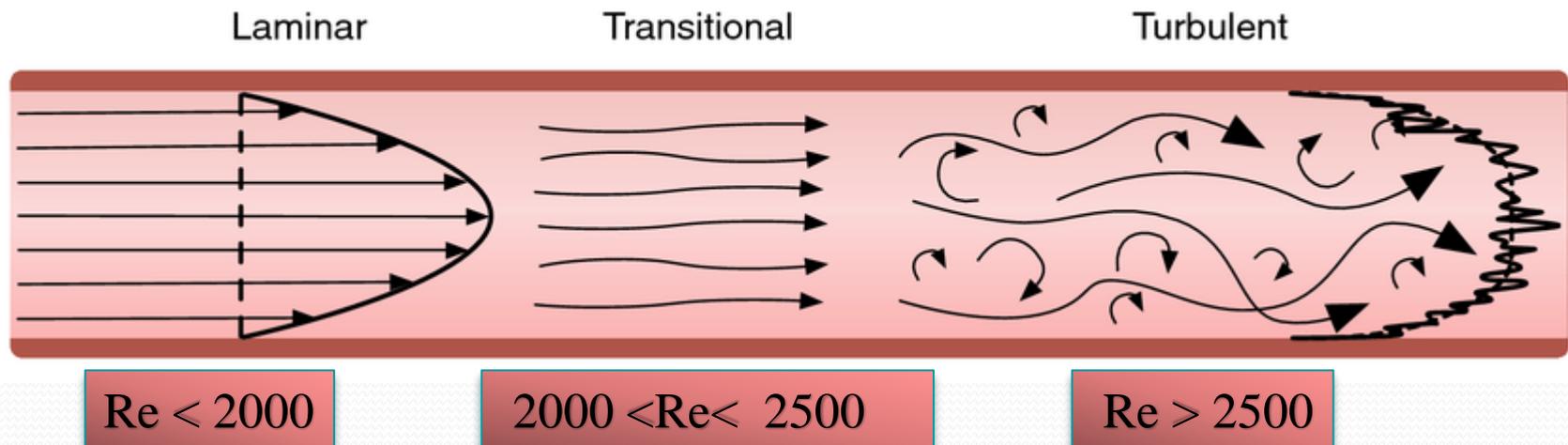
$$\frac{Ma}{\tau A} = \frac{(\rho L^3)(L/T^2)}{\mu \left(\frac{L}{T}\right) L^2} = \rho \frac{VL}{\mu}$$

With:

τ : the friction force per unit area,

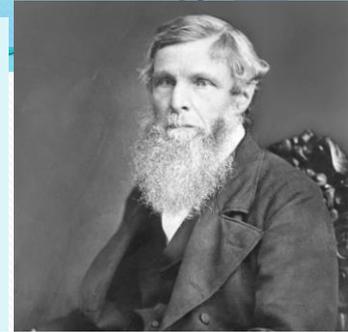
A: surface area of application of the friction force,

L: characteristic length of the flow.



I.3.2.3- Froude number

This number represents the ratio between the inertial force (**Ma**) and the forces of gravity (**Mg**).



$$\frac{Ma}{Mg} = \frac{(\rho L^3)(L/T^2)}{(\rho L^3)g} = \frac{\rho L^2 V^2}{(\rho L^3)g} = \frac{V^2}{Lg}$$

With:

g : acceleration of gravity,

L : characteristic length of the flow,

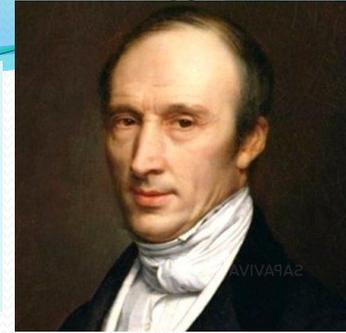
V : characteristic velocity of the flow.



The Froude number allows us to distinguish the fluvial or torrential regime from free surface flows.

I.3.2.3- Cauchy Number

It is the ratio between the inertial force (**Ma**) and the elastic forces (**EA**).



$$\frac{Ma}{EA} = \frac{(\rho L^3)(L/T^2)}{EL^2} = \frac{\rho V^2}{E}$$

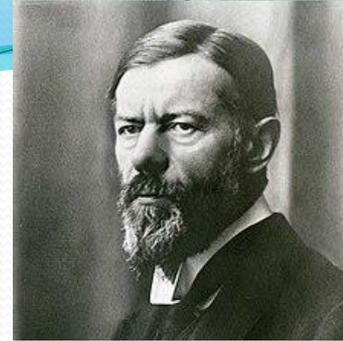
With:

- E: modulus of elasticity,
- L: characteristic length of the flow,
- V: characteristic velocity of the flow.

This number allows us to characterize the relative importance of inertial forces to hydrostatic forces in a fluid flow situation

I.3.2.5- The Weber number

It is the ratio between the inertial force (**Ma**) and the surface tension forces (**σL**).



$$\frac{Ma}{\sigma L} = \frac{(\rho L^3)(L/T^2)}{\sigma L} = \frac{\rho L V^2}{\sigma}$$

With:

σ : surface tension,

V: characteristic velocity of the flow.



The dimensionless Weber number represents the ratio of disruptive hydrodynamic forces to the stabilizing surface tension force

I.4- Basic equations

Determining the characteristics of a fluid movement consists of finding the pressure and velocity at every point. To do this, it is necessary to establish the equilibrium (or conservation) equations between the different forces acting on any volume of the fluid.

These equations are:

□ The continuity equation or equation of conservation of the mass (m) of the fluid, which is written as follows:

$$\frac{dm}{dt} = 0$$

□ The equation of the quantity of movement which reflects the balance between the sum of the external forces (F) which exert an influence on the fluid and the rate of variation of the quantity of movement of the fluid of mass (m), and which is written as follows:

$$\frac{d(m.V)}{dt} = \sum F$$

I.5-Mathematical tools

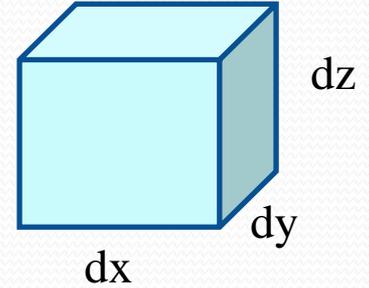
Volume element: $dV = dx \cdot dy \cdot dz$

Partial derivative: $\frac{\partial}{\partial x}$

Total derivative: $dP = \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz$

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial t}$$

Gradient of a scalar: $\overrightarrow{grad}(f) = \underline{grad}(f) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$



Gradient of a vector: $\underline{\underline{grad}}(\underline{V}) = \underline{\underline{grad}} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} & \frac{\partial V_x}{\partial z} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} & \frac{\partial V_y}{\partial z} \\ \frac{\partial V_z}{\partial x} & \frac{\partial V_z}{\partial y} & \frac{\partial V_z}{\partial z} \end{bmatrix}$

Divergence of a vector: $\text{div}(\underline{V}) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

Rotational: $\underline{\underline{Rot}}(\underline{V}) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{bmatrix}$

Chapter II : HYDROSTATIC

II.1 Definition

Hydrostatics is a branch of physics that studies the equilibrium conditions of liquids and the distribution of pressures they transmit at rest (*repos*).

The hydrostatic properties of a liquid are not constant and the main factors influencing it are the density of the liquid and the local gravity. Both of these quantities need to be known in order to determine the hydrostatic pressure of a particular liquid.

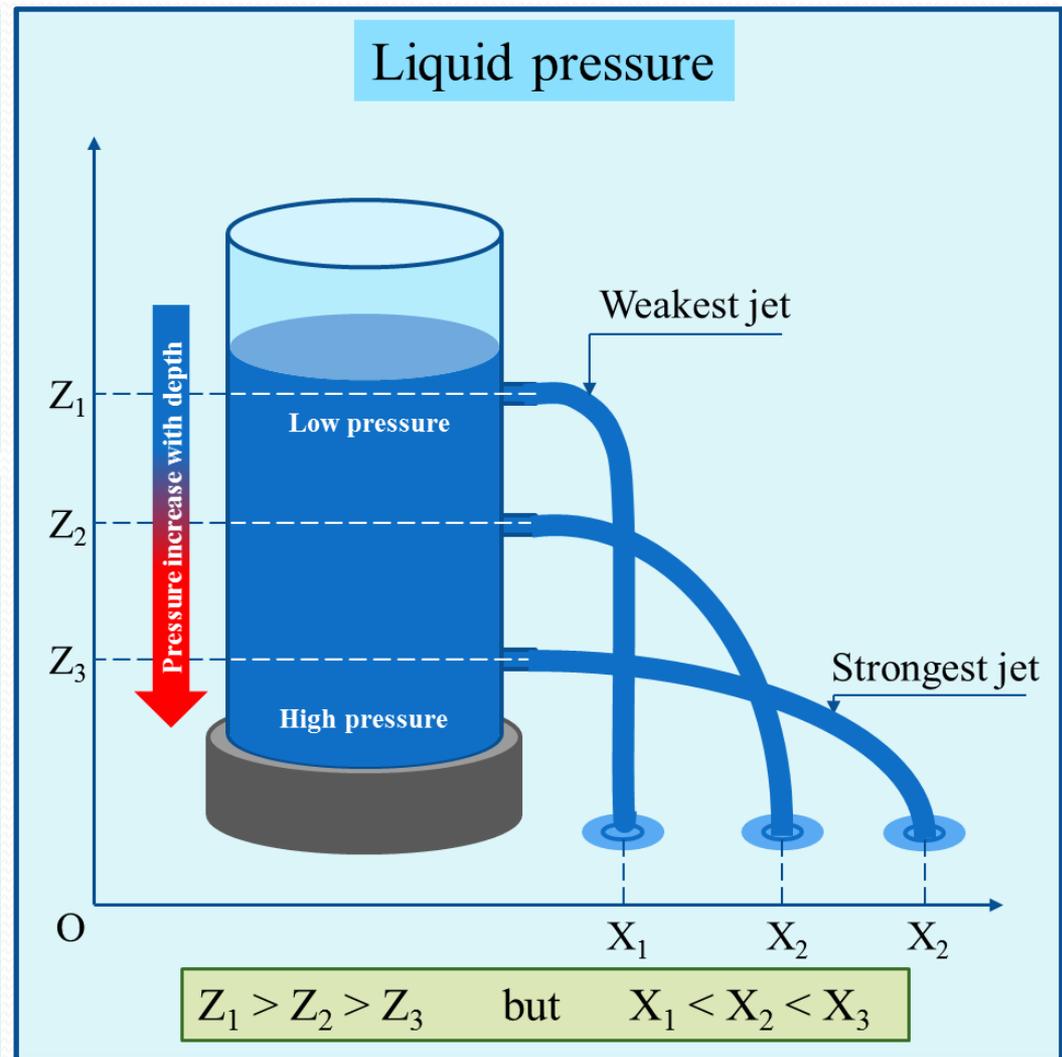


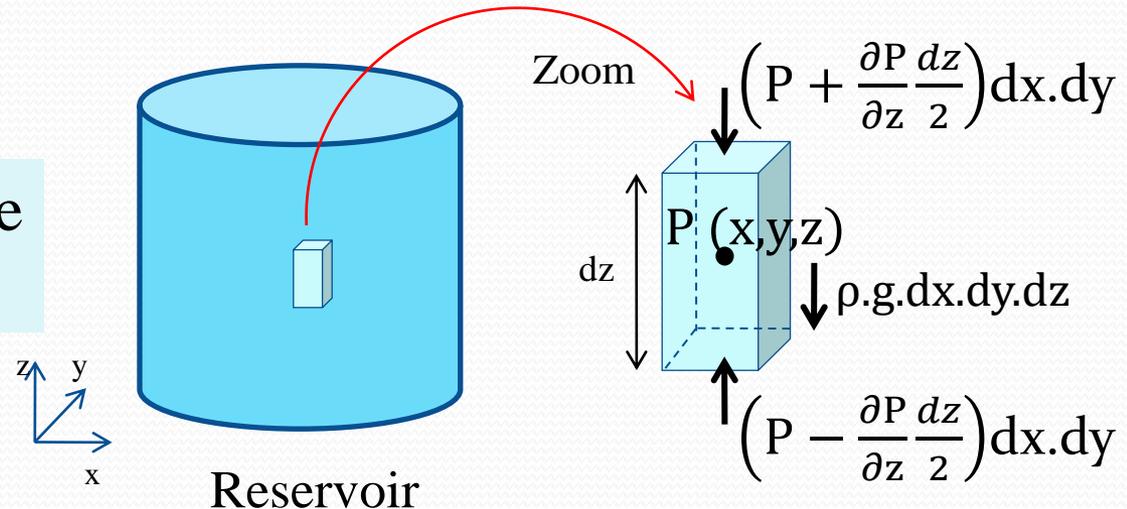
Figure II.1: Liquid pressure

II.2- Fundamental relations of hydrostatics

All the forces that act on a fluid element are of two types: **volume forces and surface forces.**

Consider a fluid at rest in a reservoir, from which we extract a small volume of parallelepiped shape with a vertical axis z . Let P be the pressure at its center.

This volume is subject to the following vertical forces:



a) Volume forces :

- **The forces of gravity:** there is only one, and it is written as follows:

$$F = \rho.g.(dx.dy.dz)$$

- **Pure acceleration forces:** are zero because the fluid is at rest ($v = 0$).
- **Convective acceleration forces:** are zero because the fluid is at rest ($v = 0$).

b) Surface forces:

➤ **Pressure forces:** in this case, there are two pressure forces; one acts on the top surface and the other on the bottom surface, and they are written as follows:

✓ Pressure forces on the bottom surface:

$$F_{pi} = \left(P - \frac{\partial P}{\partial z} \frac{dz}{2} \right) dx \cdot dy$$

✓ Pressure forces on the upper surface:

$$F_{ps} = - \left(P + \frac{\partial P}{\partial z} \frac{dz}{2} \right) dx \cdot dy$$

As the fluid is at rest, the viscosity forces and the turbulence forces are zero, because there is no relative velocity between the particles of the fluid.

In equilibrium of all forces, the hydrostatic equation is written as follows:

$$\sum F = \text{Forces d'inertie}$$

$$-\rho \cdot g \cdot (dx \cdot dy \cdot dz) + \left(P - \frac{\partial P}{\partial z} \frac{dz}{2} \right) dx \cdot dy - \left(P + \frac{\partial P}{\partial z} \frac{dz}{2} \right) dx \cdot dy = 0$$

And we shall have:

$$-\rho \cdot g - \frac{\partial P}{\partial z} = 0$$

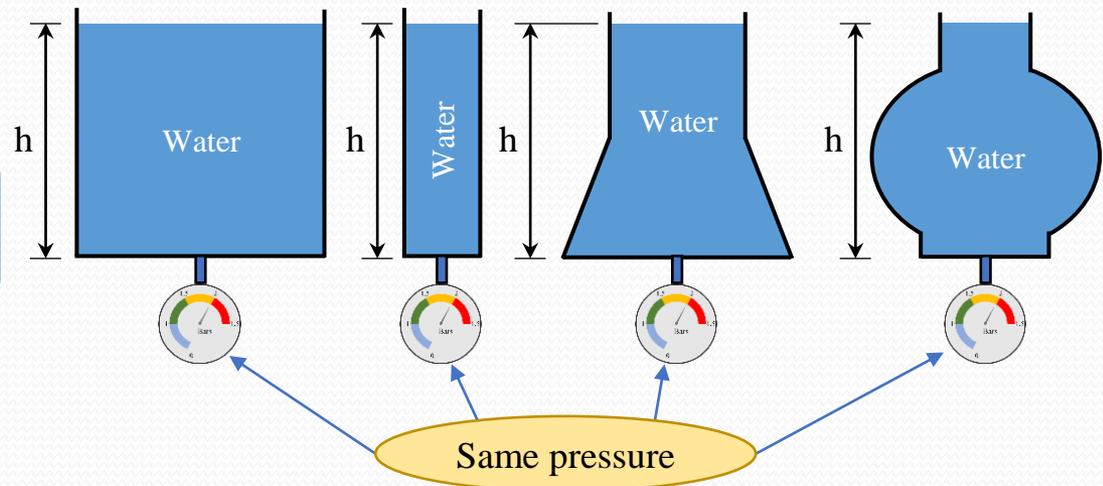
Regarding the other two directions, the equilibrium equations give:

$$\frac{\partial P}{\partial x} = 0 \quad \& \quad \frac{\partial P}{\partial y} = 0$$

By writing these three equations, we see that the pressure does not vary in the x and y directions, which means that the pressures are equal at all points on a horizontal plane. This is verified as long as we remain in the same fluid (ρ is constant).

Therefore the pressure depends on z, and we can write that:

$$dP = -\rho \cdot g \cdot dz$$



II.3- Pressure variation

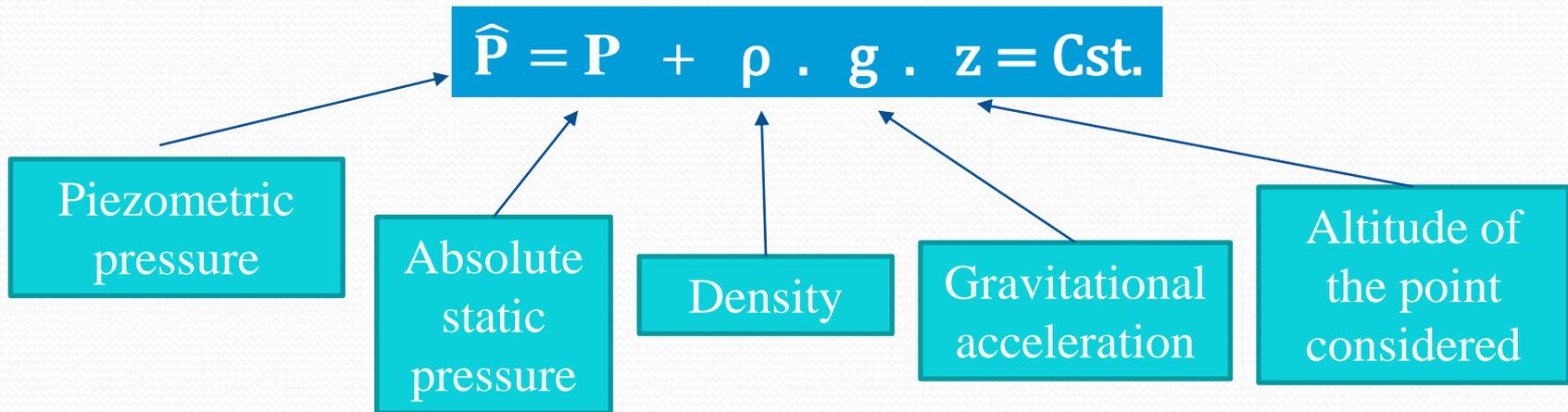
In an incompressible fluid ($\rho = \text{constant}$), from the equation:

$$dP + \rho \cdot g \cdot dz = 0$$

We will get:

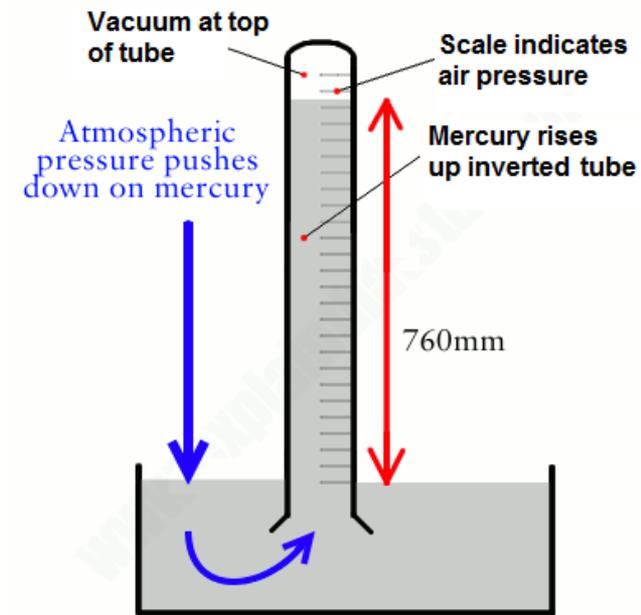
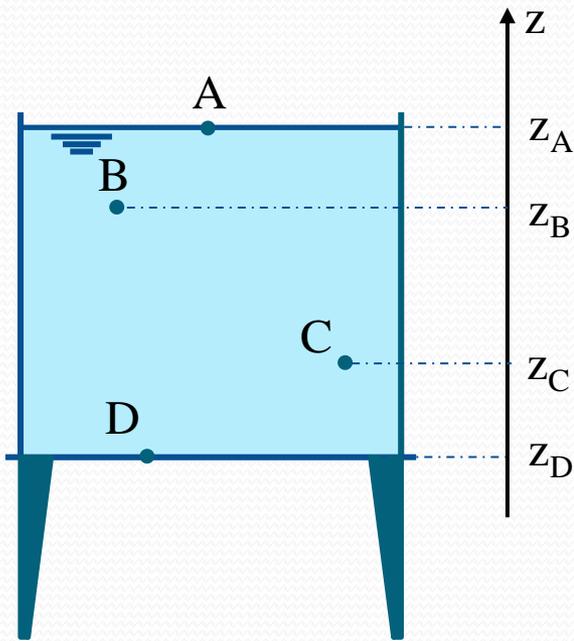
$$P + \rho \cdot g \cdot z = \text{Cst.}$$

For all fluids at rest, we get to write the follows equation:



Which leads us to say that at all points of a fluid at rest, the piezometric pressure is constant, and we will have:

$$\hat{P}_A = \hat{P}_B = \hat{P}_C = \hat{P}_D$$



Measurement units

$$1 \text{ Pa} = 1 \text{ N/m}^2$$

$$1 \text{ bar} = 10^5 \text{ Pa}$$

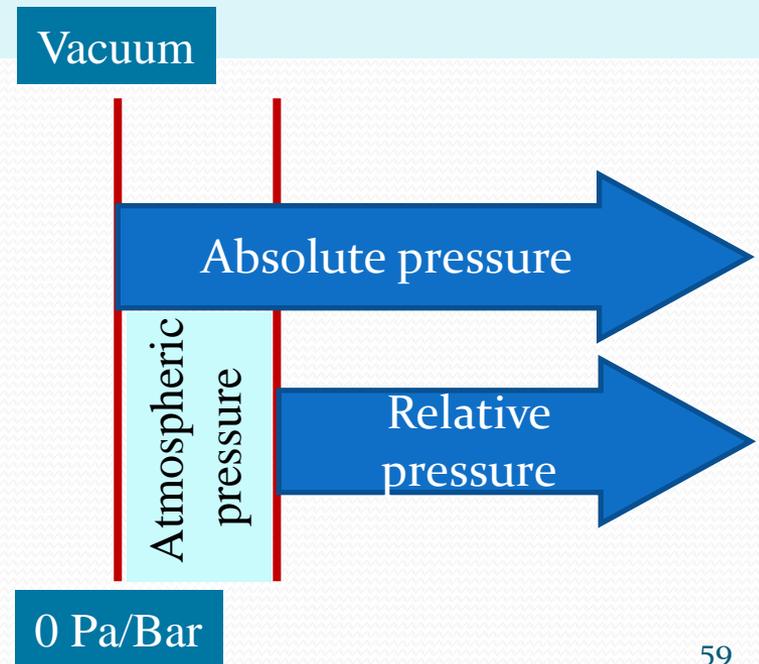
$$1 \text{ atm} = 101396 \text{ Pa} = 1,013 \text{ bar} = 10,33 \text{ mce} = 760 \text{ mm Hg}$$

Absolute pressure Vs Relative pressure

Absolute pressure is the pressure measured relative to absolute vacuum (i.e. the total absence of matter). It is always positive.

Relative pressure (Gauge pressure) is defined in relation to the atmospheric pressure existing at the time of measurement. This pressure can take a positive value if the pressure is higher than atmospheric pressure or a negative value if the pressure is lower than atmospheric pressure.

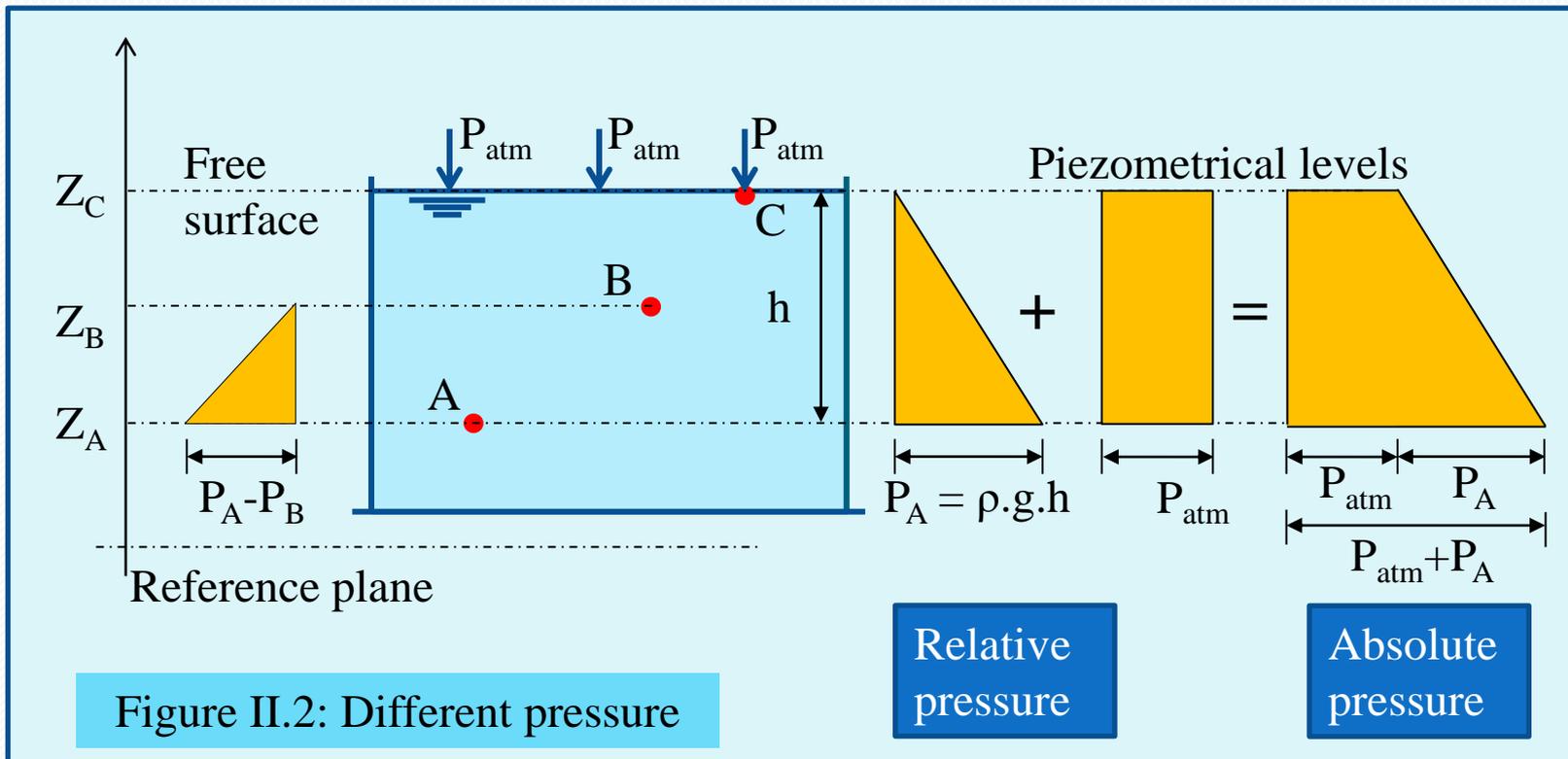
Both types of pressure physically correspond to the same pressure, they are simply expressed on scales with different "zeros".



The following relation allows us to move from one to the other :

$$P_{\text{absolute}} = P_{\text{relative}} + P_{\text{atmospheric}}$$

To illustrate this, we'll take a reservoir filled with water, the free surface of which is subject to atmospheric pressure. (P_{atm}).



If we apply the equation of hydrostatics between points A and C, we get:

$$P_A + \rho \cdot g \cdot z_A = P_C + \rho \cdot g \cdot z_C = \text{Cst}$$

So $P_A = P_C + \rho \cdot g \cdot (z_C - z_A)$

if we consider that: $z_C - z_A = h$

and knowing that $P_C = P_{\text{atm}}$

we'll have:

$$P_A = P_{\text{atm}} + \rho \cdot g \cdot h$$

This is the absolute pressure measured at point A

Pressure measuring instruments give relative pressure values (also called manometric or gauge). In our case, the relative pressure at point A is:

$$P'_A = P_A - P_{\text{atm}} = \rho \cdot g \cdot h$$

II.4. Pascal's principle and its applications

Statement



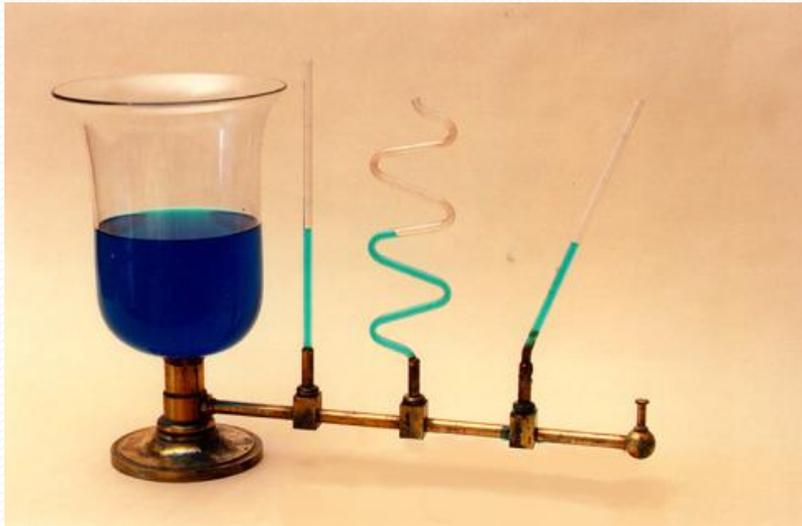
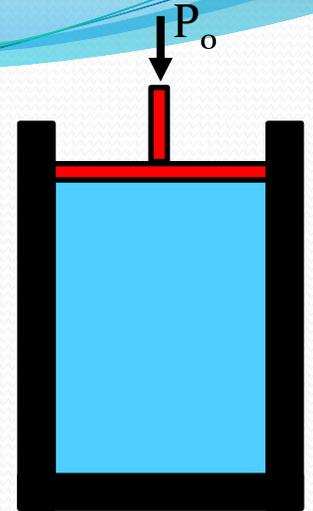
In an incompressible fluid in equilibrium, any variation in pressure at one point causes the same variation in pressure at any other point.

In other words, an applied pressure is transmitted equally to the entire mass of the liquid in all directions.

Blaise PASCAL

19 juin 1623 - 19 août 1662

If a pressure P_0 is applied to the surface of a liquid, it will be felt at any point in the liquid, even at the bottom of the container.



The principle of communicating vessels states that the pressure at a given height in a container holding a liquid is independent of the shape of the container.

Hydraulic press

$$F_1 = P_1 \cdot S_1 \quad \& \quad F_2 = P_2 \cdot S_2$$

Applying Pascal's principle, i.e.: $P_1 = P_2$, we get:

$$\frac{F_1}{S_1} = \frac{F_2}{S_2}$$

and so:

$$F_2 = S_2 \frac{F_1}{S_1} = F_1 \left(\frac{S_2}{S_1} \right) \gg \gg \gg F_1$$

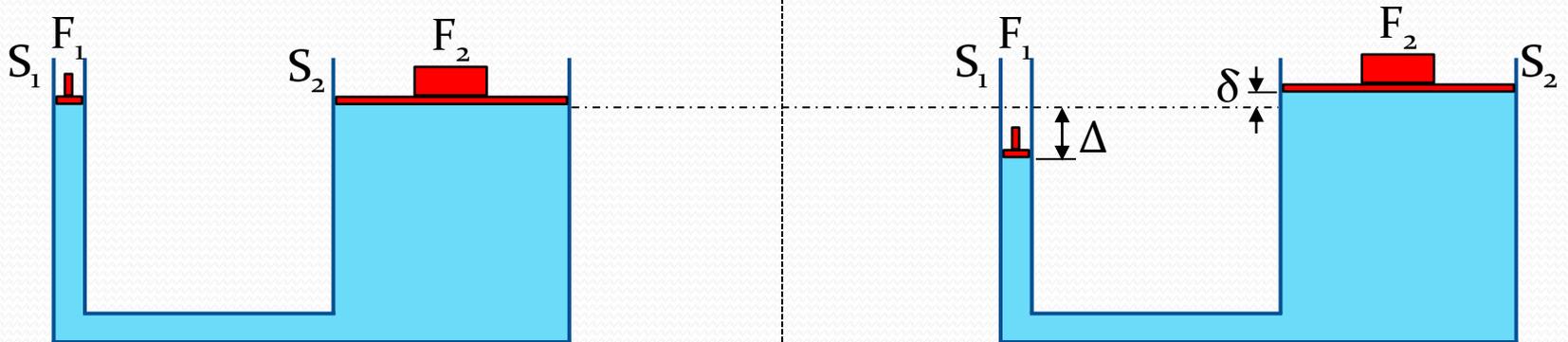


Figure II.3: Hydraulic pressure

II.5. Hydrostatic thrust (pressure) on walls

II.5.1. Definition

The hydrostatic thrust on a wall comes from the pressure forces of the fluid acting on this surface.

Characterization of fluid pressure on the surface depends on:

- ✓ the intensity: the pressure depends on the water depth h ($P = \rho.g.h$),
- ✓ the application zone: the pressure is applied to a surface element (ds),
- ✓ the direction: the pressure is always perpendicular to the surface of application.

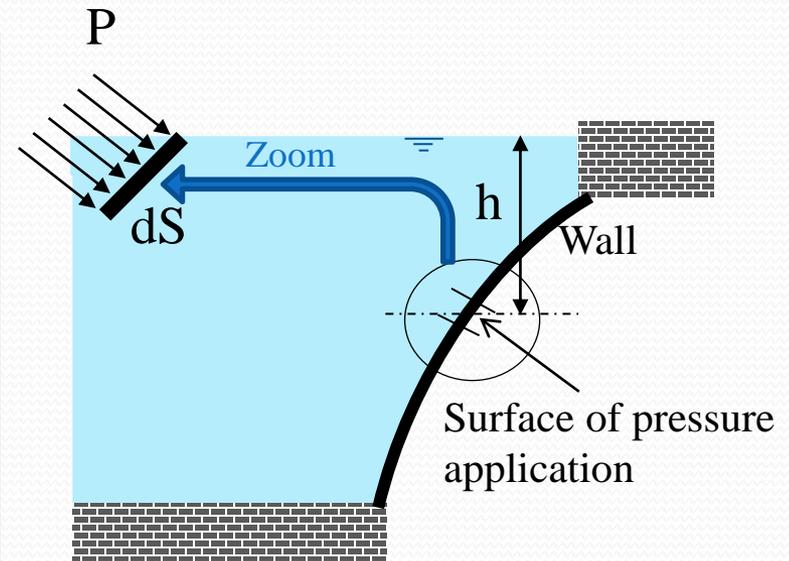


Figure II.4: Hydrostatic pressure

$$\text{Force acting on the element } dS : dF = P.dS = \rho.g.h.dS$$

II.5.2. The action of a fluid on an inclined plane wall:

The figure below represents a plane wall of any section immersed in water. We define a reference (x,y) where the axis (x) is on the free surface and (y) directed downwards.

The point $G(x_G, y_G)$ is the center of gravity of the section. The coordinate system (ξ, η) is defined as being a translation of the coordinate system (x,y) centered in G .

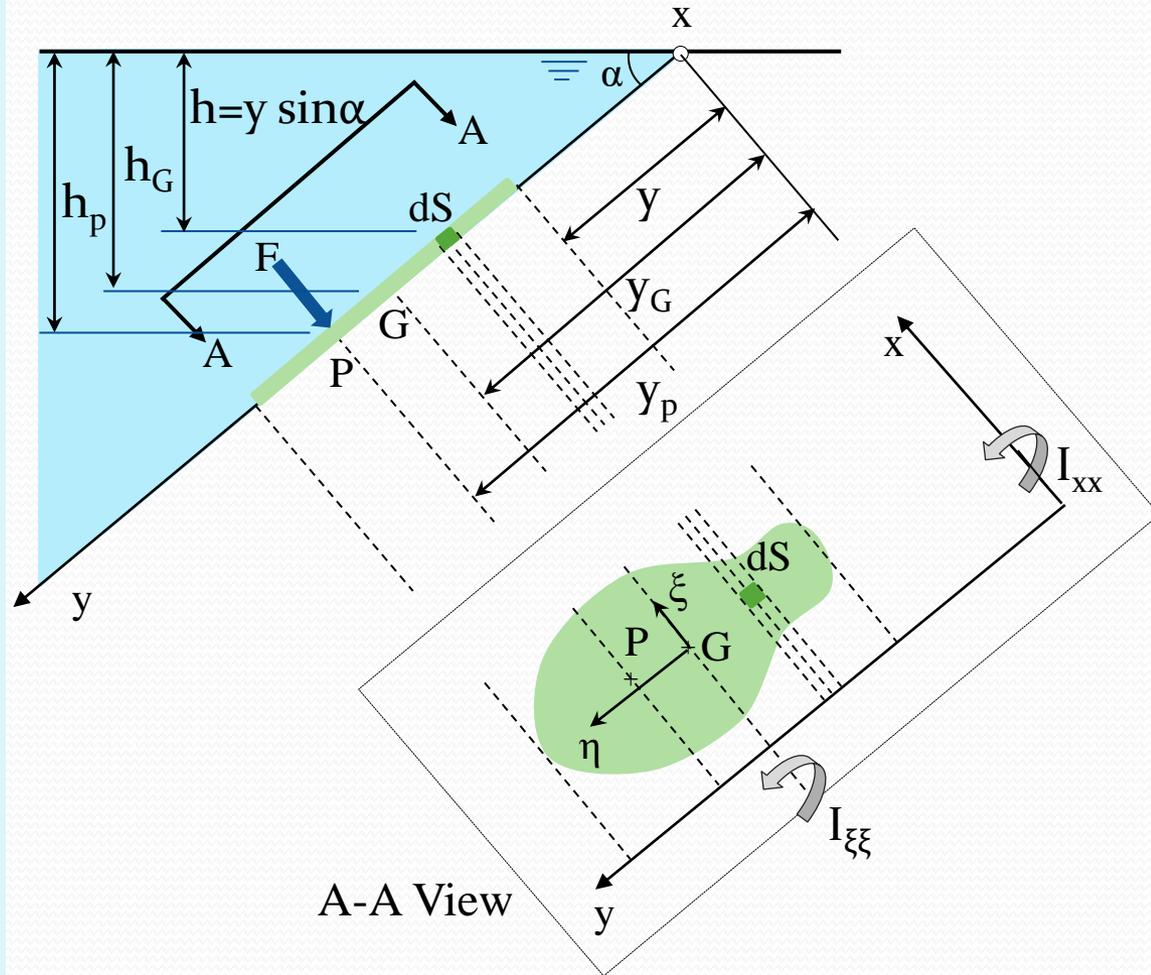


Figure II.5: Hydrostatic pressure on an inclined plane wall

The intensity of the resultant force acting on the surface S is defined by the relation:

$$F = \iint_S dF = \rho \cdot g \iint_S h dS$$

Integrating this equation gives:

$$F = \rho \cdot g \cdot h_G \cdot S$$

h_G : water height from the center of gravity of the immersed wall to the free surface.

S : immersed wall surface.

The action of the water on the plane wall is the result of the pressure forces, which is equal to the pressure at the center of gravity multiplied by the surface area of the submerged wall.

The point of application of the resultant force $P (x_p, y_p)$ is called the center of pressure or center of thrust.

The position of this point is defined by the position of the center of inertia of the elementary surfaces (dS) weighted by the pressure on each surface, which amounts to determining the equivalent moment of the pressure forces, that is to say:

$$x_p \cdot F = \iint x dF \quad \& \quad y_p \cdot F = \iint y dF$$

In the vast majority of cases, the surfaces are symmetrical with respect to the η axis, which means that $x_p = x_G$. Then only the second coordinate remains to be determined.

And so:

$$y_p = \frac{\iint y dF}{F}$$

Using: $dF = \rho g h dS$ & $F = \rho g h_G S$

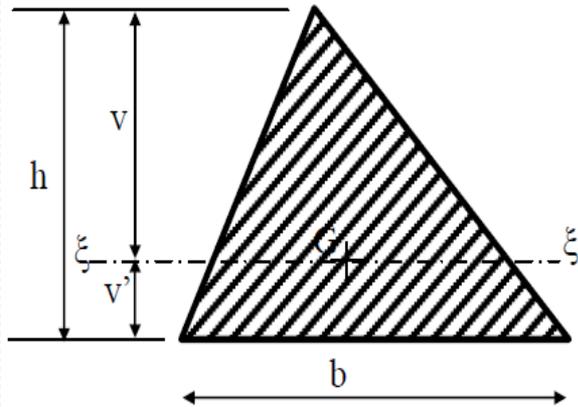
then observing that: $h = y \sin \alpha$

the following expression comes:

$$y_p = \frac{\iint y^2 dS}{\iint y dS} = y_G + \frac{I_{\xi\xi}}{y_G \cdot S}$$

The center of gravity, surface area and inertia for some plane surface shapes.

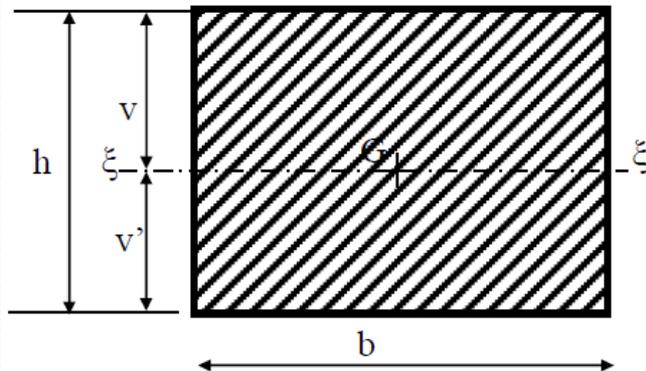
Surface



Position of center of gravity, surface area and inertia

$$v = \frac{2h}{3} ; v' = \frac{h}{3}$$

$$S = \frac{bh}{2} ; I_{\xi\xi} = \frac{bh^3}{36}$$

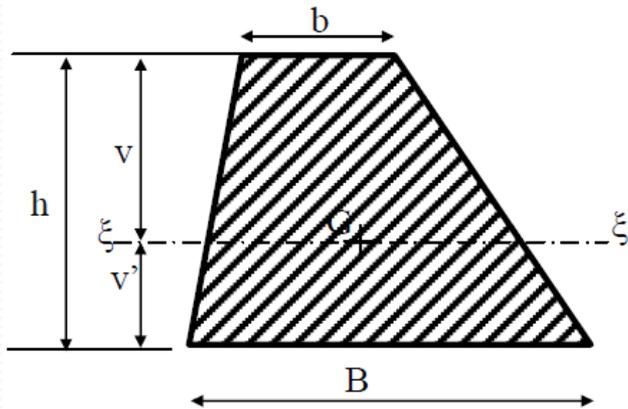


$$v = \frac{h}{2} ; v' = \frac{h}{2}$$

$$S = bh ; I_{\xi\xi} = \frac{bh^3}{12}$$

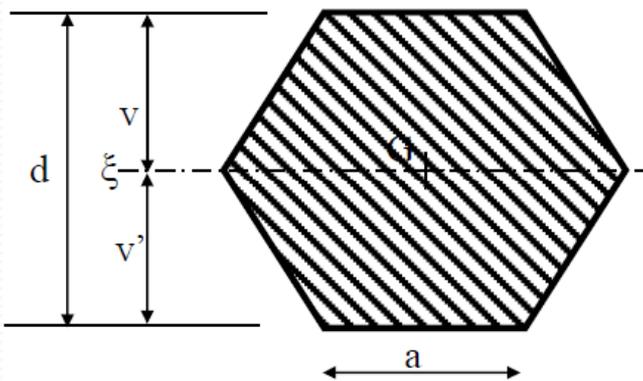
Surface

Position of center of gravity, surface area and inertia



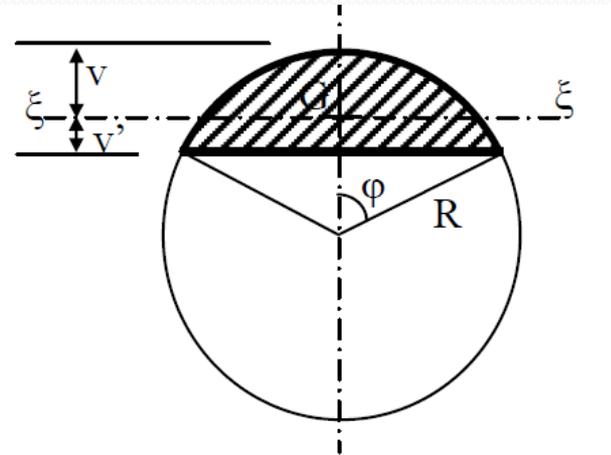
$$v = \frac{h}{3} \left(\frac{2B + b}{B + b} \right) ; v' = \frac{h}{3} \left(\frac{B + 2b}{B + b} \right)$$

$$S = \frac{h}{2} (B + b) ; I_{\xi\xi} = \frac{h^3 (B^2 + 4Bb + b^2)}{36(B + b)}$$



$$v = \frac{d}{2} = \frac{a\sqrt{3}}{2} ; v' = \frac{d}{2} = \frac{a\sqrt{3}}{2}$$

$$S = \frac{d^2 \sqrt{3}}{2} = \frac{3a^2 \sqrt{3}}{2} ; I_{\xi\xi} = \frac{5d^4}{48\sqrt{3}} = \frac{5a^4 \sqrt{3}}{16}$$

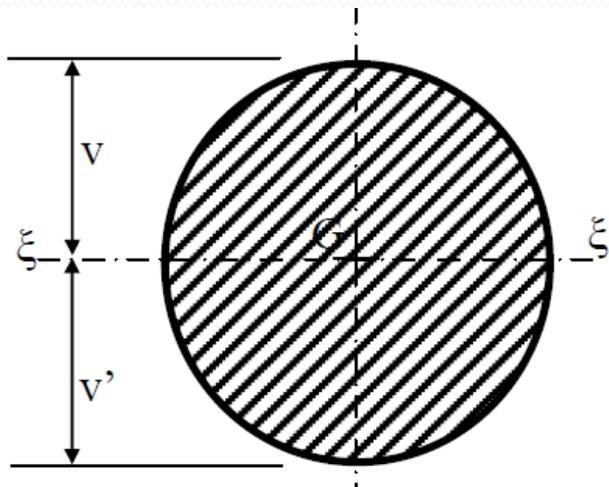


$$v = R \left(1 - \frac{4 \sin^3(\varphi)}{3(2\varphi - \sin(2\varphi))} \right)$$

$$v' = R(1 - \cos(\varphi)) - v$$

$$S = \frac{R^2}{2} (2\varphi - \sin(2\varphi))$$

$$I_{\xi\xi} = \frac{R^4}{16} (4\varphi - \sin(4\varphi)) - \frac{R^4}{9} \frac{(1 - \cos(2\varphi))^3}{(2\varphi - \sin(2\varphi))}$$



$$v = R ; v' = R$$

$$S = \pi R^2 ; I_{\xi\xi} = \frac{\pi R^4}{4}$$

II.5.2. The action of a fluid on a warped surfaces wall:

Let's take the particular case of a wall with a warped surface in the shape of a quarter circle (radius R , length l). In order to integrate, we need to switch to cylindrical coordinates, where the origin of the axes coincides with the center of the circle.

As the forces \vec{dF} don't all have the same direction, so we need to project onto the axes (\vec{Ox}, \vec{Oz}) , and then sum these projections.

$$\hat{P}_B = \hat{P}_M$$

$$\hat{P}_B = P_B + \rho \cdot g \cdot z_B$$

$$\text{with } P_B = P_{\text{atm}} \text{ et } z_B = 0$$

$$\hat{P}_M = P + \rho \cdot g \cdot z$$

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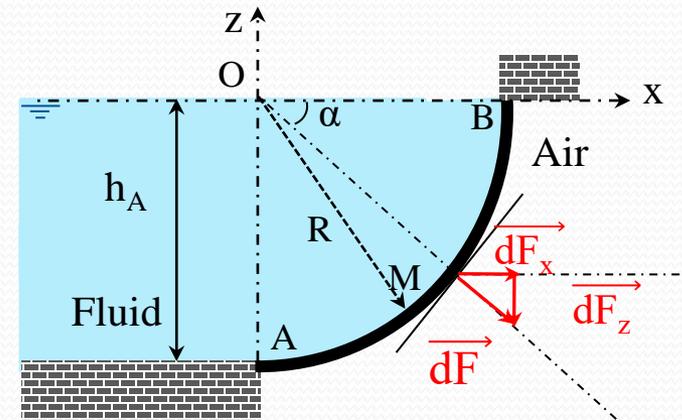


Figure II. 6: Hydrostatic pressure on a warped surfaces wall

So:

$$P_{\text{atm}} = P + \rho \cdot g \cdot z \quad \Rightarrow \quad P - P_{\text{atm}} = - \rho \cdot g \cdot z \quad (\text{avec } z < 0)$$

$$dF = (P - P_{\text{atm}}) dS = - \rho \cdot g \cdot z \cdot dS$$

If α is the angle made by the axis (OM) with the x-axis, we will have:

$$dF_x = dF \cos \alpha \quad \& \quad dF_z = dF \sin \alpha$$

$$z = R \sin \alpha \quad \text{if} \quad -\frac{\pi}{2} \leq \alpha \leq 0$$

If we integrate from 0 to $-\frac{\pi}{2}$, α is decreasing so $d\alpha < 0$ and as $dS > 0$, we can write:

$$dS = -R \cdot d\alpha \cdot l$$

which gives

$$dF_x = (-\rho \cdot g) \cdot (R \sin \alpha) \cdot (-R \cdot d\alpha \cdot l) \cos \alpha = \rho \cdot g \cdot R^2 \cdot l \sin \alpha \cos \alpha d\alpha$$

$$dF_z = (-\rho \cdot g) \cdot (R \sin \alpha) \cdot (-R \cdot d\alpha \cdot l) \sin \alpha = \rho \cdot g \cdot R^2 \cdot l \sin^2 \alpha d\alpha$$

$$\bar{F}_x = \int_0^{-\frac{\pi}{2}} \rho \cdot g \cdot R^2 \cdot l \sin \alpha \underbrace{\cos \alpha \, d\alpha}_{d(\sin \alpha)} = \rho \cdot g \cdot R^2 \cdot l \int_0^{-\frac{\pi}{2}} \sin \alpha \, d(\sin \alpha) = \rho \cdot g \cdot R^2 \cdot l \left[\frac{\sin^2 \alpha}{2} \right]_0^{-\frac{\pi}{2}}$$

$$\bar{F}_x = \rho \cdot g \frac{R^2 \cdot l}{2}$$

$$\bar{F}_z = \rho \cdot g \cdot R^2 \cdot l \int_0^{-\frac{\pi}{2}} \sin^2 \alpha \, d\alpha$$

Let's admit that:

$$X = \int_0^{-\frac{\pi}{2}} \sin^2 \alpha \, d\alpha = \int_0^{-\frac{\pi}{2}} \sin \alpha \, d(-\cos \alpha) = [-\sin \alpha \cdot \cos \alpha]_0^{-\frac{\pi}{2}} - \int_0^{-\frac{\pi}{2}} \cos^2 \alpha \, d\alpha$$

With:

$$[-\sin \alpha \cdot \cos \alpha]_0^{-\frac{\pi}{2}} = 0 \quad \& \quad -\cos^2 \alpha \, d\alpha = (1 - \sin^2 \alpha) \, d\alpha$$

$$\text{Either: } X = \int_0^{-\frac{\pi}{2}} (1 - \sin^2 \alpha) \, d\alpha = -\frac{\pi}{2} - X \quad \Rightarrow \quad X = -\frac{\pi}{4}$$

$$\bar{F}_z = \rho \cdot g \frac{R^2 \cdot l \cdot \pi}{4}$$

As:
$$\frac{R^2 \cdot l \cdot \pi}{4} = V$$

Where

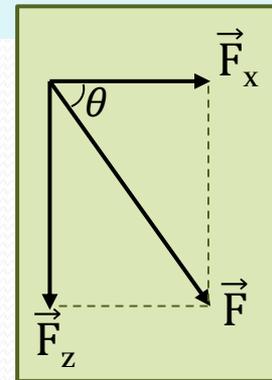
V : volume of the fluid located on the immersed wall.

$$\bar{F}_z = \rho \cdot g \cdot V = \bar{W}$$

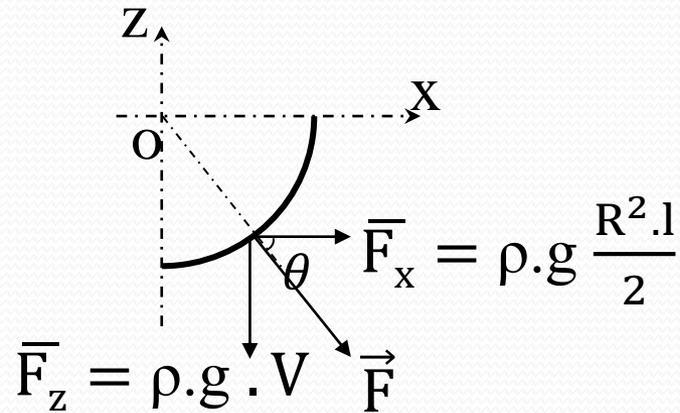
The forces along the horizontal represent the hydrostatic force. The forces along the vertical represent the weight of the water W . By making the vector composition of the forces F_x and F_z , we deduce F .

$$\|\vec{F}\| = \sqrt{F_x^2 + F_z^2}$$
$$\|\vec{F}\| = \rho \cdot g \frac{R^2 \cdot l}{2} \sqrt{1 + \frac{\pi^2}{4}}$$

Direction $\text{tg } \theta = \frac{F_z}{F_x}$



Recap



Hydrostatic thrust on the vertically projected surface which would contain a height R of fluid.

Weight of fluid contained in the quarter of the circle.

II.6. Archimede's thrust (Buoyant force)

Consider a body in equilibrium immersed in a fluid, and is subjected to forces on its six faces.

On the vertical faces, the forces cancel each other out two by two, while on the horizontal faces, we have:

- Upper surface

$$F_1 = -\rho \cdot g \cdot h_1 \cdot S_1$$

- Bottom surface

$$F_2 = \rho \cdot g \cdot h_2 \cdot S_2$$

With: $S_1 = S_2 = S$

Resultant of two forces is equal:

$$F = F_1 + F_2 = \rho \cdot g \cdot S(h_2 - h_1)$$

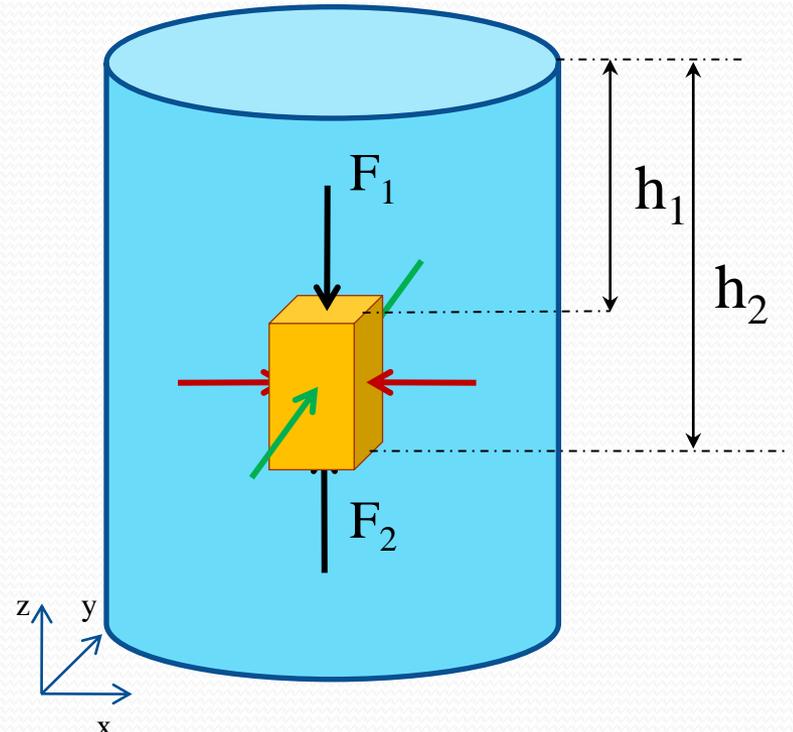
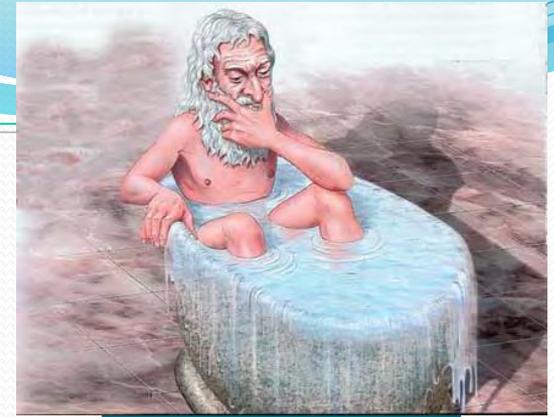


Figure II.7: Archimede's thrust

So

$$F = \rho_{\text{fluid}} \cdot g \cdot V_{\text{solid}}$$

This force is called Archimedes' thrust.



287-212 BC

Archimedes' principle: any body immersed in a fluid in equilibrium is subject to a vertical upward thrust from the fluid, equal to the weight of the volume of fluid displaced and applied to the center of mass of this volume (center of buoyancy).

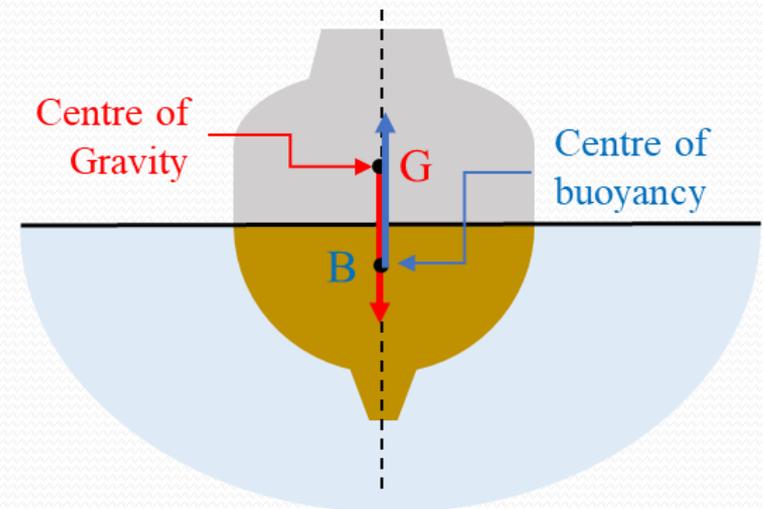


Figure II.8: Centre of gravity Vs centre of buoyancy

Consequences:

- if $F = P$, the body is in equilibrium in the fluid – their respective densities are identical;
- if $F > P$, the body rises to the surface until there is an equilibrium of forces – the density of the body is lower than that of the liquid;
- if $F < P$, the body sinks to the bottom of the tank and exerts a force $P - F$ on the bottom of the tank – the density of the body is higher than that of the liquid

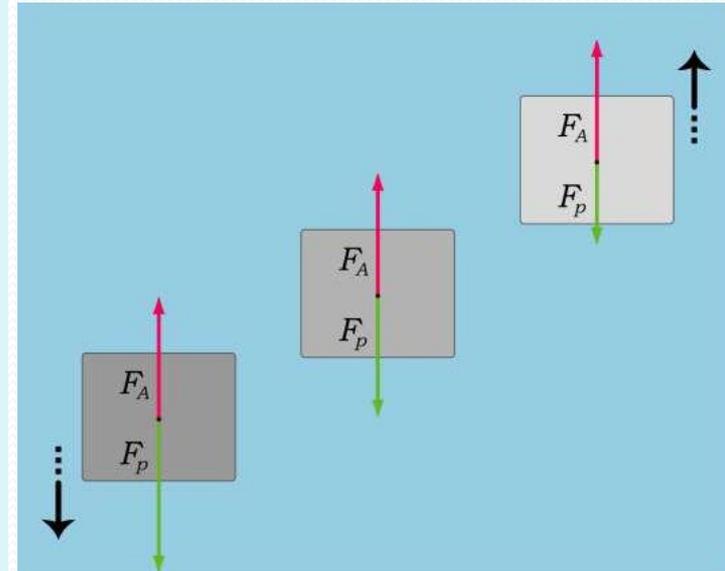
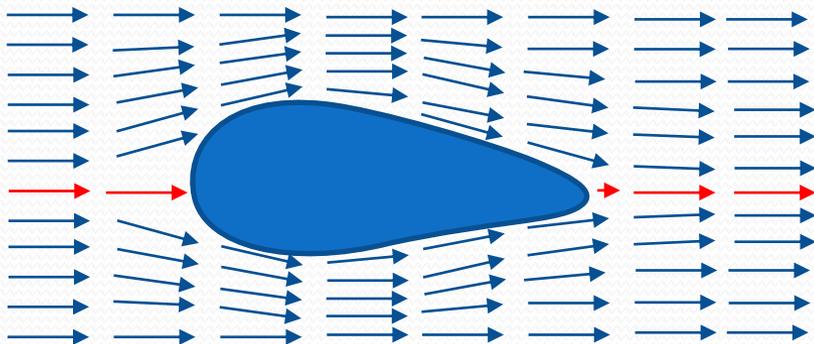


Figure II.9: Weight Vs Archimedes' thrust

CHAPTER III : KINEMATICS OF FLUID

III.1. Introduction

Fluid kinematics describes the motion of the fluid (using the concepts of streamlines and velocity fields) and its consequences without considering the nature of the forces causing the motion. Only the relationships between the positions of the fluid particles and time are taken into account.



Velocity field of fluid flow around an obstacle.

The description of the moving fluid can be done by two methods.



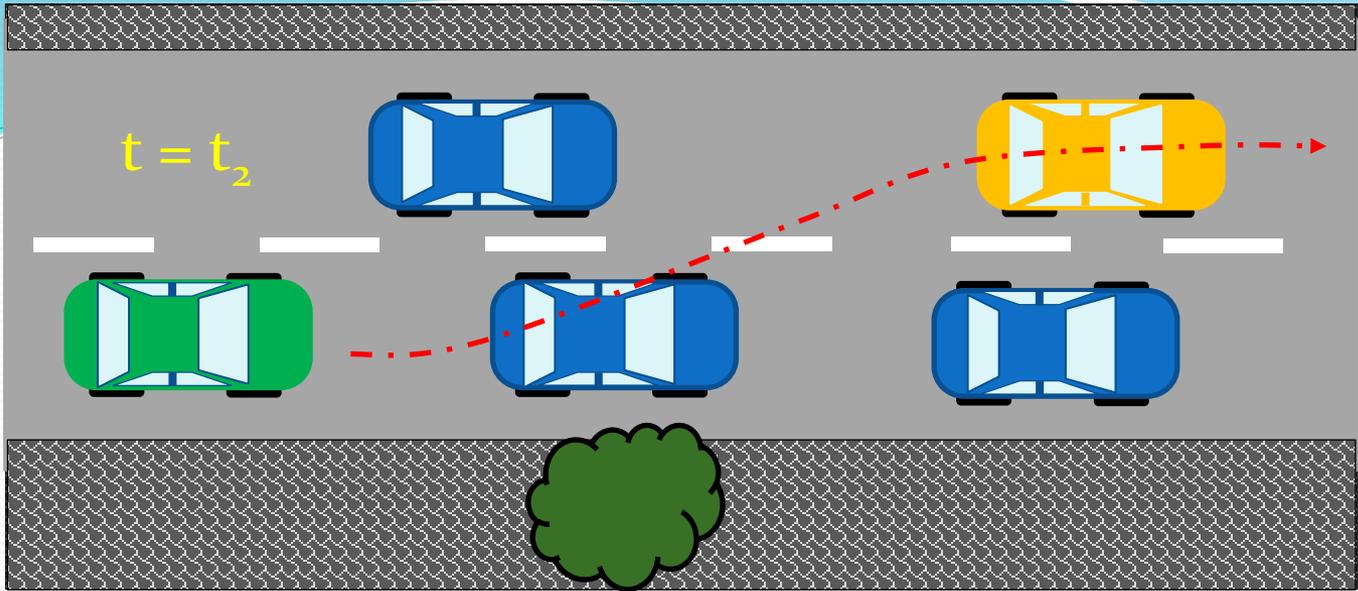
Joseph-Louis Lagrange
25 janvier 1736/10 avril 1813

We can choose to follow the fluid particles in their movement. \Rightarrow Lagrangian description



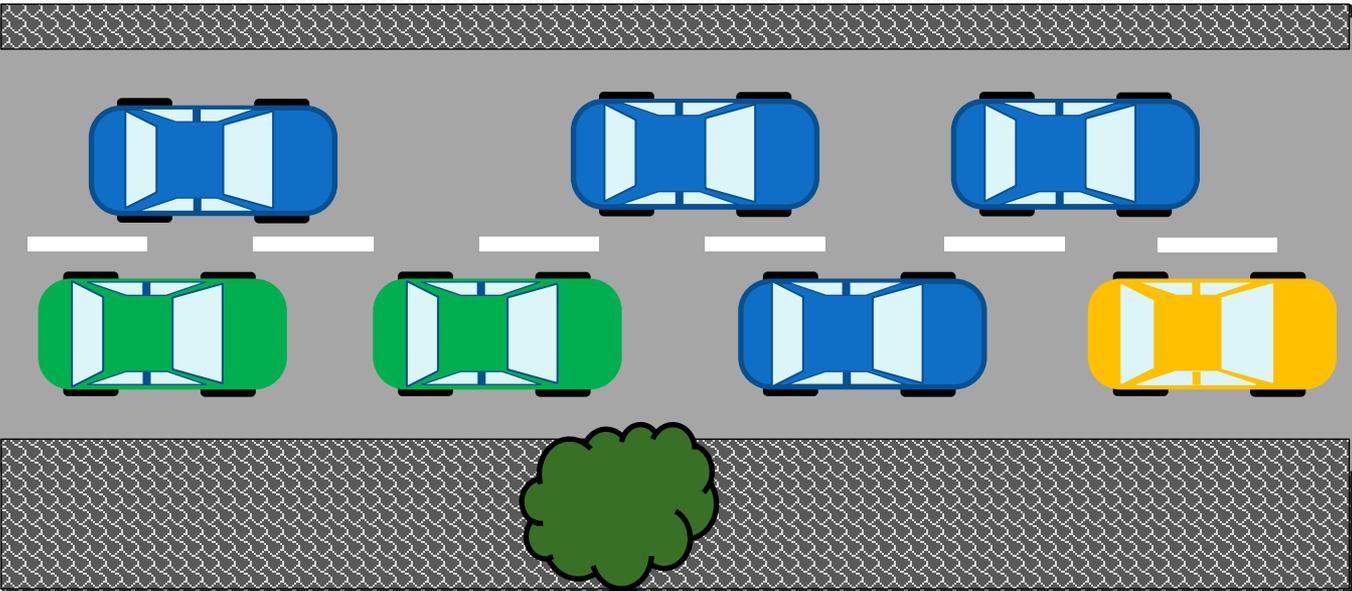
Leonhard Euler
15 avril 1707 /le 18 septembre 1783

We can take a snapshot at a given instant of the velocity field of all fluid particles. \Rightarrow Eulerian description



By following the yellow car with your eyes only, we are in Lagrangian formalism.

By observing how cars circulate around the tree, we are in Eulerian formalism.



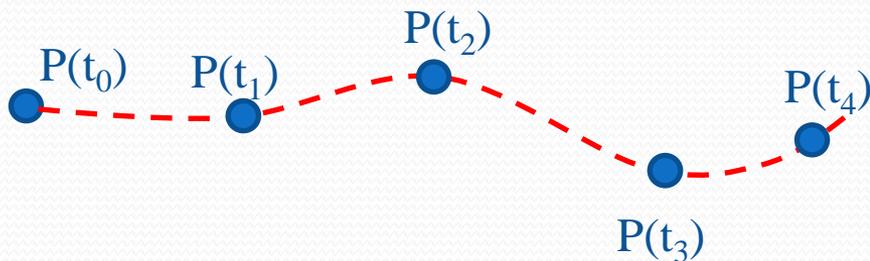
III.2. Lagrangian description

Lagrange's method considers that the fluid particle is individualized and is followed in its movements. And so, we can express the coordinates of a point **M** of the fluid mass as a function of time and the initial position of the point considered.

$$M \begin{cases} x = f(x_0, y_0, z_0, t) \\ y = \varphi(x_0, y_0, z_0, t) \\ z = \psi(x_0, y_0, z_0, t) \end{cases}$$

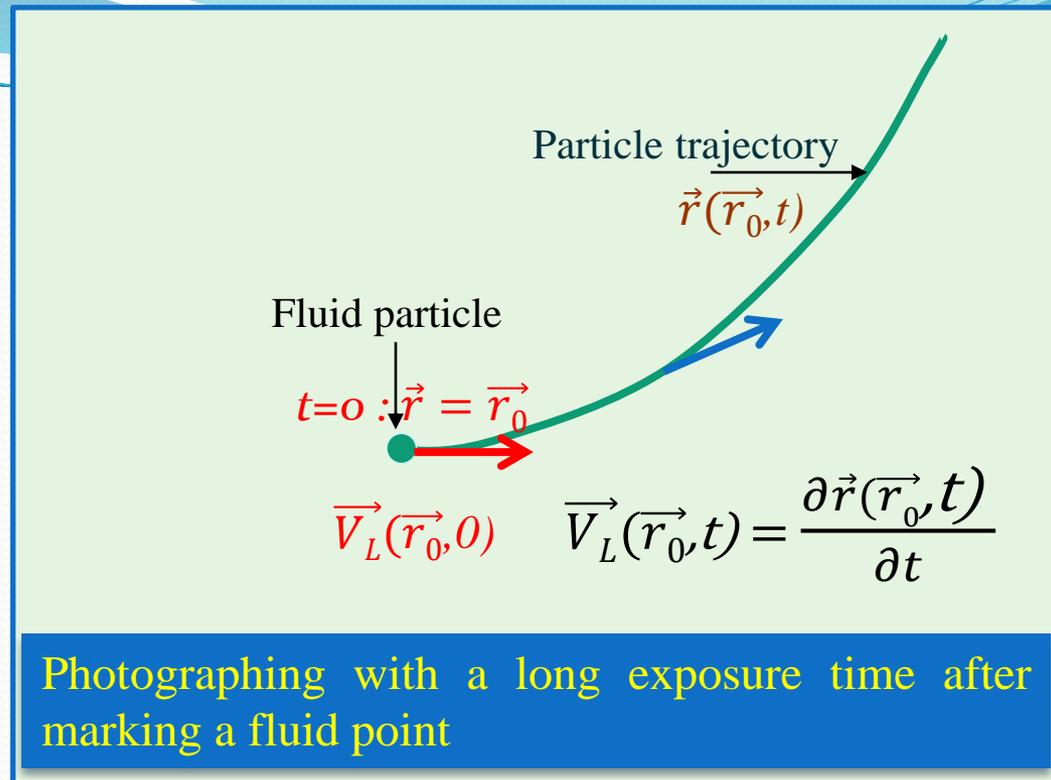
x, y and z are the Lagrange variables

The trajectory of a fluid particle is defined as the path followed by this particle over time, i.e. the set of successive positions of this particle during its movement.



Trajectory (path line) of a fluid particle P.

In the Lagrangian description, the position of the fluid particle M is identified by the vector $\vec{r}_M(t)$. The Lagrange variables are the time t and the position of M at a reference time $t_0 : \vec{r}_M(t_0)$.



The trajectory of the fluid particle is obtained mathematically by time integration of the Lagrangian velocity field $\vec{V}_L(\vec{r}_0, t)$

$$\vec{r}(t) = \vec{r}_0 + \int_{t_0}^t \vec{V}(\vec{r}_0, t') dt'$$

Trajectory of the fluid particle

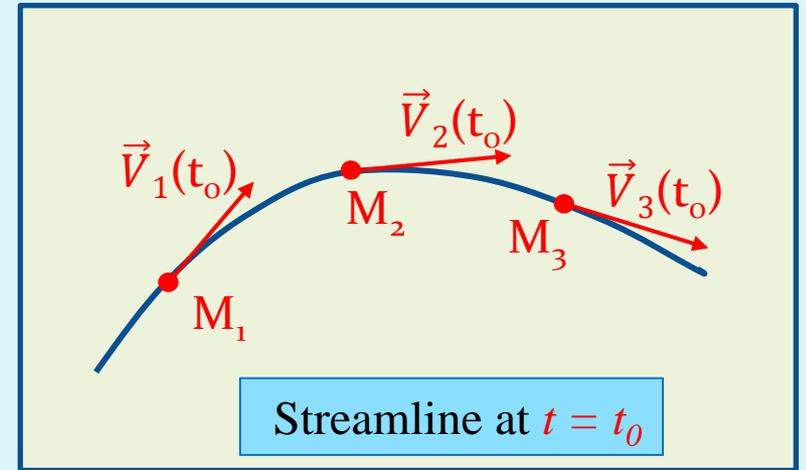
III.3. Eulerian description

Euler's method considers a fixed point in space and studies, as a function of time, what happens at this point. It will therefore be a question of determining the velocity of the fluid particles that pass successively through this point, as a function of time. The velocity \vec{V} being determined by its three components (u, v, w) on three axes OX , OY and OZ , with :

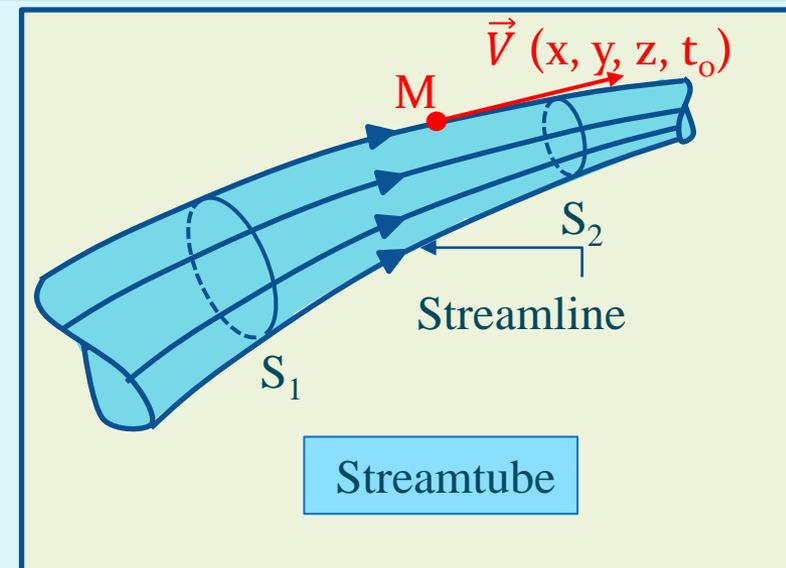
$$\vec{V} \begin{cases} u = f(x, y, z, t) \\ v = \varphi(x, y, z, t) \\ w = \psi(x, y, z, t) \end{cases}$$

u, v and w are the Euler variables

- Streamlines are curves whose tangent direction at each of its points is the direction of the velocity vector at time t_0 .



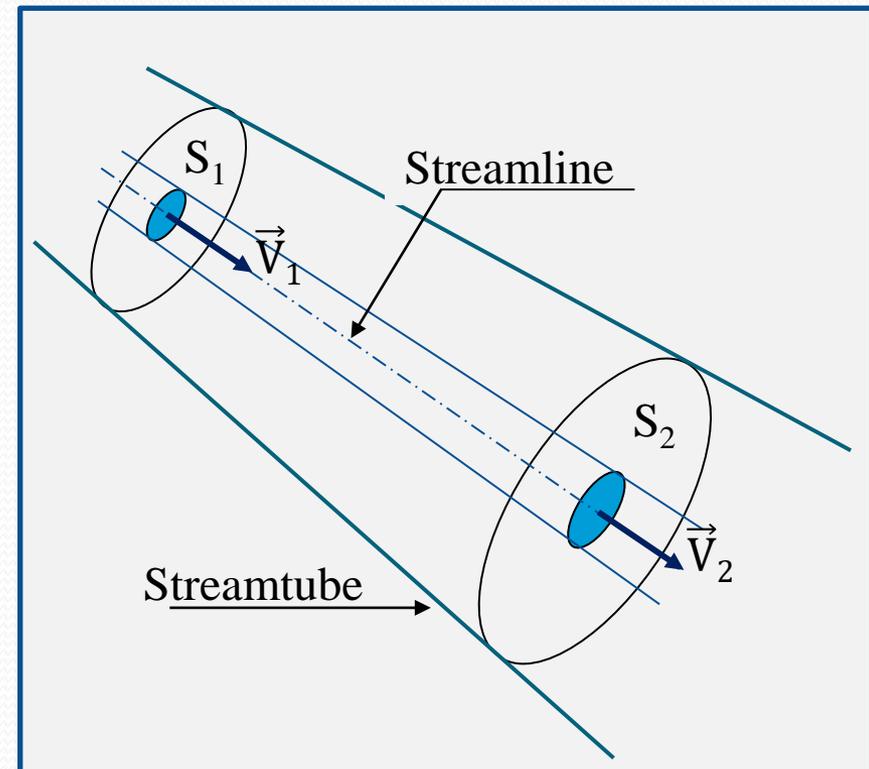
- A streamtube is the set of current lines resting on a closed contour.



As long as the velocity vector \vec{V} (u, v, w) is tangent to the streamline, the differential equations of the streamlines are written as follows :

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

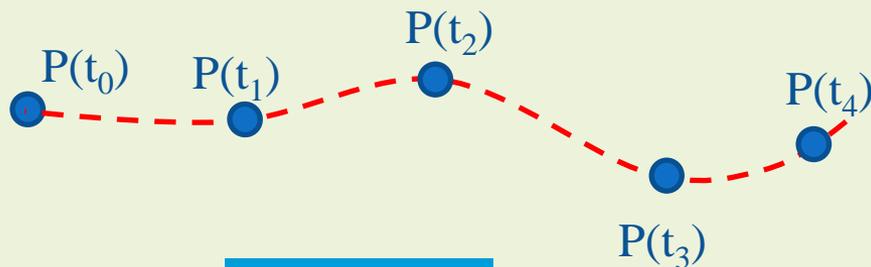
Along a streamline



Important Note

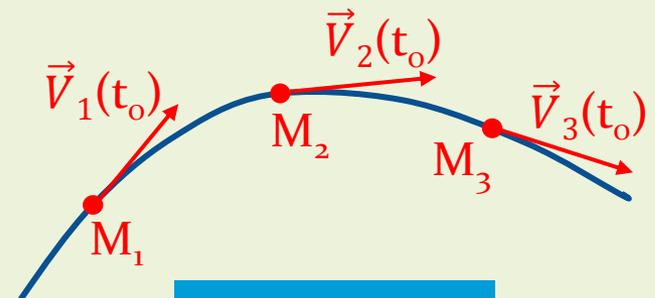
- It is imperative to know that path line and streamline are two fundamentally different notions.
- It appears that the streamline is relative to the same instant but groups together different particles while the path line, which refers to the same particle, is a curve parameterized in time..

Same particle, different times

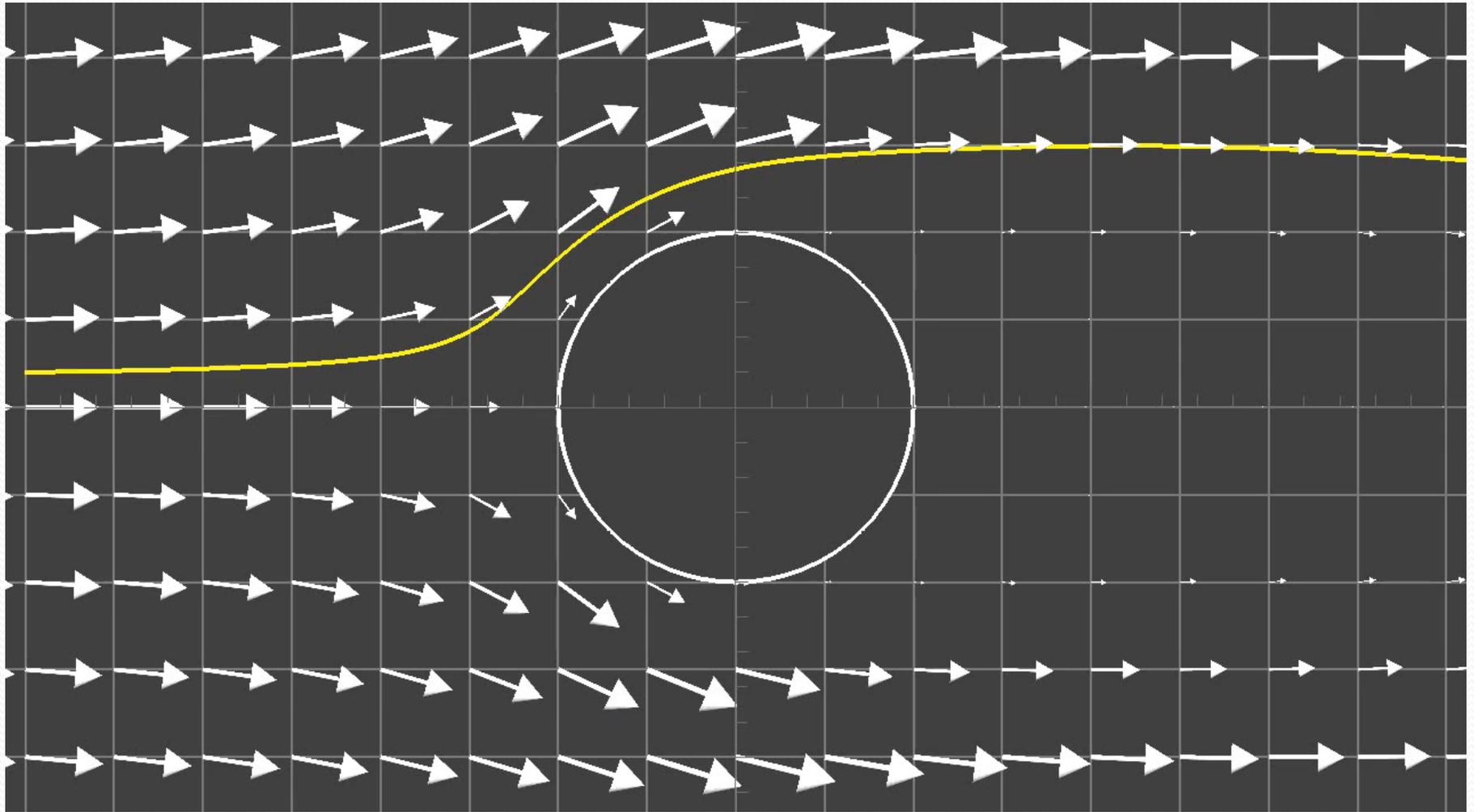


Path line

Same time, different particles

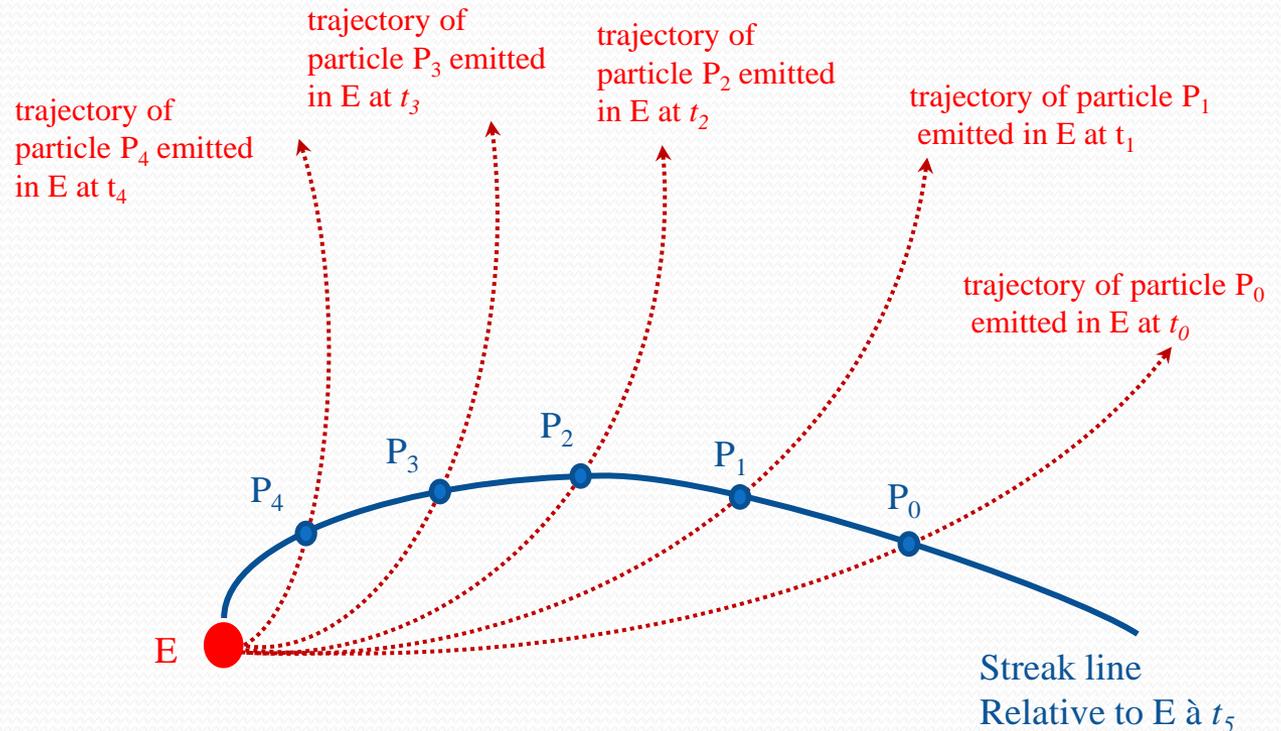


Streamline



III.4. Streak Lines

All particles having passed through the same point E are located at time t on a curve called the "Streak line" relative to point E at time t .



Streak line

The streak line is made up of all the positions occupied, at a given instant, by the particles passing through a given point at successive previous instants.

In fact, the streak line is a curve which can easily be demonstrated experimentally: the most explicit example is the coloring source within a flow of translucent fluid, where the colored net thus produced corresponds to an streak line.

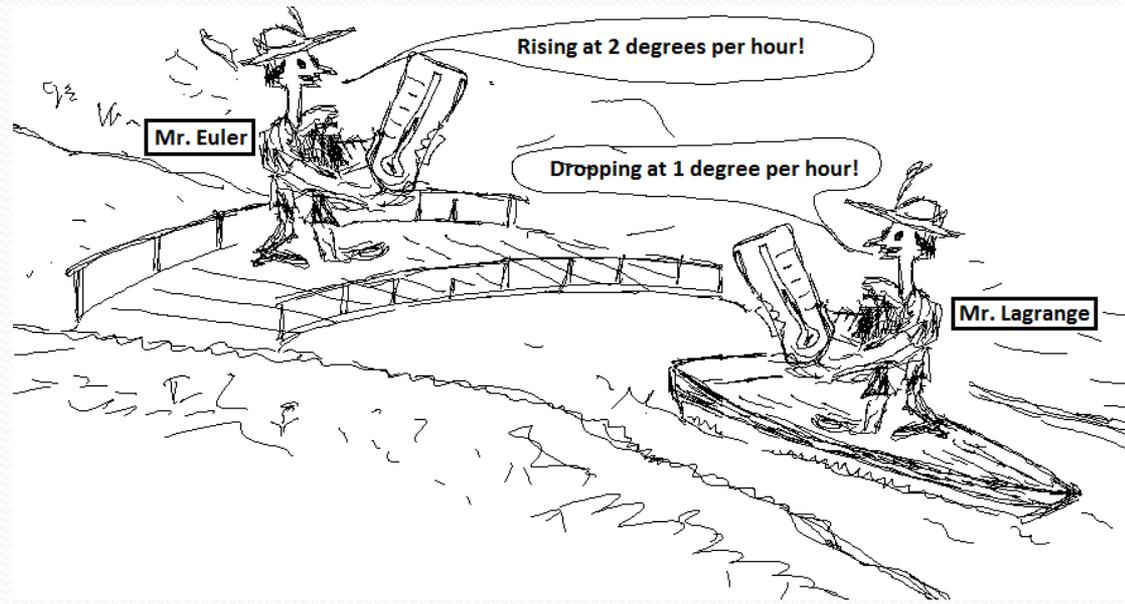


III.5. Comparison and application areas of Lagrange and Euler descriptions

The following table summarizes, in a very explicit manner, all of these two descriptions:

Descriptions	Lagrange 	Euler 
Variables	x_0, y_0, z_0, t	x, y, z, t
Unknowns	$x = f(x_0, y_0, z_0, t)$ $y = \varphi(x_0, y_0, z_0, t)$ $z = \psi(x_0, y_0, z_0, t)$	$u = f(x, y, z, t)$ $v = \varphi(x, y, z, t)$ $w = \psi(x, y, z, t)$

Both descriptions are used in the field of mechanics. In rigid solid mechanics, a priori, the Lagrangian formalism describes the physical phenomenon better.



In fluid mechanics, in the general case, it is not necessary to follow the trajectory and relative positions of the particle with respect to time. It is more appropriate to fix a given point in space, and to study, at each instant of the different particles passing through this point, the temporal variations of the velocity and pressure. For this reason the Eulerian formalism is more adequate.

III.6. Stationary or permanent flow

A flow is said to be stationary (or permanent), if all the fields of local quantities (speed, pressure, etc.) do not depend explicitly on time (t). In other words, the partial time derivatives of Eulerian fields are zero.

$$\vec{V}(\vec{r}, t) = \vec{V}(\vec{r})$$
$$u = u(x, y, z) ; v = v(x, y, z) ; w = w(x, y, z)$$

In a more explicit manner, in a stationary flow, the streamlines at time t do not depend on time t_1 , and they remain fixed over time.

And we will have:

$$\frac{dx}{u(x,y,z)} = \frac{dy}{v(x,y,z)} = \frac{dz}{w(x,y,z)} = dt$$

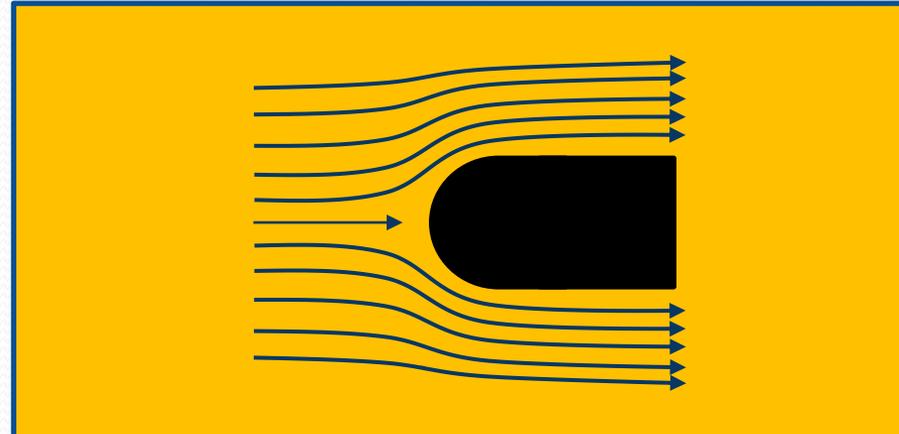
Path line equation

$$\frac{dx}{u(x,y,z)} = \frac{dy}{v(x,y,z)} = \frac{dz}{w(x,y,z)}$$

Streamline equation

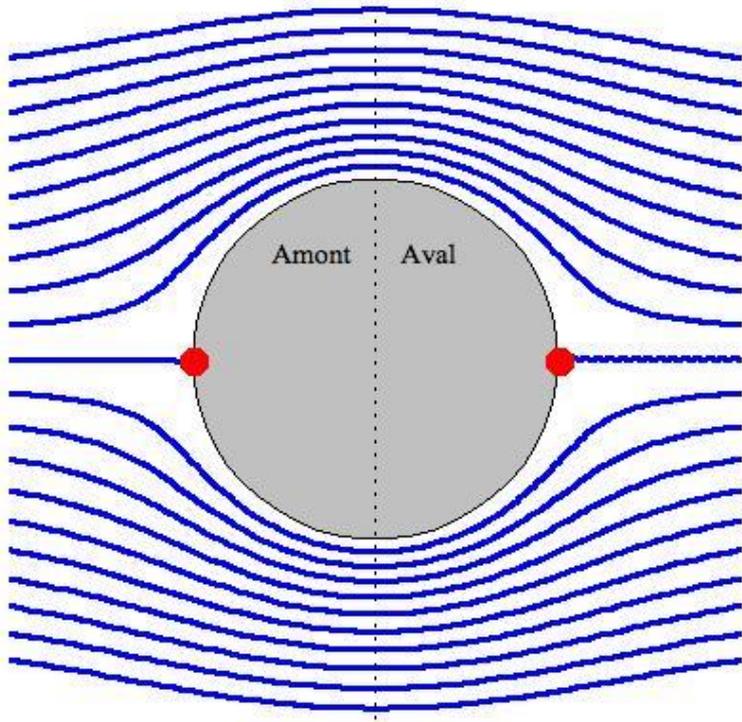
In fact, the different particles emitted from the same point over time have the same trajectories: these therefore represent at the same time the emission lines.

Furthermore, the local velocity vector (independent of time) is tangent at each point to the trajectories which therefore also represent the current lines.

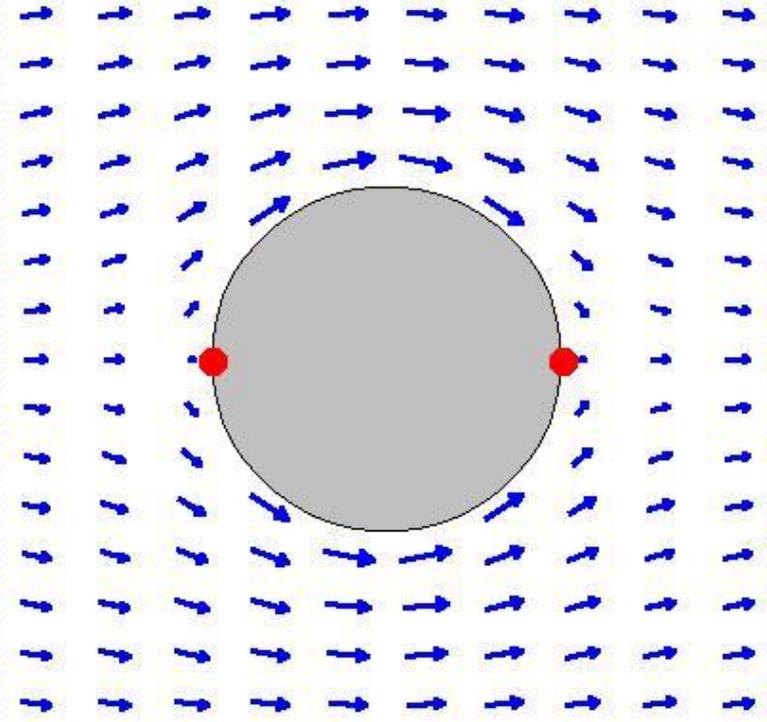


Permanent flow around a stationary obstacle.

The three line systems coincide.



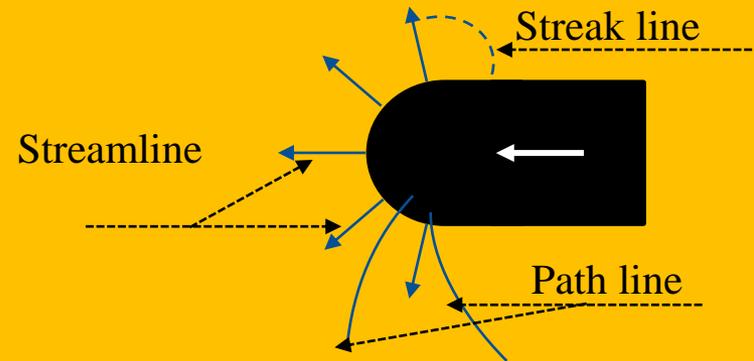
Air flow around a disk.
The streamlines perfectly match the shape of the obstacle, even downstream of the flow.



Velocity field of the flow around a disk.

In red are the stagnation points, places where the flow velocity is zero.
Source <http://images.math.cnrs.fr/>

On the other hand, in the case of an unstationary (non-permanent) flow, the three systems of lines are generally distinct, and the correspondence between them is difficult to study. In this case, we are generally only interested in the streamlines inside the fluid.

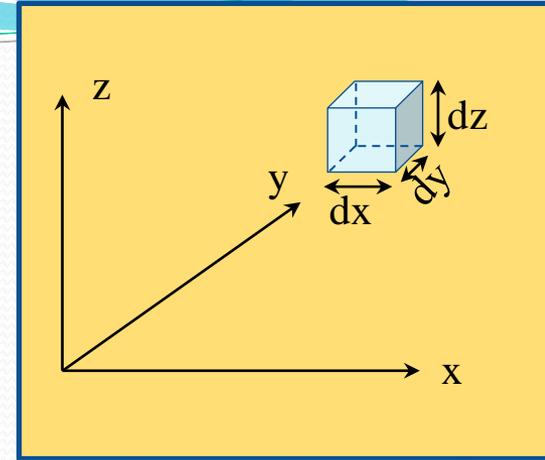


Unstationary flow.
The obstacle is moving in a still body of water.

III.7. Mass flow and volume flow

III.7.1. Mass flow

Consider an elementary parallelepiped of volume fluid $dx dy dz$.



The mass flow rate q_m is the elementary mass d_m of fluid which crosses an elementary straight surface dS during a time interval dt .

So:

$$q_m = \iint_S \frac{dm}{dt} = \iint_S \frac{\rho dx dy dz}{dt} = \rho \iint_S dS \frac{dx}{dt} \quad ; \quad \left(\frac{dx}{dt} = v\right)$$

After integration we get:

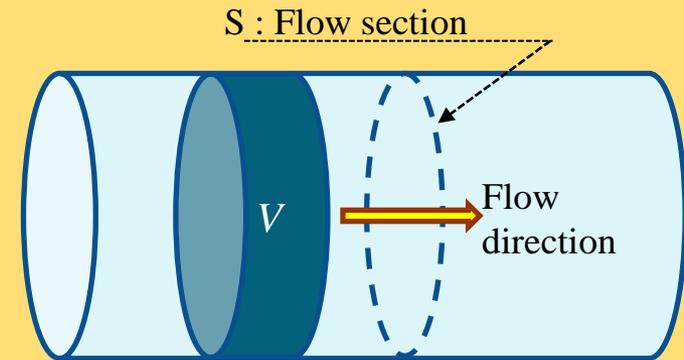
$$q_m = \rho v S \quad [kg/s]$$

v : average velocity of fluid flow (m/s)

ρ : density (kg/m³)

III.7.1. Volume flow

The volume flow Q_v is the elementary volume dV of fluid which passes through an elementary straight surface dS during a time interval dt .



What will be the time t required for the volume V of the fluid to pass through the section S ?

$$Q_v = \iiint_V \frac{dV}{dt} = \iiint_V \frac{dx \, dy \, dz}{dt} = \iint_V dS \frac{dx}{dt}; \quad \left(\frac{dx}{dt} = v\right)$$

After integration we get:

$$Q_v = v \cdot S \quad [m^3/s]$$

v : average velocity of fluid flow (m/s)

V : volume of fluid (m^3).

S : the area of the cross section (m^2).

A possible origin of the letter Q would be the Latin word **quantitas** (genitive **quantitatis**), which means quantity.

CHAPTER **IV**: THE HYDRODYNAMIC OF PERFECT FLUIDS

The purpose of hydrodynamics is to study the movements of liquids as a function of the forces that give rise to them. Among these forces, the viscosity forces only intervene for real fluids. This remark leads to the distinction between real and perfect liquids. The latter have the particularity of not having viscosity and not developing turbulence.

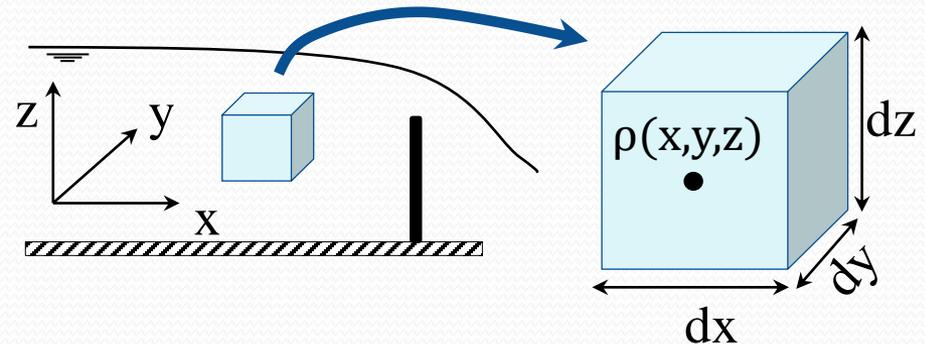
In all that follows, we will work in **Euler variable**, that is to say that we will study at each point of the flow the speed and the pressure of the fluid as a function of time.



IV.1 – Continuity Equation

The continuity equation expresses that the fluid is continuous, and that mass is conserved during flow, i.e., that there can be no external input or removal of matter in any part of the fluid.

In other words, the conservation of mass means that the change in the fluid mass of a volume element dv over time dt is equal to the difference between the incoming fluid mass and the outgoing fluid mass in this same volume.



Mass variation between t and $t+dt$

The mass of fluid contained in the volume $dv=dx.dy.dz$ is equal at time t to:

$$\rho.dx.dy.dz$$

After a time dt in this same volume, the mass is equal to:

$$\left(\rho + \frac{\partial\rho}{\partial t} dt\right)dx.dy.dz$$

We therefore observe a variation in mass of:

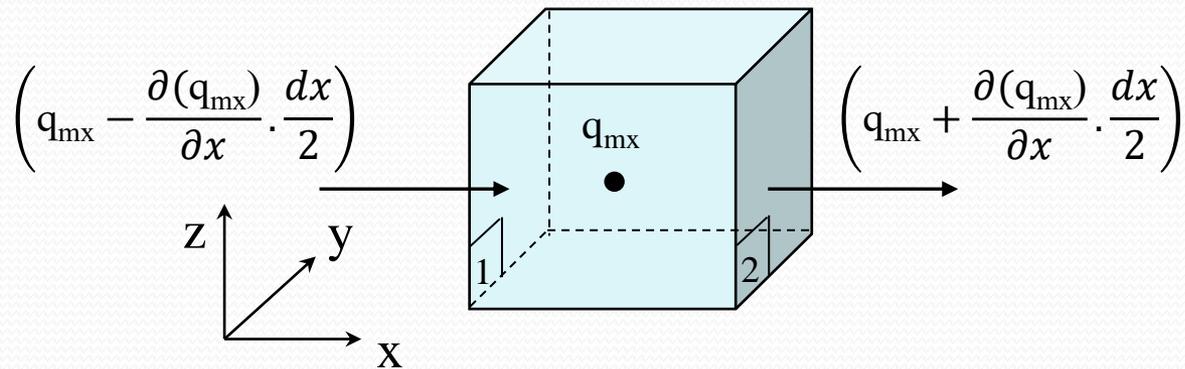
$$\frac{\partial\rho}{\partial t} dt.dx.dy.dz$$

Variation in mass of fluid during the duration dt between what enters and what exits:

The mass flow (q_m) is defined by:

$$q_{mx} = \rho \cdot Q_{vx} = \rho \cdot u \cdot dy \cdot dz$$

With q_{mx} : mass flow along the x axis



The mass of fluid entering through face **1** (along **x**) during time **dt** is:

$$\left(q_{mx} - \frac{\partial(q_{mx})}{\partial x} \cdot \frac{dx}{2} \right) \cdot dt = \left(\rho \cdot u \cdot dy \cdot dz - \frac{\partial(\rho \cdot u \cdot dy \cdot dz)}{\partial x} \cdot \frac{dx}{2} \right) \cdot dt$$

The mass of fluid exiting through face **2** (along **x**) during time **dt** is:

$$\left(q_{mx} + \frac{\partial(q_{mx})}{\partial x} \cdot \frac{dx}{2} \right) \cdot dt = \left(\rho \cdot u \cdot dy \cdot dz + \frac{\partial(\rho \cdot u \cdot dy \cdot dz)}{\partial x} \cdot \frac{dx}{2} \right) \cdot dt$$

The difference in mass between what enters and what exits in the volume dv (following x), is worth:

$$- \frac{\partial \rho u}{\partial x} dx \cdot dy \cdot dz \cdot dt$$

We have the same along the other axes (**y** and **z**) with $V = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$:

$$-\frac{\partial \rho v}{\partial y} dx \cdot dy \cdot dz \cdot dt$$

;

$$-\frac{\partial \rho w}{\partial z} dx \cdot dy \cdot dz \cdot dt$$

Writing that the change in mass of fluid of an element of volume **dv** during a time **dt** is equal to the mass of fluid entering this volume minus the mass of fluid exiting, which gives:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z}$$

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho V) = 0$$

This is the continuity equation of a conservative fluid.

Case of incompressible fluids:

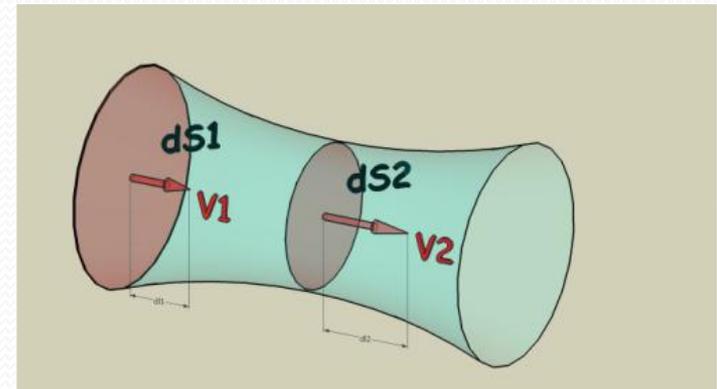
In the case where the fluid is in permanent motion ($\frac{\partial}{\partial t} = 0$), the density ρ is independent of time and if, in addition, we are in the presence of an incompressible fluid, where ρ is independent of x , y and z , the continuity equation will become :

$$\text{div}(\mathbf{V}) = 0$$

The physical interpretation of this equation is as follows: the flow rates Q entering and leaving through any volume filled with the fluid must be equal.

$$Q_1 = Q_2 \quad \Rightarrow \quad S_1 V_1 = S_2 V_2$$

The velocity is inversely proportional to the passage section.

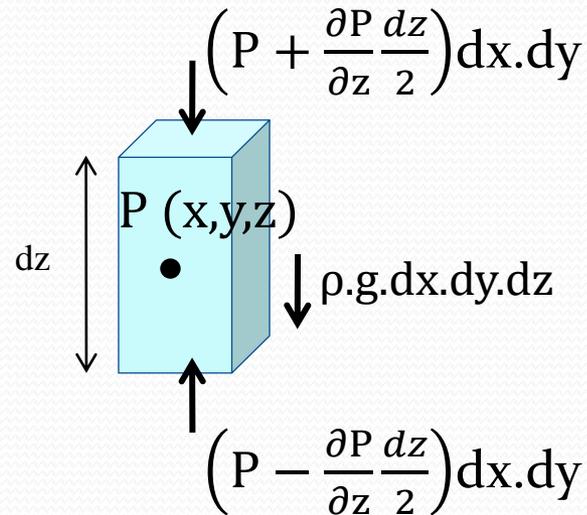
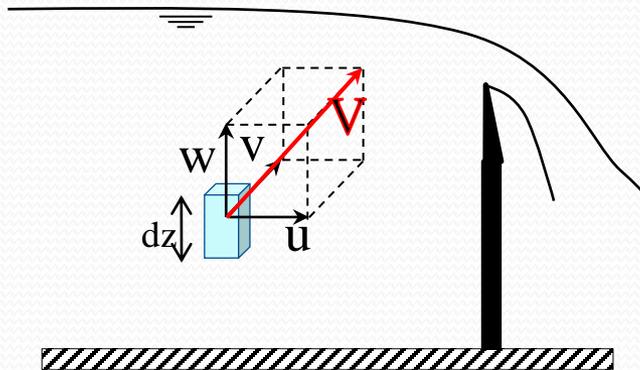


IV.2 – General equations of motion: Euler's equations

For perfect fluids, it is assumed that the viscosity is zero and therefore turbulence cannot be developed.



1707 -1783



Reasoning, initially, along the vertical (z), the forces which act on this volume element $dv = dx.dy.dz$, are:

Volume forces:

➤ **Gravitational forces:** there is only one, and it is written as follows:

$$F = -\rho.g.(dx.dy.dz) = -\rho.g.dv$$

➤ **Inertial forces:**

$$\rho \frac{dw}{dt} dv = \rho \left(\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} \right) dv$$

- Pure acceleration forces: $\rho \frac{\partial \rho w}{\partial t} dx.dy.dz$
- Convective acceleration forces: $\rho \cdot \underline{\text{grad}}(w) \cdot \underline{V} \cdot dx.dy.dz$

b) Surface forces:

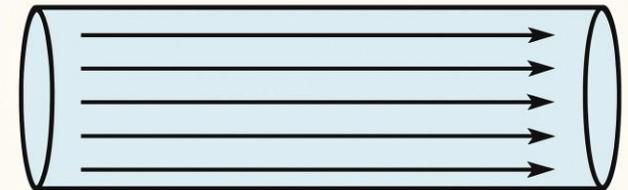
➤ Pressure forces :

✓ Bottom surface: $\left(P - \frac{\partial P}{\partial z} \frac{dz}{2} \right) dx \cdot dy$

✓ Upper surface: $- \left(P + \frac{\partial P}{\partial z} \frac{dz}{2} \right) dx \cdot dy$

➤ Viscosity forces : 0

➤ Friction forces due to turbulence: 0



Ideal flow of a perfect
(inviscid) liquid

In equilibrium of all forces, the equation of the hydrodynamics of perfect fluids is written as follows:

$$\Sigma F = \text{Inertial forces}$$

i.e. :

$$-\rho \cdot g \cdot (\cancel{dx} \cdot \cancel{dy} \cdot \cancel{dz}) + \left(\cancel{P} - \frac{\partial P}{\partial z} \frac{\cancel{dz}}{2} \right) \cancel{dx} \cdot \cancel{dy} - \left(\cancel{P} + \frac{\partial P}{\partial z} \frac{\cancel{dz}}{2} \right) \cancel{dx} \cdot \cancel{dy} =$$

$$\rho \frac{\partial \rho w}{\partial t} \cancel{dx} \cdot \cancel{dy} \cdot \cancel{dz} + \rho \cdot \underline{\text{grad}}(w) \cdot \underline{V} \cdot \cancel{dx} \cdot \cancel{dy} \cdot \cancel{dz}$$

$$\boxed{-\rho \cdot g - \frac{\partial P}{\partial z} = \rho \frac{\partial \rho w}{\partial t} + \rho \cdot \underline{\text{grad}}(w) \cdot \underline{V}}$$

By writing the equation along the three axes, we will have:

$$\left\{ \begin{array}{l} -\frac{1}{\rho} \frac{\partial P}{\partial x} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\ -\frac{1}{\rho} \frac{\partial P}{\partial y} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\ -g - \frac{1}{\rho} \frac{\partial P}{\partial z} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \end{array} \right.$$

Which means that Euler's equation is written as follows:

$$\frac{\partial \underline{V}}{\partial t} + \underline{\text{grad}}(\underline{V}) \cdot \underline{V} + \frac{1}{\rho} \underline{\text{grad}}(p + \rho \cdot g \cdot z) = 0$$

In hydrodynamics, the characteristics of an incompressible fluid's motion depend on four unknowns: u , v , w , and P , so at least four equations are required.

- Euler's equation gives three equations.
- The mass conservation equation gives the fourth.

This would allow the system of equations to be solved.

IV.3 – Momentum Theorem:

The fundamental principle of dynamics is that any (temporal) variation in momentum results from the application of forces.

$$\boxed{\frac{d(m.V)}{dt} = \sum F}$$



Therefore, we can write a general relation of the equation of conservation of momentum, for a tube of fluid volume V fixed in space, limited by a surface S , of the following form:

$$\iint_S \rho \vec{V} (\vec{V} \cdot \vec{n}_{\text{ext}}) dS = \Sigma \vec{F}_{\text{ext}}$$

\vec{n}_{ext} : unit normal to the surfaces exterior to the current tube;

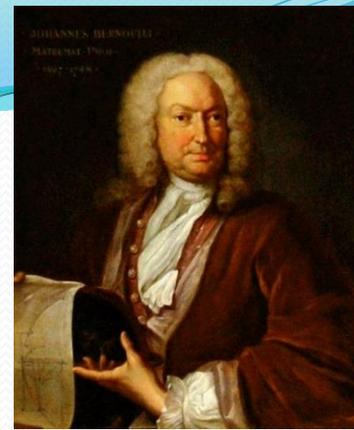
\vec{F}_{ext} : the sum of external forces;

\vec{V} : fluid velocity in the section S .

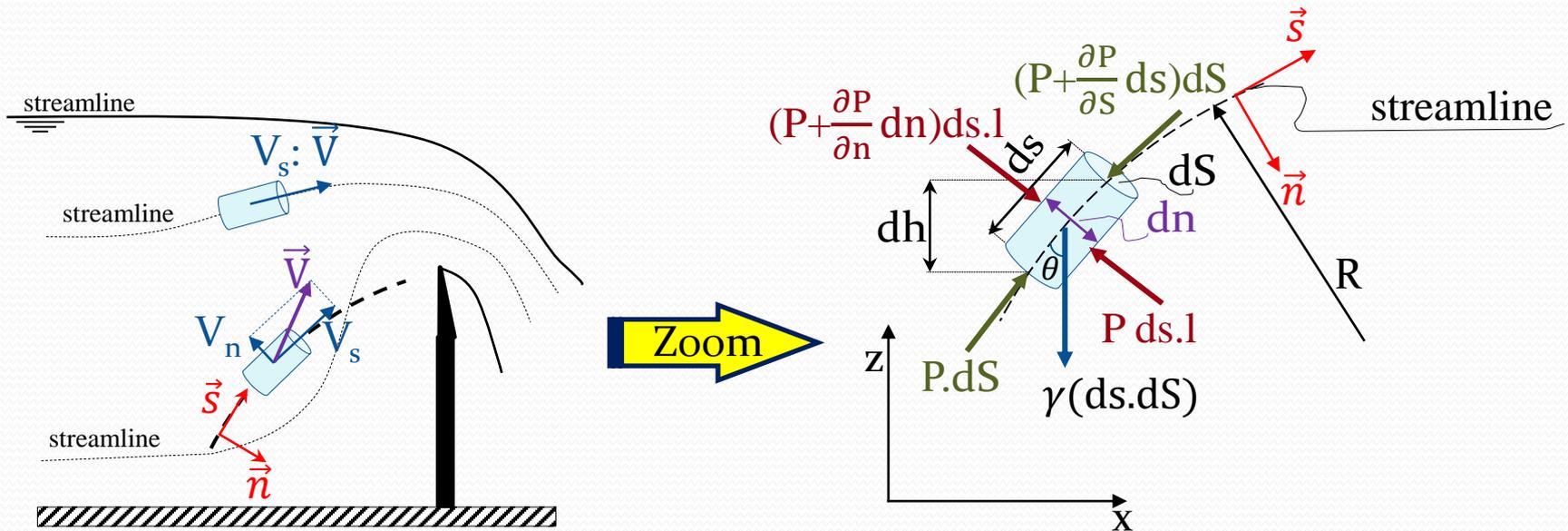
N.B.: As we are in the presence of a perfect fluid \vec{V} is uniform. And S is formed of several surfaces

VI.4 – Bernoulli equations

The system of intrinsic equations consists of writing the steady-state Euler equations ($\frac{\partial}{\partial t} = 0$) in a particular frame of reference. This frame is constituted by the streamlines for the vector $\underline{\underline{t}}$ and by the vector $\underline{\underline{n}}$ such that $\underline{\underline{v}}$ is perpendicular to $\underline{\underline{n}}$.



1667 - 1748



$$\gamma = \rho \cdot g$$

With calling \vec{s} the unit vector of the tangent to the trajectory, we have:

$$\vec{V} = V \cdot \vec{s} \quad \text{and} \quad \frac{d\vec{V}}{dt} = \frac{dV}{dt} \vec{s} + V \frac{d\vec{s}}{dt}$$

$$\frac{d\vec{s}}{dt} = \frac{d\vec{s}}{ds} \cdot \frac{ds}{dt} = \frac{\vec{n}}{R} V$$

R : curvature radius

\vec{n} : the vector perpendicular to \vec{s}

V : the intensity of a fluid particle.

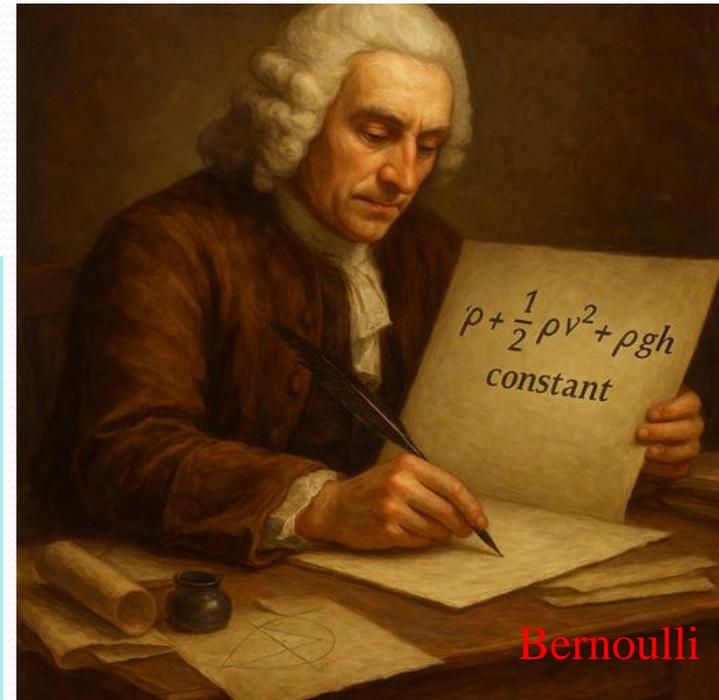
$$V \frac{\partial V}{\partial s} = - \frac{1}{\rho} \frac{\partial}{\partial s} (\rho \cdot g \cdot z + P) \quad \text{along } \vec{s}$$

$$V \cdot \frac{V}{R} = - \frac{1}{\rho} \frac{\partial}{\partial n} (\rho \cdot g \cdot z + P) \quad \text{along } \vec{n}$$

The Bernoulli equation is obtained by integrating the first equation while remaining on the streamline, and is written as follows:

$$\rho \cdot \frac{v^2}{2} + \rho \cdot g \cdot z + P = \text{Cste}$$

- The quantity $\left\{ \rho \cdot \frac{v^2}{2} + \rho \cdot g \cdot z + P \right\}$ is constant along a streamline.
- Bernoulli's equation is valid at any point of the incompressible fluid in permanent motion.



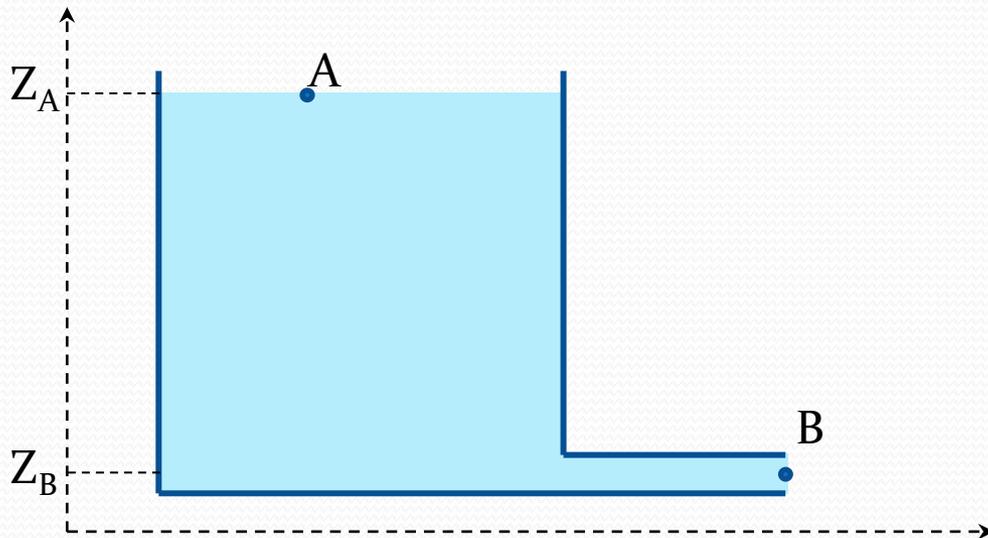
$$\rho \cdot \frac{V_A^2}{2} + \rho \cdot g \cdot z_A + P_A = \rho \cdot \frac{V_B^2}{2} + \rho \cdot g \cdot z_B + P_B$$

With : $P_A = P_B = P_{atm}$ and $V_A \lll V_B$ car $S_A \ggg S_B$

$$\rho \cdot g \cdot z_A = \rho \cdot \frac{V_B^2}{2} + \rho \cdot g \cdot z_B \Rightarrow V_B = \sqrt{2g(z_A - z_B)}$$

$$V_B = \sqrt{2gh}$$

This equation is known as Torricelli's formula.



mabessedik@yahoo.fr



Torricelli
1608-1647

IV.5 – Hydraulic head

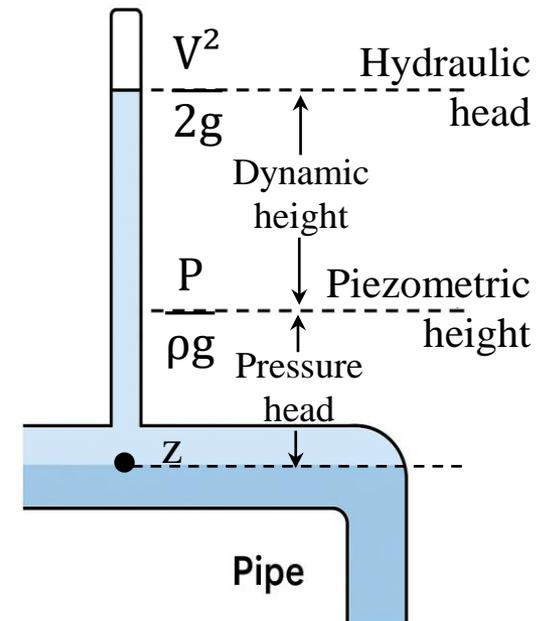
As we mentioned earlier, according to Bernoulli's equation the hydraulic head is constant along a streamline.

This hydraulic head is expressed as follows:

$$H \text{ [m]} = z + \frac{V^2}{2g} + \frac{P}{\rho g}$$

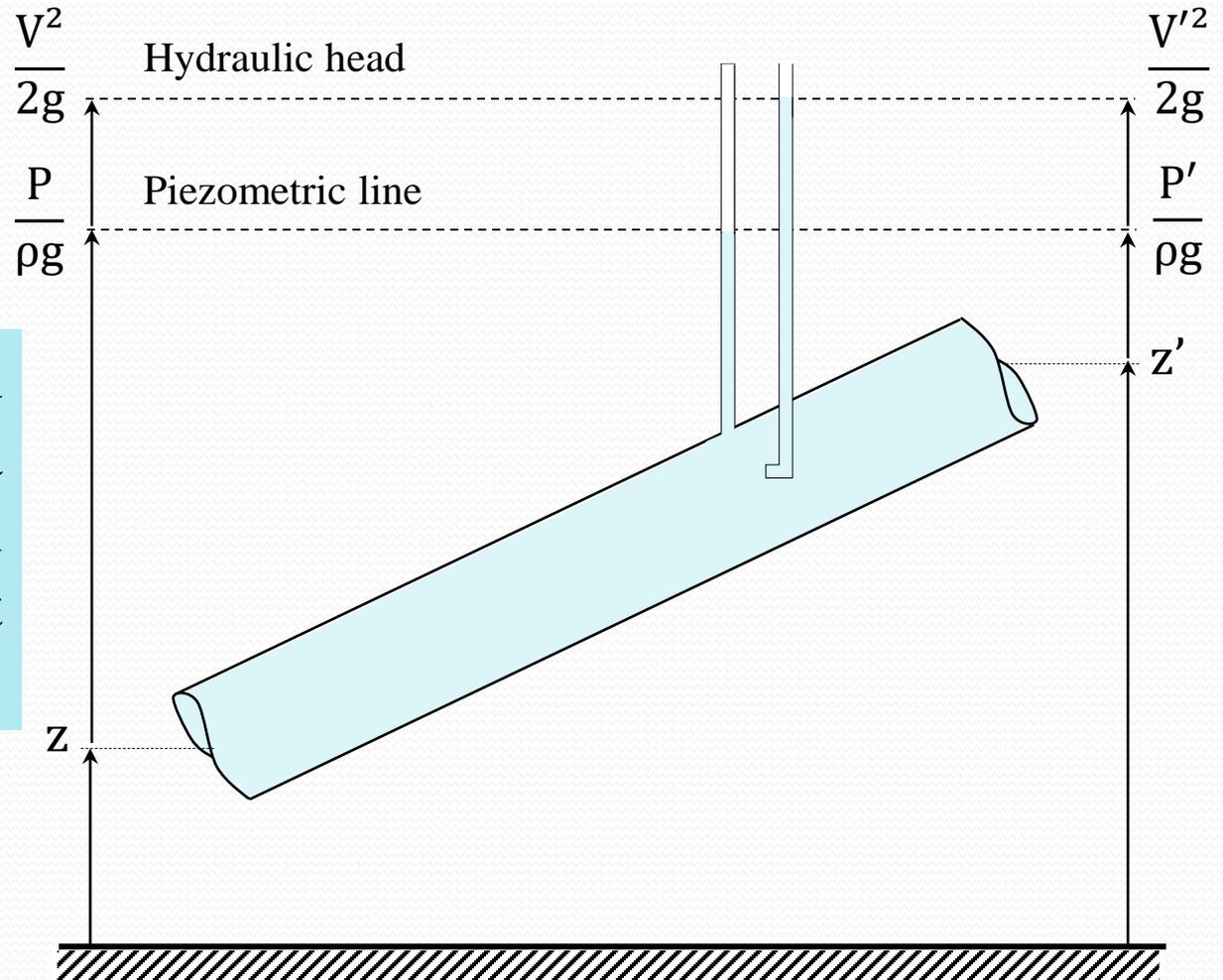
With:

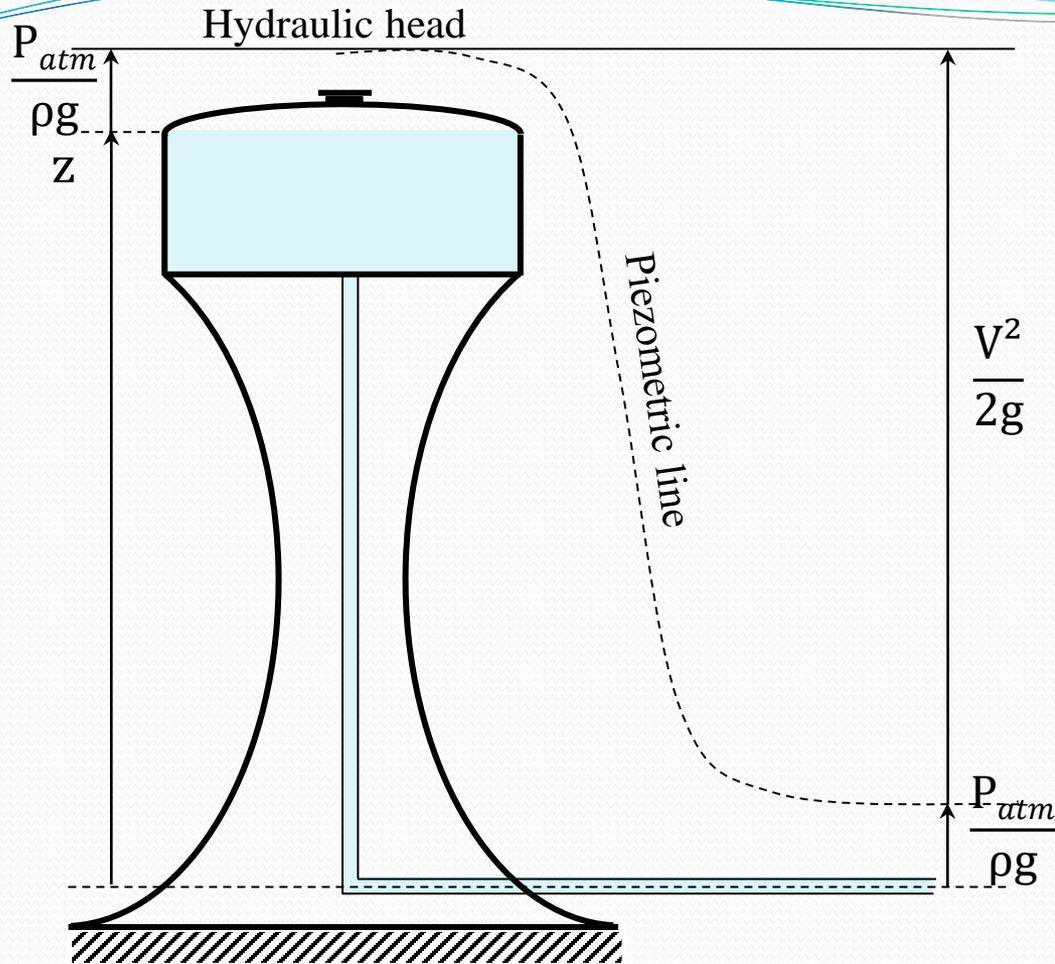
- $\frac{P}{\rho g}$: Pressure head
 - z : Altitude of point
 - $\frac{V^2}{2g}$: Dynamic height
- } Piezometric height



a) Graphical representation

Hydraulic head and piezometric line for a constant section pipeline for perfect fluid.





Load line and piezometric line for a water tower open to a pipe to the atmosphere.

CHAPITRE **V** :

HYDRODYNAMICS OF REAL FLUIDS

V.1 – Real fluids and turbulence

In the previous chapter, we assumed that fluids were perfect (zero viscosity and turbulence) and that, consequently, there was no friction either between the molecules of the fluid or on the walls of the pipes. In reality, however, a perfect liquid cannot exist..

In a real fluid flow, there are two types of frictional forces:

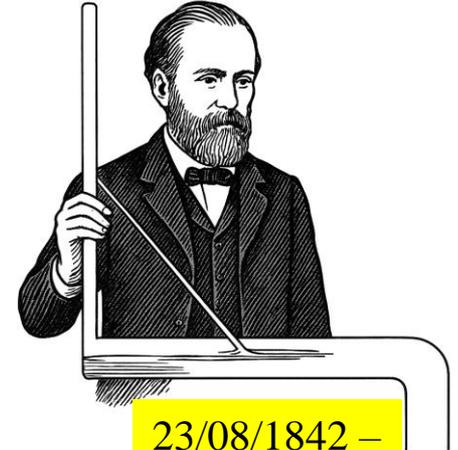
- intermolecular friction in the fluid,
- friction between the fluid and objects (pipe walls, etc.).

These frictional forces are called **viscosity forces**..

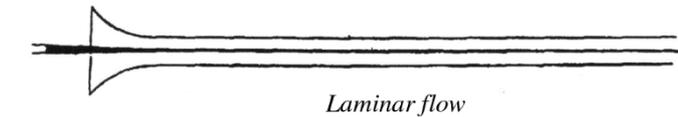
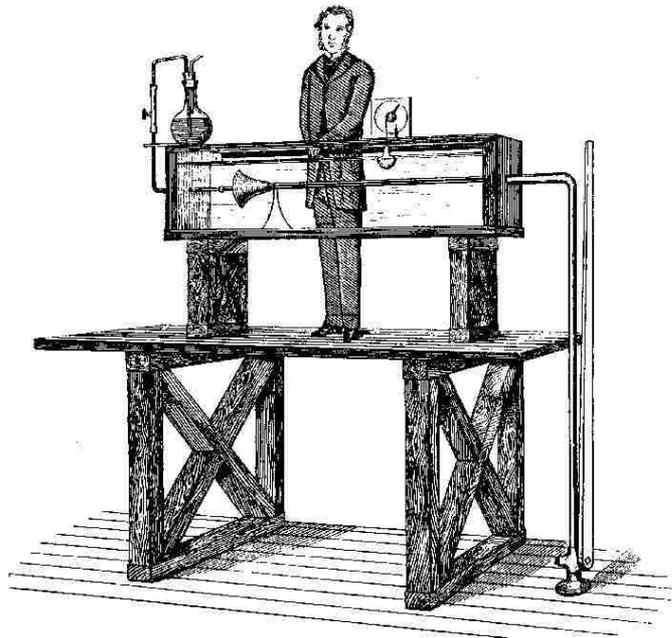
REYNOLDS NUMBER

Experiments carried out by Osborne Reynolds in 1883, when a liquid was flowing through a straight cylindrical pipe into which a stream of colored liquid was also flowing, showed the existence of two distinct types of flow regime: **Laminar** and **Turbulent**.

Osborne Reynolds



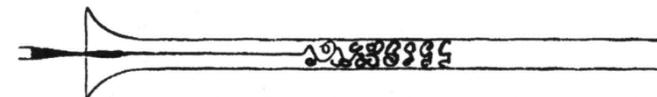
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Laminar flow

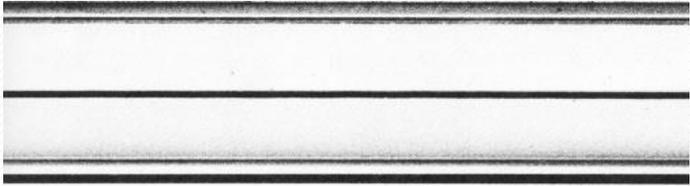


Turbulent flow



Turbulent flow (observed with an electric spark)

In the presence of viscosity forces there are three types of flows:



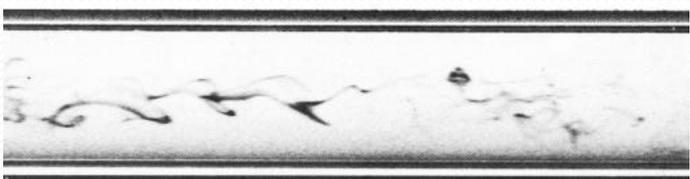
Laminar flow: all particles passing through the same point have the same trajectory. This is a regular flow.

$Re < 2000$



Transient flow: also called intermediate regime. The limit between these different types of flow is obviously difficult to grasp.

$2000 < Re < 2500$



Turbulent flow: vortices form, particles follow chaotic trajectories.

- Smooth turbulent flow if $2000 < Re < 100000$
- Rough turbulent flow if $Re > 100000$

Using various fluids with different viscosities, varying the flow rate and the pipe diameter, Reynolds showed that the parameter that determines whether the flow is laminar or turbulent is a dimensionless number called the Reynolds number given by the following expression:

$$R_e = \frac{V \cdot d}{\nu}$$

- V : Average flow velocity through the section considered (m/s)
- d : Pipe diameter or width of the fluid stream (m).
- ν : Kinematic viscosity of the fluid (m^2/s).

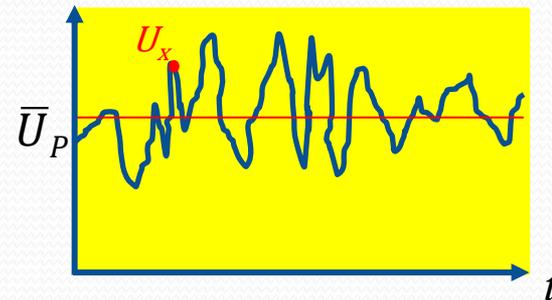
The transition from one regime to another is caused by several factors:

- The average velocity of the fluid particles;
- The viscosity and density of the fluid;
- The geometric characteristics of the pipe (diameter).

The main characteristic of turbulent movement is that the velocities and pressures at a given point fluctuate randomly over time.



a: Laminar flow

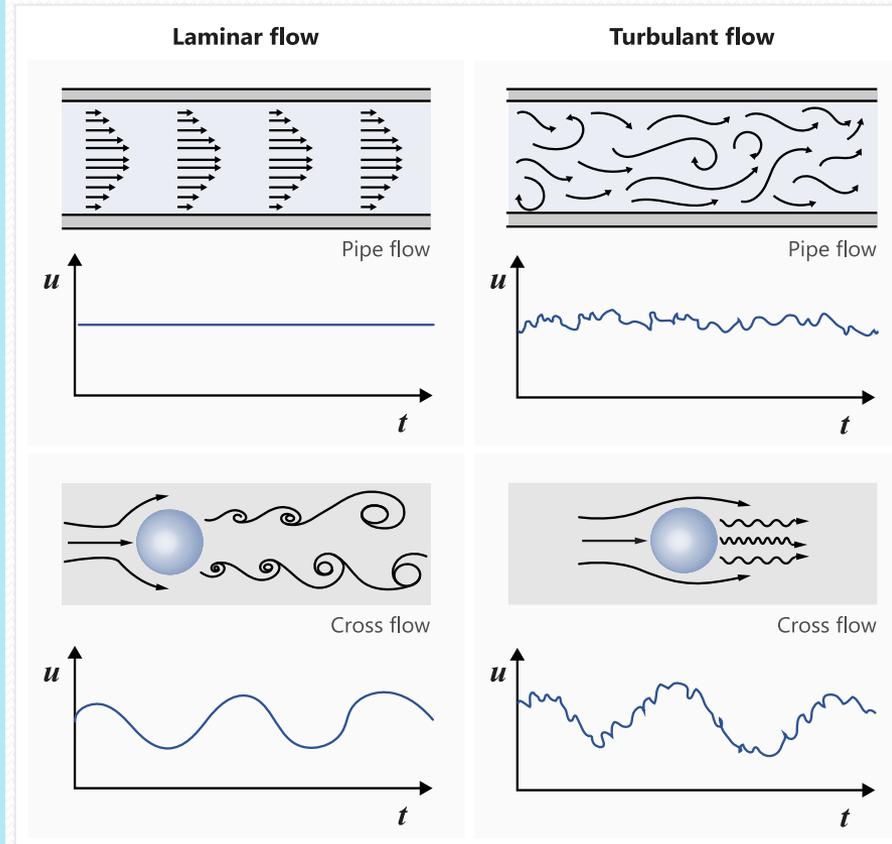


b: Turbulent flow

Variation of one of the velocity components for (a) laminar and (b) turbulent flows at a point P.

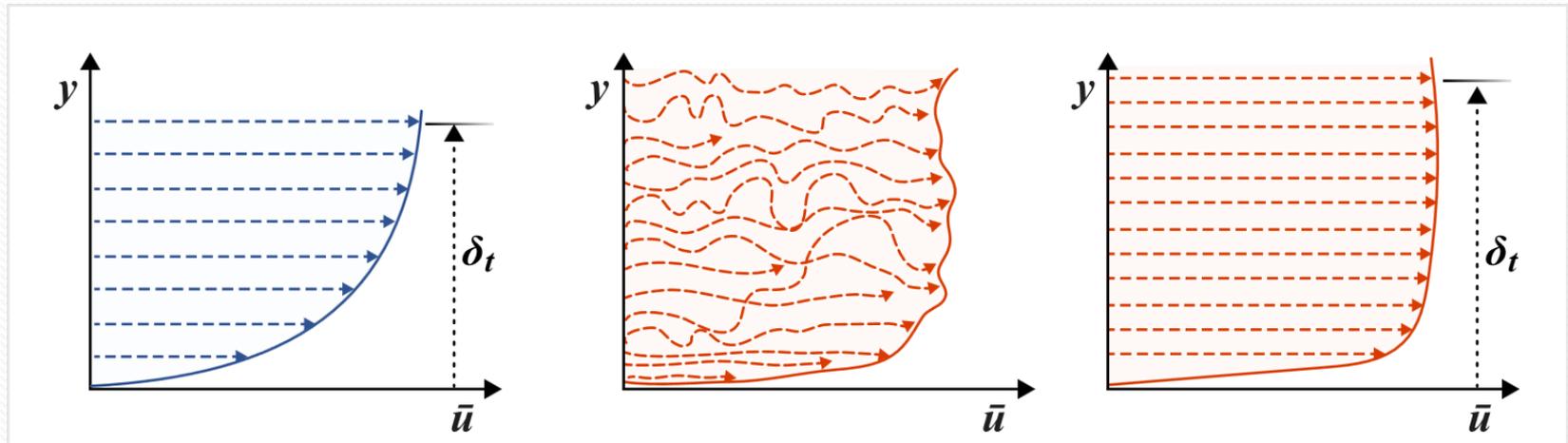
Turbulence can be created by viscosity forces on the walls (parietal turbulence) or by the different layers of the flow, which are at different speeds from one another.

This figure shows the time history of local velocity variations for laminar flows and turbulent flows, respectively. For the pipe flow cases, laminar flows have nearly constant velocity, while turbulent flows have random or chaotic velocity fluctuation patterns. For a cylinder in cross flow, laminar flows produce a sine wave velocity pattern at a downstream location of the cylinder, while the turbulent flows have similar wave patterns but with embedded fluctuations.



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The agitation in turbulent flow is due to these fluctuations, which give a velocity profile that is more uniform on average than in the case of laminar flow in a tube.



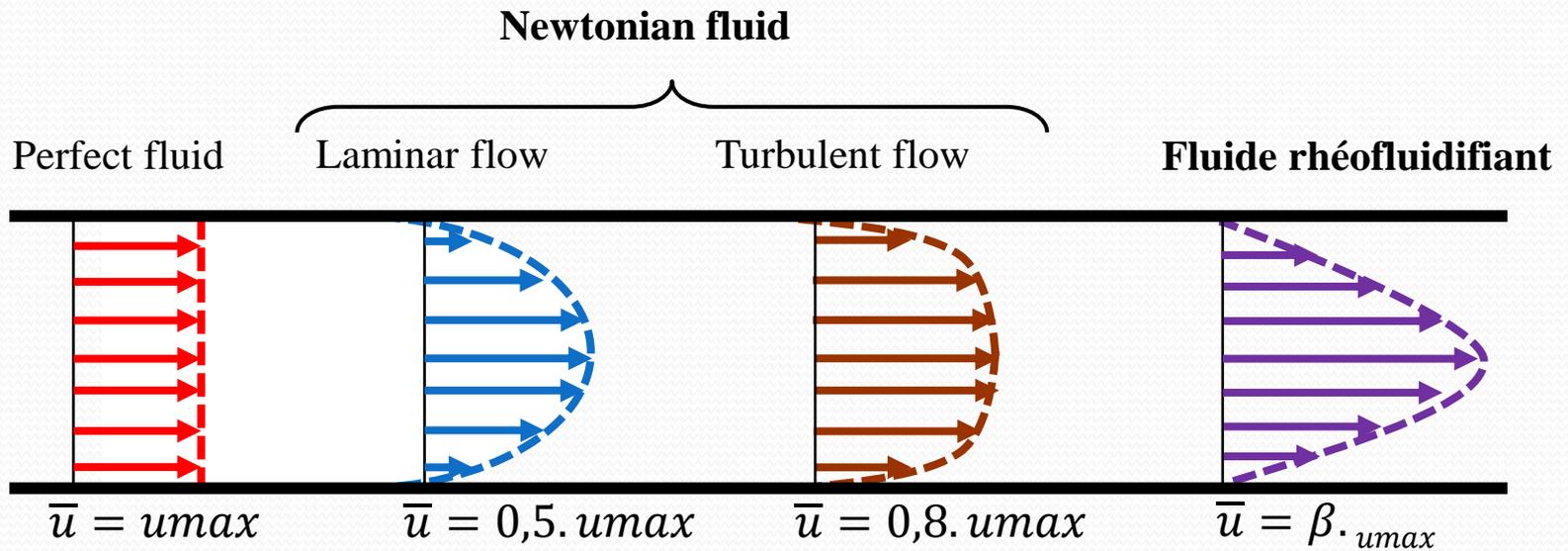
Laminar velocity profile

Turbulent instantaneous velocity profile

Turbulent time-averaged velocity profile

Time averaged velocity profiles for laminar and turbulent Flows

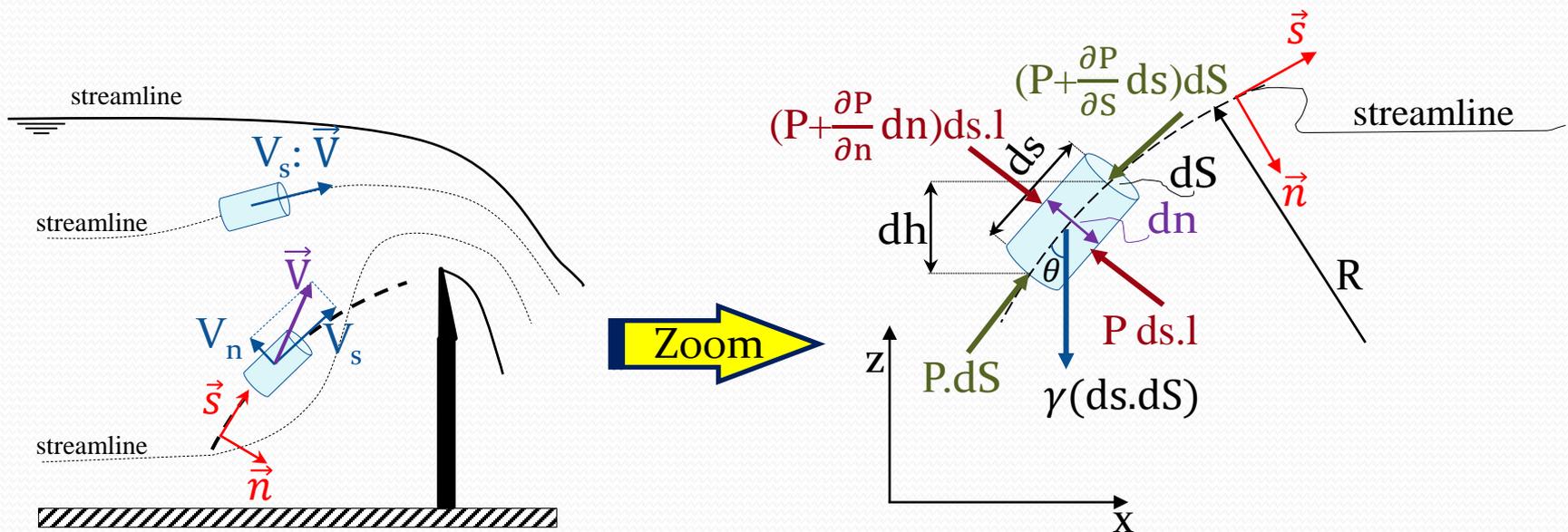
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Comparison of velocity profiles in a tube with different types of flow.

V.2 – L'équation de Bernoulli pour les fluides réels

We have seen that in the case of a real fluid in steady state, other forces intervene, notably the forces due to friction, which cause a dissipation of mechanical energy into thermal energy.



Along a streamline, mechanical energy decreases due to the increase in thermal energy; this phenomenon is called friction loss in a liquid.

After integration between two points on the same streamline, the Bernoulli equation is written:

$$\frac{P_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{P_B}{\rho g} + \frac{V_B^2}{2g} + z_B + j_{A-B}$$

- j_{A-B} : is the pressure loss between A and B (m)

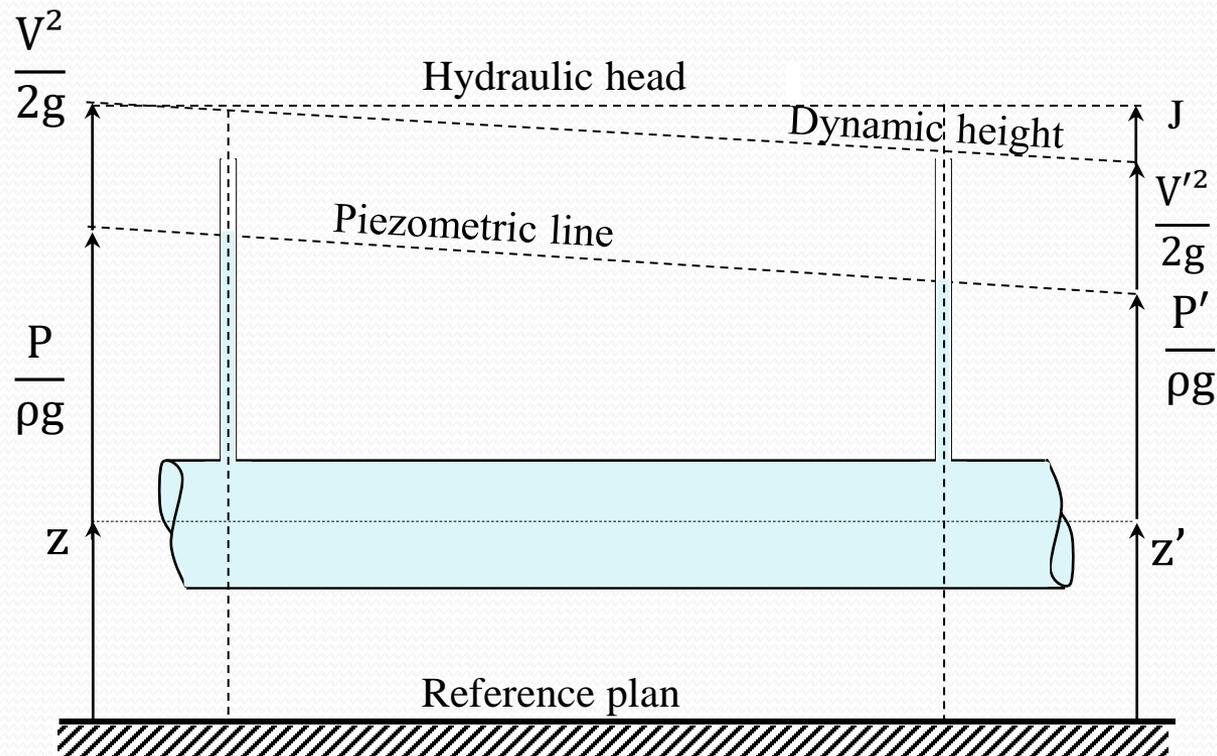


Fig. V. : Graphical representation of the load line and the piezometric line for a constant section pipeline for a real fluid

V.3 – Evaluation des pertes de charge

The flow of a real fluid in a pipe is subject to frictional forces related to viscosity that will tend to slow it down. These represent the head losses along the flow.

In real fluid hydrodynamics, there are two types of pressure drop:

- **Singular pressure loss** which occurs when the uniform flow is locally disturbed ,
- **Linear pressure loss** representing the energy lost between two points.

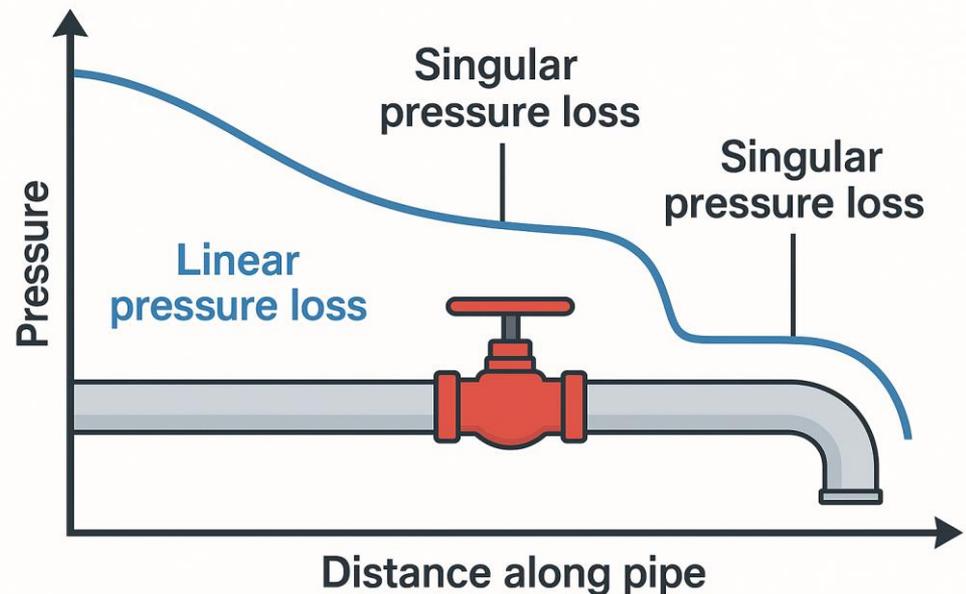


Fig. V. : linear and singular pressure loss

V.3.1– Singular pressure loss

When a flow in a pipe undergoes sudden variations (in cross-section or direction), so-called singular pressure losses occur. These are inevitably present in all installations, and cause mechanical (corrosion) and hydrodynamic problems.

Singular pressure losses can be caused by:

- a change in pipe cross-section (convergent, divergent),
- a change in direction (elbow),
- a branch or connection,
- a flow measurement and control device...

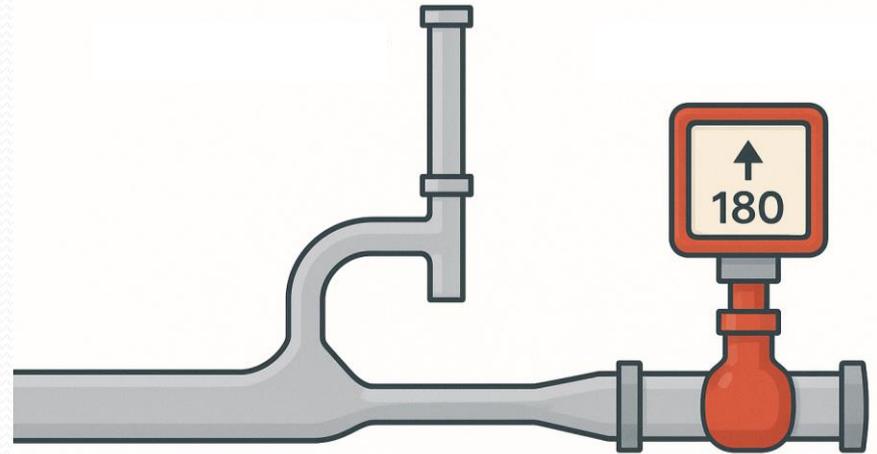


Fig. V : Singular pressure losses

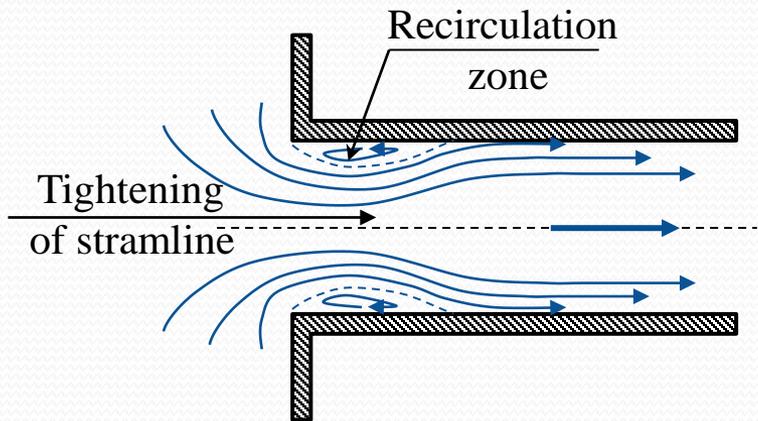


Fig V. : Curve of streamline, which detaches from the wall.

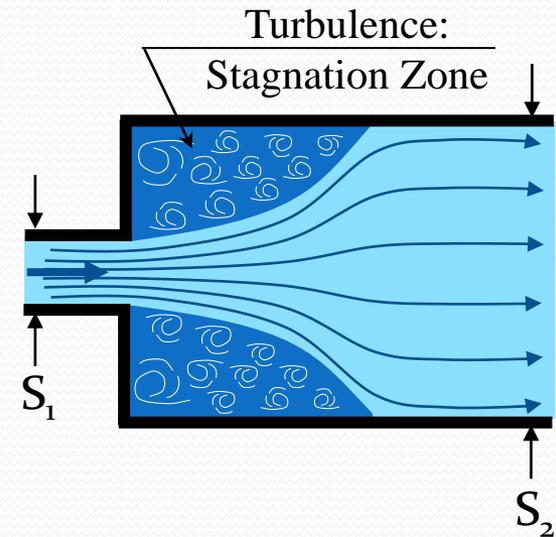


Fig. V.: Formation of recirculation and/or stagnation zones.

The singular pressure losses are expressed by the following relationship:

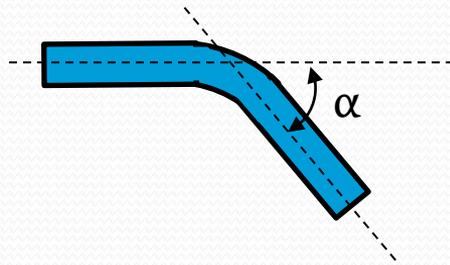
$$J = \zeta \cdot \frac{V^2}{2g}$$

With:

- ζ : Coefficient (without unit) of pressure losses. It depends on the nature and geometry of the shape accident.

The values of ζ are given by the manufacturers in their catalogs.

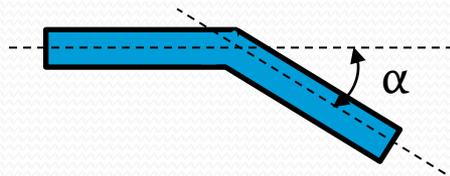
Rounded elbow



Values of ζ

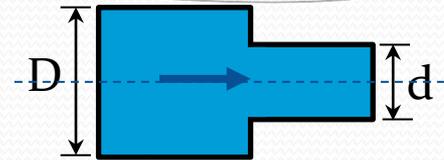
α	$\frac{r}{D} = \frac{\text{curvature radius}}{\text{inner diameter}}$				
	1	2	3	4	5
22,5	0,05	0,05	0,05	0,05	0,05
30	0,07	0,06	0,06	0,06	0,05
45	0,14	0,10	0,09	0,08	0,08
60	0,19	0,12	0,11	0,10	0,09
90	0,21	0,14	0,12	0,11	0,09

Sharp elbow



α	22,5	30	45	60	90
ζ	0,07	0,11	0,24	0,47	1,13

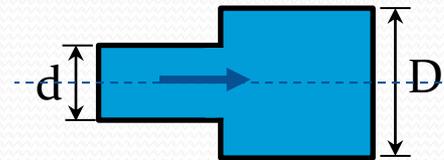
Sudden contraction



$\frac{d}{D}$	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
ζ	0,50	0,48	0,45	0,43	0,40	0,36	0,31	0,24	0,17	0,09	0,00

In the case of a tank discharging into a pipeline, it is necessary to take $\zeta = 0,50$.

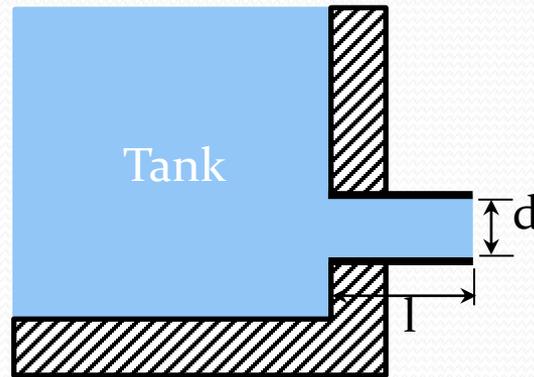
Sudden enlargement



$\frac{d}{D}$	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
ζ	1,00	0,98	0,92	0,83	0,71	0,56	0,41	0,26	0,13	0,04	0,00

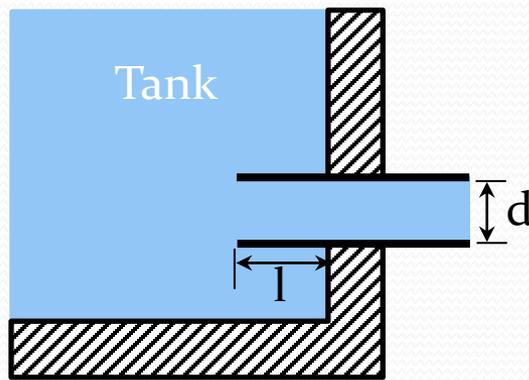
In the case of a pipeline discharging into a tank, it is necessary to take $\zeta = 0,50$.

Flow nozzle



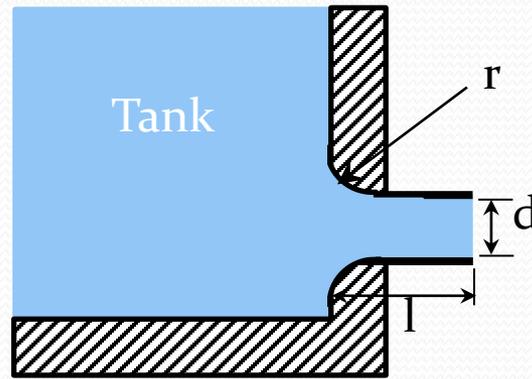
For $2.d < l < 5.d$
 $\zeta = 1,5$

Outlet with
protrusion inside
the tank



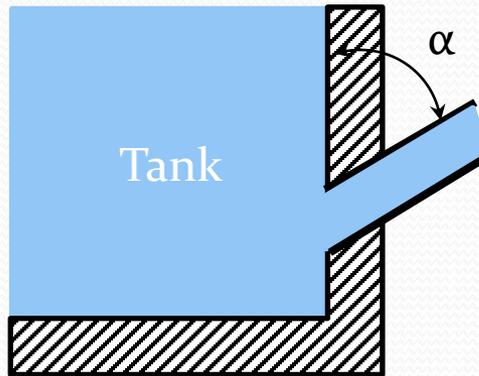
For a protrusion $l > \frac{d}{2}$
 $\zeta = 1,0$

Rounded edge
connector



For $\frac{r}{d} > 0,18$ $\zeta = 0,05$

Oblique cylindrical
connection



α	20°	30°	45°	60°	70°	80°	90°
ζ	0,96	0,91	0,81	0,70	0,63	0,56	0,50

V.3.1– Linear pressure loss

Linear pressure losses are generated by friction along straight lengths of pipes. They depend:

- the type of flow and the quality of the tube (λ)
- the diameter of the pipe ($1/D$)
- the dynamic pressure ($v^2/2g$)
- the length of the pipe.

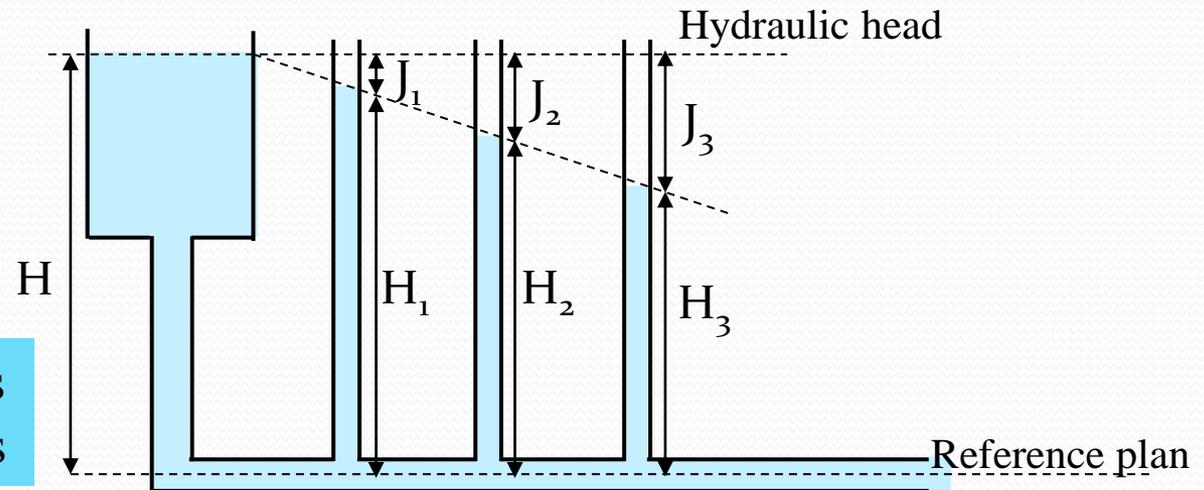


Fig V. : Linear pressure losses along straight lengths of pipes

Due to the difficulty of solving the Navier-Stokes equation system, these pressure losses are generally calculated by empirical equations of the type:

$$\Delta H = L.J = \lambda \frac{L}{D} \frac{V^2}{2g} (mce)$$

With:

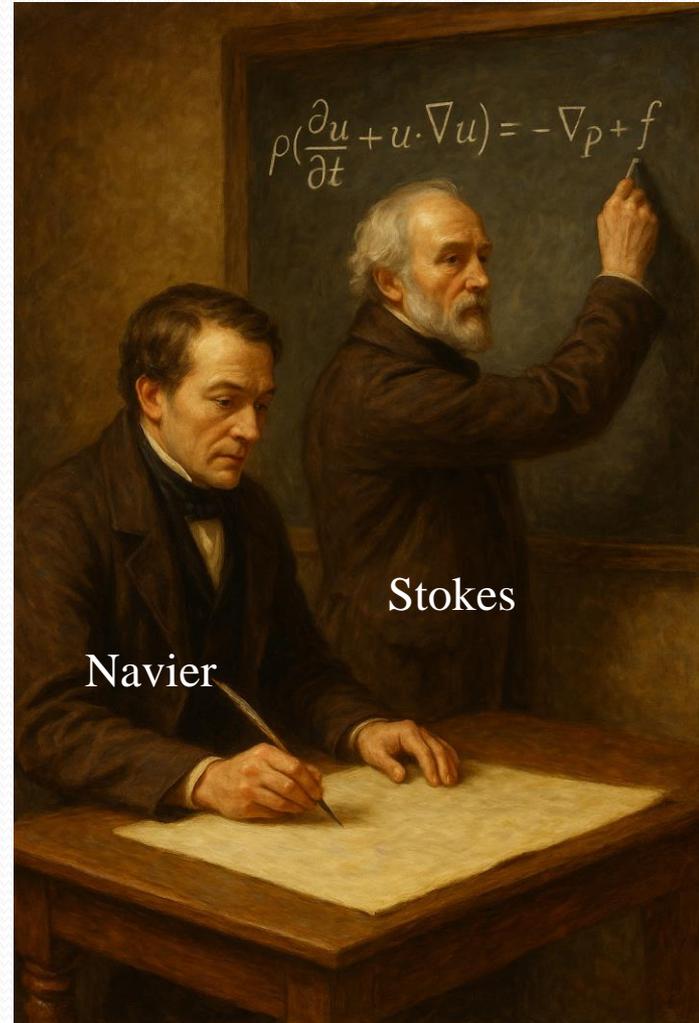
λ : pressure loss coefficient. It is dimensionless and depends on the Reynolds number and wall roughness.

L: pipe length.

D: diameter.

V: average flow velocity.

J: pressure drop per unit length.



Navier

Stokes

In laminar flow

In laminar flow, only viscosity forces are involved in the calculation of the coefficient λ . Given that the velocity is very low, the surface condition of the wall has no impact on linear pressure losses.

So we have:

$$\lambda = \frac{64}{R_e} \quad \text{and as} \quad R_e = \frac{V \cdot D}{\nu}$$

$$\Delta H = L \frac{32}{g} \frac{V \cdot V}{D^2}$$

- V: Average velocity of the flow through the section considered (m/s)
- D: Pipe diameter or fluid stream width (m).
- ν : Kinematic viscosity of the fluid (m²/s).
- g: Gravity (m/s²).

In turbulent flow

The irregularities (roughness) of the internal surface of the pipe have a direct effect on the friction forces, which exerts a definite influence on the flow. A rough surface can be considered as consisting of a series of elementary protuberances. They are characterized by a certain height k (sometimes designated by ϵ).

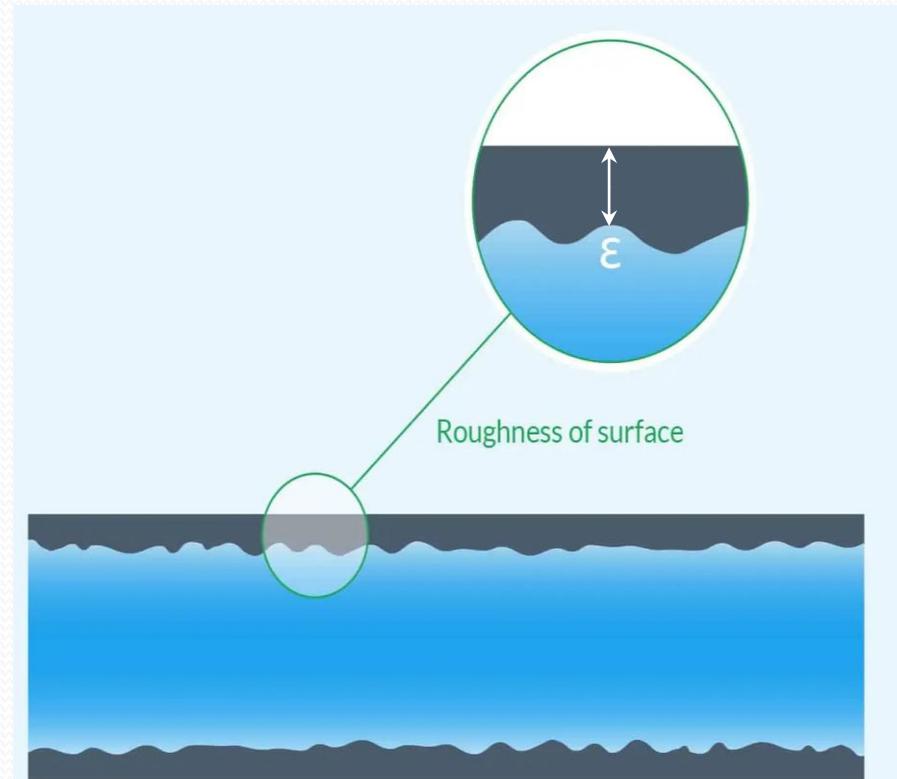


Fig V. : Roughness of internal surface

In the case of rough turbulent regimes, the pressure losses depend only on the friction of the fluid on the asperities of the pipe and therefore on the relative roughness: k/D (ϵ/D).

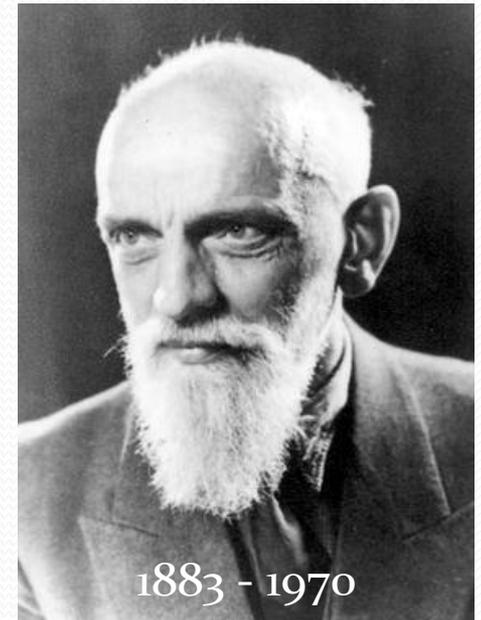
Methods for calculating the pressure loss coefficient

In the turbulent regime, there are several methods for calculating the pressure loss coefficient.

Blasius formula

For a domain where: $4000 < Re < 100000$, we can use the Blasius formula as an approximate formula:

$$\lambda = (100.R_e)^{-0,25}$$



Nikuradse formula

This formula is used for rough turbulent flows:

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10} \left[\frac{k}{3,7D} \right]$$

Its area of use is for

$$\frac{k \cdot Re \sqrt{\lambda}}{D} \geq 200$$

So λ only depends on k/D .



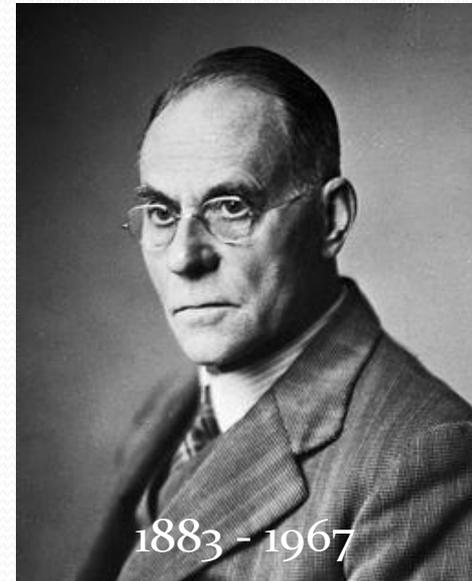
Colebrook formula

This is the most recognized formula for turbulent flows, for $4000 < Re < 10^8$:

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10} \left[\frac{k}{3,7D} + \frac{2,51}{Re \cdot \sqrt{\lambda}} \right]$$

This formula is implicit and can therefore only be solved using successive approximations.

This implicit relationship is difficult to exploit analytically and is most often represented on a graph.



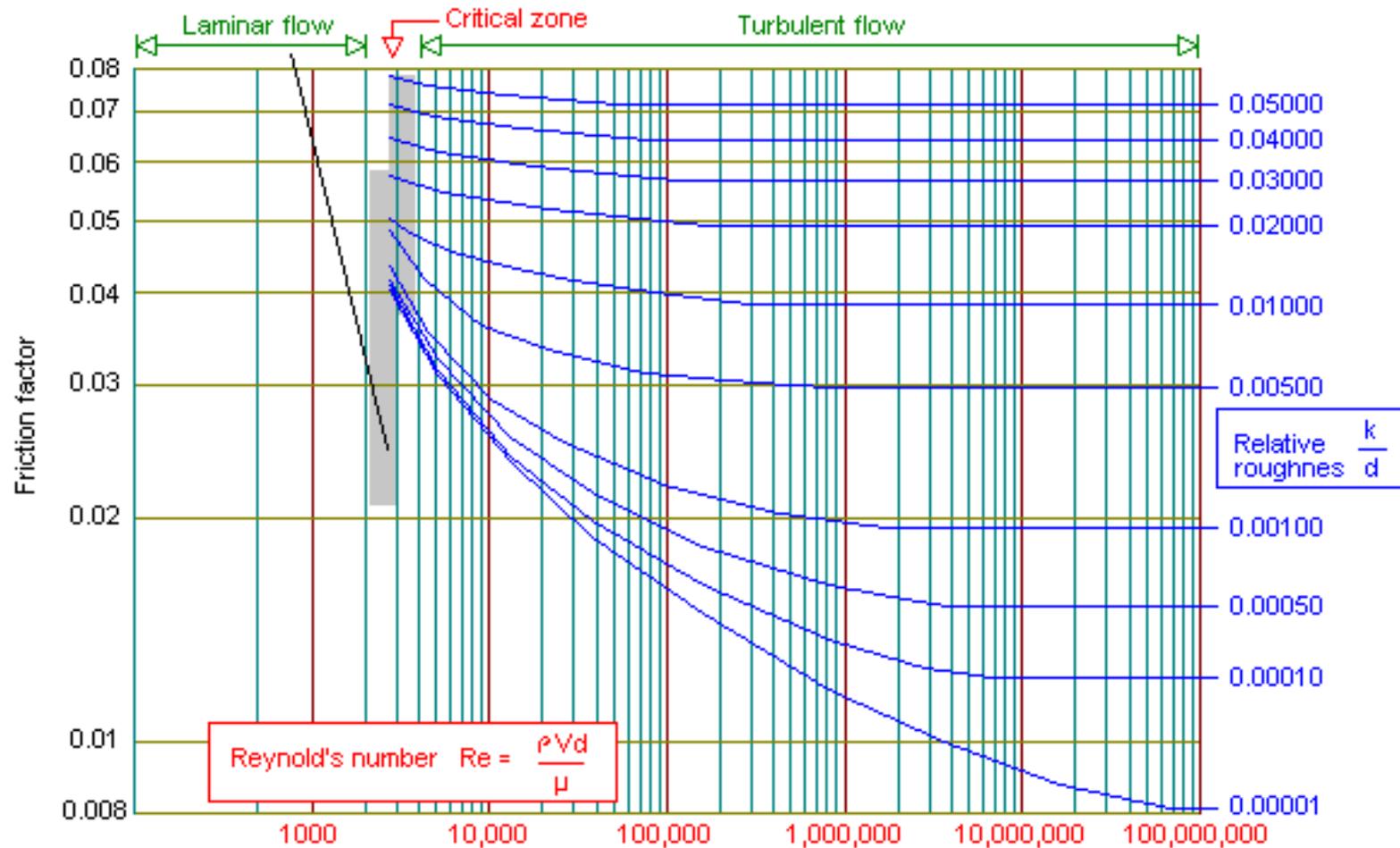


Fig. V. : Moody-Stanton diagram relating the regular pressure drop coefficient λ to the Reynolds number Re and the relative roughness ϵ/D .

Swamee-Jain formula

This equation is an explicit formula used to quickly calculate λ for turbulent flow in a pipe. It's very useful because it avoids iterative methods like solving the implicit Colebrook equation.

The Swamee-Jain equation is:

$$\lambda = \frac{0.25}{\left[\log_{10} \left(\frac{\varepsilon}{3.7D} + \frac{5.74}{Re^{0.9}} \right) \right]^2}$$

where:

- λ : pressure loss coefficient
- ε : pipe roughness height (m)
- D : pipe diameter (m)
- $Re = \text{Reynolds number}$ (dimensionless)

Important Notes: this formula is valid for turbulent flow ($Re > 4000$)