Corrigé du TD N°4 : Interpolation polynomiale

Exercice 1

$P(x) = \sum_{i=0}^{n-1} a_i L_i(x)$ $\begin{cases} a_i = f_i \\ L_i = \prod_{j=0, j\neq i}^{n-1} \frac{x-x_j}{x_i-x_j} \end{cases}$

On veut démontrer que $P(x_k) = a_k$ pour $i = 0, \dots, n-1$

$P(x_k) = \sum_{i=0}^{n-1} a_i L_i(x_k)$ et $L_i(x_k) = \prod_{j=0, j\neq i}^{n-1} \frac{x_k-x_j}{x_i-x_j}$

- Si $k=j \Rightarrow \frac{x_k-x_j}{x_i-x_j} = 0 \Rightarrow \boxed{L_i(x_k) = 0}$ si $k\neq i$
- Si $k=i \Rightarrow \frac{x_k-x_j}{x_k-x_j} = 1 \Rightarrow \boxed{L_i(x_k) = 1}$

Donc $P(x_k) = \sum_{i=0}^{n-1} a_i L_i(x_k)$: dans cette somme tous les produits sont nuls sauf pour $k=i \Rightarrow \boxed{\boldsymbol{P(x_k) = a_k}}$

Exercice 2

1/ Le polynôme de Lagrange : $P(x) = \sum_{i=0}^{n-1} a_i L_i(x)$; $L_i = \prod_{j=0, j\neq i}^{n-1} \frac{x-x_j}{x_i-x_j}$ et $a_i = f_i$

	x_0	x_1	x_2	x_3
xi	0	2	4	6
f(xi)=fi	0	4	0	4

$$L_0 = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-2)(x-4)(x-6)}{(0-2)(0-4)(0-6)} = -\frac{(x-2)(x-4)(x-6)}{48} ;$$

$$L_1 = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-4)(x-6)}{(2-0)(2-4)(2-6)} = \frac{(x-0)(x-4)(x-6)}{16} ;$$

$$L_2 = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-2)(x-6)}{(4-0)(4-2)(4-6)} = -\frac{(x-0)(x-2)(x-6)}{16} ;$$

$$L_3 = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-2)(x-4)}{(6-0)(6-2)(6-4)} = \frac{(x-0)(x-2)(x-4)}{48}$$

$$P(x) = a_0 L_0(x) + a_1 L_1(x) + a_2 L_2(x) + a_3 L_3(x) \Leftrightarrow P(x) = 0.L_0(x) + 4.L_1(x) + 0.L_2(x) + 4.L_3(x)$$

$$\boxed{P(x) = \frac{20}{3}x - 3x^2 + \frac{1}{3}x^3}$$

2/ Le polynôme de Newton :

$$P(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \dots + f[x_0,x_1,\dots,x_{n-1}](x-x_0)(x-x_1)\dots(x-x_{n-2})$$

$$P(x) = \sum_{i=0}^{n-1} f[x_0,x_1,\dots,x_i] \prod_{j=0}^{i-1}(x-x_j)$$

$$f[x_0] = f(x_0) ;\ f[x_0,x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} ;\ ;\ f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$$

$x_0=0$	***$f[x_0]=0$***	***$f[x_0,x_1] = \frac{4-0}{2-0} = 2$***		
$x_1=2$	*$f[x_1]=4$*	*$f[x_1,x_2] = \frac{0-4}{4-2} = -2$*	***$f[x_0,x_1,x_2] = \frac{-2-2}{4-0} = -1$***	***$f[x_0,x_1,x_2,x_3] = \frac{1+1}{6-0} = \frac{1}{3}$***
$x_2=4$	*$f[x_2]=0$*	*$f[x_2,x_3] = \frac{4-0}{6-4} = 2$*	*$f[x_1,x_2,x_3] = \frac{2+2}{6-2} = 1$*	
$x_3=6$	*$f[x_3]=4$*			

$$\Rightarrow P(x) = 0 + 2(x-x_0) - 1(x-x_0)(x-x_1) + \frac{1}{3}(x-x_0)(x-x_1)(x-x_2)$$
$$= 0 + 2(x-0) - 1(x-0)(x-2) + \frac{1}{3}(x-0)(x-2)(x-4)$$

$$\Rightarrow \boxed{\boldsymbol{P(x) = \frac{20}{3}x - 3x^2 + \frac{1}{3}x^3}}$$

Exercice 3

1/ Le polynôme de Newton :

Points	x	Y=*f(x_i)*
1	1	0
2	1,5	1
3	2	2
4	2,5	-1,5

$x_0=1$	**$f[x_0]=0$**	**$f[x_0,x_1]=\frac{1-0}{1,5-1}=2$**	**$f[x_0,x_1,x_2]=\frac{2-2}{2-1}=0$**	**$f[x_0,x_1,x_2,x_3]=\frac{-9-0}{2,5-1}=-6$**
$x_1=1,5$	$f[x_1]=1$	$f[x_1,x_2]=\frac{2-1}{2-1,5}=2$	$f[x_1,x_2,x_3]=\frac{-7-2}{2,5-1,5}=-9$	
$x_2=2$	$f[x_2]=2$	$f[x_2,x_3]=\frac{-1,5-2}{2,5-2}=-7$		
$x_3=2,5$	$f[x_3]=-1,5$			

$$\Rightarrow P(x)=0+2(x-x_0)+0(x-x_0)(x-x_1)-6(x-x_0)(x-x_1)(x-x_2)$$
$$=0+2(x-1)+0(x-1)(x-1,5)-6(x-1)(x-1,5)(x-2)$$

$$\Rightarrow \boxed{\boldsymbol{P(x)=16-37x+27x^2-6x^3}}$$

2/

x_i	0,1	0,2	0,3	0,4	0,5
$f(x_i)$	1,4	1,56	1,76	2,00	2,28

$x_0=0,1$	**$f[x_0]=1,4$**	**$f[x_0,x_1]=\frac{1,56-1,4}{0,2-0,1}=1,6$**	**$f[x_0,x_1,x_2]=\frac{2-1,6}{0,3-0,1}=2$**	**$f[x_0,x_1,x_2,x_3]=\frac{2-2}{0,4-0,1}=0$**	**$f[x_0,x_1,x_2,x_3,x_4]=0$**
$x_1=0,2$	$f[x_1]=1,56$	$f[x_1,x_2]=\frac{1,76-1,56}{0,3-0,2}=2$	$f[x_1,x_2,x_3]=\frac{2,4-2}{0,4-0,2}=2$	$f[x_1,x_2,x_3,x_4]=\frac{2-2}{0,4-0,1}=0$	
$x_2=0,3$	$f[x_2]=1,76$	$f[x_2,x_3]=\frac{2-1,76}{0,4-0,3}=2,4$	$f[x_2,x_3,x_4]=\frac{2,8-2,4}{0,5-0,3}=2$		
$x_3=0,4$	$f[x_3]=2$	$f[x_3,x_4]=\frac{2,28-2}{0,5-0,4}=2,8$			
$x_4=0,5$	$f[x_4]=2,28$				

$$\Rightarrow P(x)=1,4+1,6(x-x_0)+2(x-x_0)(x-x_1)+0(x-x_0)(x-x_1)(x-x_2)+0(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$
$$=1,4+1,6(x-0,1)+2(x-0,1)(x-0,2)+0(x-0,1)(x-0,2)(x-0,3)+0(x-0,1)(x-0,2)(x-0,3)(x-0,4)$$

$$\Rightarrow \boxed{\boldsymbol{P(x)=1,28+x+2x^2}}$$

Exercice 4

$$f(x) = |x|$$

x_i	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x_i)=\lvert x\rvert$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1

La base de Lagrange :

$$\begin{cases}
L_0 = \dfrac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} = \dfrac{\left(x+\frac{1}{2}\right)(x-0)\left(x-\frac{1}{2}\right)(x-1)}{\left(-1+\frac{1}{2}\right)(-1-0)\left(-1-\frac{1}{2}\right)(-1-1)} = \dfrac{2x\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)(x-1)}{3} \\
L_1 = \dfrac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} = \dfrac{(x+1)(x-0)\left(x-\frac{1}{2}\right)(x-1)}{\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}-0\right)\left(-\frac{1}{2}-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)} = -\dfrac{8x(x+1)\left(x-\frac{1}{2}\right)(x-1)}{3} \\
L_2 = \dfrac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} = \dfrac{(x+1)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)(x-1)}{(0+1)\left(0+\frac{1}{2}\right)\left(0-\frac{1}{2}\right)(0-1)} = \dfrac{(x+1)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)(x-1)}{4} \\
L_3 = \dfrac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} = \dfrac{(x+1)\left(x+\frac{1}{2}\right)(x-0)(x-1)}{\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-0\right)\left(\frac{1}{2}-1\right)} = -\dfrac{8x(x+1)\left(x+\frac{1}{2}\right)(x-1)}{3} \\
L_4 = \dfrac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} = \dfrac{(x+1)\left(x+\frac{1}{2}\right)(x-0)\left(x-\frac{1}{2}\right)}{(1+1)\left(1+\frac{1}{2}\right)(1-0)\left(1-\frac{1}{2}\right)} = \dfrac{2x(x+1)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}{3}
\end{cases}$$

$$P(x) = a_0L_0(x) + a_1L_1(x) + a_2L_2(x) + a_3L_3(x) + a_4L_4(x)$$

$$\Leftrightarrow P(x) = L_0(x) + \tfrac{1}{2}L_1(x) + 0.L_2(x) + \tfrac{1}{2}L_3(x) + L_4(x) \Rightarrow \boxed{P(x) = \tfrac{4}{3}x^2\left(\tfrac{7}{4} - x^2\right)}$$

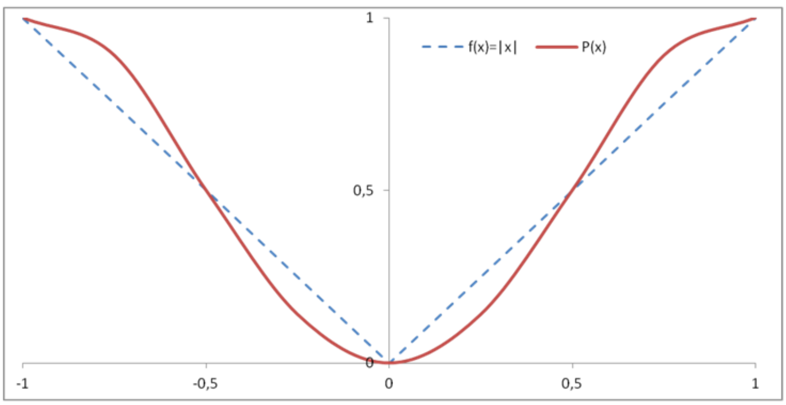
La base de Newton :

$x_0=-1$	$f[x_0]=1$	$f[x_0,x_1] = \frac{-0,5}{-0,5+1} = -1$	$f[x_0,x_1,x_2] = \frac{-1+1}{0+1} = 0$	$f[x_0,x_1,x_2,x_3] = \frac{2-0}{0,5+1} = \frac{4}{3}$	$f[x_0,x_1,x_2,x_3,x_4] = \frac{-\frac{4}{3}-\frac{4}{3}}{1+1} = -\frac{4}{3}$
$x_1=-\frac{1}{2}$	$f[x_1]=\frac{1}{2}$	$f[x_1,x_2] = \frac{-0,5}{0+0,5} = -1$	$f[x_1,x_2,x_3] = \frac{1+1}{0+1} = 2$	$f[x_1,x_2,x_3,x_4] = \frac{0-2}{1+0,5} = -\frac{4}{3}$	
$x_2=0$	$f[x_2]=0$	$f[x_2,x_3] = \frac{0,5}{0,5} = 1$	$f[x_2,x_3,x_4] = \frac{1-1}{1-0} = 0$		
$x_3=\frac{1}{2}$	$f[x_3]=\frac{1}{2}$	$f[x_3,x_4] = \frac{0,5}{0,5} = 1$			
$x_4=1$	$f[x_4]=1$				

$$P(x) = 1 - (x-x_0) + 0(x-x_0)(x-x_1) + \frac{4}{3}(x-x_0)(x-x_1)(x-x_2) - \frac{4}{3}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$= 1 - (x+1) + 0(x+1)\left(x+\tfrac{1}{2}\right) + \tfrac{4}{3}(x+1)\left(x+\tfrac{1}{2}\right)(x-0) - \tfrac{4}{3}(x+1)\left(x+\tfrac{1}{2}\right)(x-0)\left(x-\tfrac{1}{2}\right)$$

$$\Rightarrow \boxed{\boldsymbol{P(x) = \tfrac{4}{3}x^2\left(\tfrac{7}{4} - x^2\right)}}$$



Exercice 5

Un polynôme de degré 2 est de la forme : $P(x) = a_0 + a_1 x + a_2 x^2$ alors $E(a_0, a_1, a_2) = \sum_{i=1}^{m=5} \left(y_i - (a_0 + a_1 x + a_2 x^2)\right)^2$ est une fonction de $a_0, a_1\ et\ a_2$, et donc $\nabla E(a_0, a_1, a_2) = 0$ s'écrit sous la forme explicite :

$$\begin{cases} \frac{\partial E}{\partial a_0} = 0 \\ \frac{\partial E}{\partial a_1} = 0 \\ \frac{\partial E}{\partial a_2} = 0 \end{cases} \Leftrightarrow \begin{cases} \sum_{i=1}^{m=5}\left(y_i - (a_0 + a_1 x_i + a_2 {x_i}^2)\right) \times 2(-1) = 0 \\ \sum_{i=1}^{m=5}\left(y_i - (a_0 + a_1 x_i + a_2 {x_i}^2)\right) \times 2(-x_i) = 0 \\ \sum_{i=1}^{m=5}\left(y_i - (a_0 + a_1 x_i + a_2 {x_i}^2)\right) \times 2(-{x_i}^2) = 0 \end{cases} \Leftrightarrow \begin{cases} 5a_0 + a_1 \sum_{i=1}^{5} x_i + a_2 \sum_{i=1}^{5} {x_i}^2 = \sum_{i=1}^{5} y_i \\ a_0 \sum_{i=1}^{5} x_i + a_1 \sum_{i=1}^{5} {x_i}^2 + a_2 \sum_{i=1}^{5} {x_i}^3 = \sum_{i=1}^{5} x_i y_i \\ a_0 \sum_{i=1}^{5} {x_i}^2 + a_1 \sum_{i=1}^{5} {x_i}^3 + a_2 \sum_{i=1}^{5} {x_i}^4 = \sum_{i=1}^{5} {x_i}^2 y_i \end{cases}$$

	x_i	y_i	${x_i}^2$	${x_i}^3$	${x_i}^4$	$x_i y_i$	${x_i}^2 y_i$
	-1	-1,5	1	-1	1	1,5	-1,5
	-0,5	0	0,25	-0,125	0,0625	0	0
	0	0,25	0	0	0	0	0
	0,5	0	0,25	0,125	0,0625	0	0
	1	0	1	1	1	0	0
$\sum$	**0**	**-1,25**	**2,5**	**2**	**2,125**	**1,5**	**-1,5**

$$\Rightarrow \begin{cases} 5a_0 + 2{,}5a_2 = -1{,}25 \\ 2{,}5a_1 + 2a_2 = 1{,}5 \\ 2{,}5a_0 + 2a_1 + 2{,}125a_2 = -1{,}5 \end{cases} \Rightarrow \boxed{\begin{cases} \boldsymbol{a_0 = -1,68} \\ \boldsymbol{a_1 = -1,69} \\ \boldsymbol{a_2 = 2,86} \end{cases}}$$

Donc le polynôme P(x) s'écrit :

$$\boxed{\boldsymbol{P(x) = -1,68 - 1,69\, x + 2,86\, x^2}}$$