TD n2 : Intégrales et calcul des primitives (première partie).

I) Intégrales indéfinies :

Exercice 01 :

(1) $\displaystyle\int \frac{x^2+1}{\sqrt[3]{x}}\,dx = \int (x^2+1)\,x^{-1/3}\,dx$

$$= \int \left(x^{5/3} + x^{-1/3}\right) dx$$

$$= \frac{1}{\frac{5}{3}+1}\,x^{5/3+1} + \frac{1}{-\frac{1}{3}+1}\,x^{-\frac{1}{3}+1} + C, \quad C \in \mathbb{R}$$

$$= \frac{3}{8}\,x^{8/3} + \frac{3}{2}\,x^{2/3} + C.$$

(2) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int \frac{1}{x}(\ln x)^2\,dx = \frac{(\ln x)^3}{3} + C, \quad C \in \mathbb{R}.$

(On peut poser $u = \ln x$, on obtient $\displaystyle\int u'(x)\,u^2(x)\,dx = \frac{u^3(x)}{3} + c$).

(3) $\displaystyle\int \cos(3x+2)\,dx = \frac{1}{3}\int 3\cos(3x+2)\,dx$

On pose $u(x) = 3x+2$, alors on obtient

$$\frac{1}{3}\int 3\cos(3x+2)\,dx = \frac{1}{3}\int u'(x)\cos(u(x))\,dx$$

$$= \frac{1}{3}\sin(u(x)) + C, \quad C \in \mathbb{R}$$

$$= \frac{1}{3}\sin(3x+2) + C.$$

(4) $\displaystyle\int (\operatorname{sh} x)^2\,dx = \int \left(\frac{e^{x}-e^{-x}}{2}\right)^2 dx = \frac{1}{4}\int \left(e^{2x} + e^{-2x} - 2\right) dx$

$$= \frac{1}{4}\left[\frac{e^{2x}}{2} + \frac{e^{-2x}}{-2} - 2x\right] + C, \quad C \in \mathbb{R}.$$

$$= \frac{1}{4}\left[\frac{e^{2x} - e^{-2x}}{2} - 2x\right] + C$$

$$= \frac{1}{4}\operatorname{sh}(2x) - \frac{x}{2} + C.$$

5) $\displaystyle\int \frac{dx}{x \ln x^2} = \frac{1}{2}\int \frac{dx}{x \ln|x|} = \frac{1}{2}\int \frac{\frac{1}{x}\,dx}{\ln|x|}$.

On pose $u = \ln|x|$, alors on obtient

$$\frac{1}{2}\int \frac{u'(x)}{u(x)}\,dx = \frac{1}{2}\ln|u(x)| + C, \quad C\in\mathbb{R}$$

$$= \frac{1}{2}\ln\left|\ln|x|\right| + C.$$

6) $\displaystyle\int \frac{e^{x}}{e^{-x}+e^{x}}\,dx = \int \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{1+e^{2x}}\,dx$

On pose $u = 1+e^{2x}$, on obtient.

$$\frac{1}{2}\int \frac{u'(x)}{u(x)}\,dx = \frac{1}{2}\ln|u(x)| + C, \quad C\in\mathbb{R}$$

$$= \frac{1}{2}\ln(1+e^{2x}) + C.$$

Exercice 02 : 1) $\displaystyle\int \frac{\ln x}{x^n}\,dx, \quad n \neq 1.$

On sait que $\displaystyle\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx$.

Donc, si on pose

$u(x) = \ln x$, $\quad u'(x) = \frac{1}{x}$

$v'(x) = \frac{1}{x^n} = x^{-n}$, $\quad v(x) = \frac{x^{-n+1}}{-n+1}$, alors

$$\int \frac{\ln x}{x^n}\,dx = \frac{x^{1-n}}{1-n}\ln x - \int \frac{x^{-n}}{1-n}\,dx$$

$$= \frac{x^{1-n}}{1-n}\ln x + \frac{1}{n-1}\int x^{-n}\,dx = \frac{x^{1-n}}{1-n}\ln x + \frac{1}{n-1}\left[\frac{x^{-n+1}}{-n+1}\right] + C, \quad C\in\mathbb{R}$$

$$= \frac{x^{1-n}}{1-n}\ln x - \frac{x^{-n+1}}{(n-1)^2} + C.$$

(2) $\int (x^2-x)\sin x\,dx = ?$

$u(x) = x^2-x \quad , \quad u'(x) = 2x-1$

$v'(x) = \sin x \quad , \quad v(x) = -\cos x$

Ainsi, $\int (x^2-x)\sin x\,dx = -(x^2-x)\cos x + \int (2x-1)\cos x\,dx$

une deuxième intégration par parties, on pose

$u(x) = 2x-1 \quad , \quad u'(x) = 2$

$v'(x) = \cos x \quad , \quad v(x) = \sin x.$

On a :

$$\int (2x-1)\cos x\,dx = (2x-1)\sin x - 2\int \sin x\,dx$$

$$= (2x-1)\sin x + 2\cos x + C, \qquad C \in \mathbb{R}.$$

Donc, finalement,

$$\int (x^2-x)\sin x\,dx = (x-x^2)\cos x + (2x-1)\sin x + 2\cos x + C.$$

(3) $\int \frac{\ln(1+x^2)}{x^2}\,dx = ?$

$u(x) = \ln(1+x^2) \quad , \quad u'(x) = \frac{2x}{1+x^2}$

$v'(x) = \frac{1}{x^2} \quad , \quad v(x) = -\frac{1}{x}.$

On a :

$$\int \frac{\ln(1+x^2)}{x^2}\,dx = -\frac{\ln(1+x^2)}{x} + \int \frac{2}{1+x^2}\,dx$$

$$= -\frac{\ln(1+x^2)}{x} + 2\,\mathrm{arctg}\,x + C, \qquad C \in \mathbb{R}.$$

(4) $\int e^x \cos x\,dx = ?$

On pose

$u(x) = e^x$, $u'(x) = e^x$

$v'(x) = \cos x$, $v(x) = \sin x$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

une deuxième intégration par partie, donne

$u(x) = e^x$, $u'(x) = e^x$

$v'(x) = \sin x$, $v(x) = -\cos x$

Donc,

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + \int e^x \cos x dx$$

Ainsi,

$$\int e^x \cos x \, dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

Ceci implique que

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\Rightarrow \int e^x \cos x \, dx = \frac{e^x}{2} (\sin x + \cos x) + C.$$

Exercice 03 : (1) $\int \frac{dx}{x^2+x+1}$.

On a : $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4} \left[\left(\frac{2x+1}{\sqrt{3}} \right)^2 + 1 \right]$

On pose

$$u = \frac{2x+1}{\sqrt{3}} \Rightarrow du = \frac{2}{\sqrt{3}} dx \Rightarrow dx = \frac{\sqrt{3}}{2} du.$$

Ainsi, $\int \frac{dx}{x^2+x+1} = \int \frac{\frac{\sqrt{3}}{2} du}{\frac{3}{4}(u^2+1)} = \frac{2}{\sqrt{3}} \operatorname{arctg} u + c$, $c \in \mathbb{R}$.

$$\int \frac{dx}{x^2+x+1} = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + c.$$

② $\displaystyle\int \frac{dx}{x^2+x\sqrt{2x-x^2}} = ?$ $\quad x \in]0,2]$.

On pose $x = \frac{1}{u} \Rightarrow dx = \frac{-1}{u^2}du$.

$$\int \frac{dx}{x^2+x\sqrt{2x-x^2}} = \int \frac{-\frac{1}{u^2}du}{\frac{1}{u^2}+\frac{1}{u}\sqrt{\frac{2}{u}-\frac{1}{u^2}}} = \int \frac{-\frac{1}{u^2}du}{\frac{1}{u^2}+\frac{1}{u}\sqrt{\frac{2u^2-u}{u^3}}}$$

$$= \int \frac{-\frac{1}{u^2}du}{\frac{1}{u^2}+\frac{1}{u^2}\sqrt{\frac{2u^2-u}{u}}} = \int \frac{-du}{1+\sqrt{2u-1}}$$

On pose $t = \sqrt{2u-1} \Rightarrow dt = \frac{du}{\sqrt{2u-1}}$.

Ainsi, $\displaystyle\int \frac{-du}{1+\sqrt{2u-1}} = \int -\frac{\sqrt{2u-1}\,dt}{1+t} = -\int \frac{t}{1+t}dt$

$$= -\int \frac{t+1-1}{t+1}dt = -\left[\int dt - \int \frac{dt}{1+t}\right]$$

$$= -t + \ln|1+t| + C, \quad C \in \mathbb{R}.$$

$$= -\sqrt{2u-1} + \ln\left(1+\sqrt{2u-1}\right) + C.$$

$$= -\sqrt{\frac{2}{x}-1} + \ln\left(1+\sqrt{\frac{2}{x}-1}\right) + C.$$

③ $\displaystyle\int \sqrt{a^2+x^2}\,dx$.

Si $a=0$, alors on a :

$$\int \sqrt{x^2}\,dx = \int |x|\,dx = -\frac{x|x|}{2} + c, \quad c \in \mathbb{R}.$$

Si $a \neq 0$, alors :

$$\int \sqrt{a^2+x^2}\,dx = |a| \int \sqrt{1+\left(\frac{x}{a}\right)^2}\,dx$$

posons $\frac{x}{a} = \operatorname{sh} t \Rightarrow x = a \operatorname{sh} t$

$\Rightarrow dx = a \operatorname{ch} t\, dt.$

Ainsi $\int \sqrt{a^2+x^2}\,dx = a|a| \int \sqrt{1+\operatorname{sh}^2 t}\, \operatorname{ch} t\, dt$

$$= a|a| \int \sqrt{\operatorname{ch}^2 t}\, \operatorname{ch} t\, dt$$

$$= a|a| \int \operatorname{ch}^2 t\, dt, \qquad \operatorname{ch} t \geq 0, \forall t \in \mathbb{R}.$$

$$= \frac{a|a|}{2} \int (\operatorname{ch}(2t)+1)\, dt$$

$$= \frac{a|a|}{2} \left[\frac{1}{2} \operatorname{sh}(2t) + t\right] + c, \quad c \in \mathbb{R}$$

$$= \frac{a|a|}{2} (\operatorname{sh} t \operatorname{ch} t + t) + c$$

$$= \frac{a|a|}{2} \left(\operatorname{sh} t \sqrt{1+\operatorname{sh}^2 t} + t\right) + c$$

$$= \frac{a|a|}{2} \left(\frac{x}{a} \sqrt{1+\left(\frac{x}{a}\right)^2} + \operatorname{arg ch}\left(\frac{x}{a}\right)\right) + c$$

$$= \frac{x}{2} \times \sqrt{x^2+a^2} + \frac{a|a|}{2} \operatorname{arg sh}\left(\frac{x}{a}\right) + c.$$

(4) $$\int \frac{dx}{2\sin^2 x + 3\cos^2 x} = \int \frac{\frac{dx}{\cos^2 x}}{2\frac{\sin^2 x}{\cos^2 x} + 3} = \int \frac{\frac{1}{\cos^2 x}}{2\operatorname{tg}^2 x + 3}\, dx$$

Posons $u = \operatorname{tg} x \Rightarrow du = \dfrac{dx}{\cos^2 x}$.

Ainsi,
$$\int \frac{dx}{2\sin^2 x + 3\cos^2 x} = \int \frac{du}{2u^2+3} = \frac{1}{3}\int \frac{du}{\frac{2}{3}u^2+1}$$
$$= \frac{1}{3}\int \frac{du}{\left(\frac{\sqrt{2}}{\sqrt{3}}u\right)^2+1}.$$

On pose $t = \sqrt{\frac{2}{3}}\,u \Rightarrow dt = \sqrt{\frac{2}{3}}\,du$. Alors
$$\int \frac{dx}{2\sin^2 x+3\cos^2 x} = \frac{1}{3}\sqrt{\frac{3}{2}}\int \frac{dt}{1+t^2} = \frac{1}{3}\sqrt{\frac{3}{2}}\operatorname{arctg} t + c,\ c\in\mathbb{R}$$
$$= \frac{1}{3}\sqrt{\frac{3}{2}}\operatorname{arctg}\left(\sqrt{\frac{2}{3}}\operatorname{tg} x\right) + c.$$

Exercice 04:

(1) $\displaystyle\int \frac{x^2}{x^2-1}\,dx$.

$$\frac{x^2}{x^2-1} = \frac{x^2-1+1}{x^2-1} = 1 + \frac{1}{x^2-1} \quad \text{et}$$
$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$
$$= \frac{Ax+A+Bx-B}{x^2-1} = \frac{(A+B)x+A-B}{x^2-1}$$

par identification, $\begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \begin{cases} A = 1/2 \\ B = -1/2 \end{cases}$.

Ainsi,
$$\int \frac{x^2}{x^2-1}dx = \int dx + \frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{dx}{x+1}$$
$$= x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c,\ c\in\mathbb{R}.$$

② $\displaystyle\int \frac{2\,dx}{x(x^2+1)} = ?$

$$\frac{2}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1} = \frac{ax^2+a+bx^2+cx}{x(x^2+1)}$$

par identification,

$$\begin{cases} a+b=0 \\ a=2 \\ c=0 \end{cases} \Rightarrow \begin{cases} a=2 \\ b=-2 \\ c=0 \end{cases}$$

Ainsi,

$$\int \frac{2\,dx}{x(x^2+1)} = \int \frac{2\,dx}{x} - \int \frac{2x\,dx}{x^2+1}$$

$$= 2\ln|x| - \ln(x^2+1) + C; \quad C \in \mathbb{R}.$$

③ $\displaystyle\int \frac{dx}{x(x^2+x+1)^2} = ?$

$$\frac{1}{x(x^2+x+1)^2} = \frac{a}{x} + \frac{bx+c}{x^2+x+1} + \frac{dx+e}{(x^2+x+1)^2}$$

$$= \frac{ax^2+ax+a+bx^2+cx}{x(x^2+x+1)} + \frac{dx+e}{(x^2+x+1)^2}$$

$$= \frac{(ax^2+ax+a+bx^2+cx)(x^2+x+1) + (dx+e)x}{x(x^2+x+1)^2}$$

$$= \frac{ax^4+ax^3+ax^2+bx^4+cx^3+ax^3+ax^2+ax+bx^3+cx^2+ax^2+ax+bx^2+cx + dx^2+ex}{x(x^2+x+1)^2}$$

par identification

$$\begin{cases} a+b=0 \\ 2a+b+c=0 \\ 3a+b+c+d=0 \\ 2a+c+e=0 \\ a=1 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=-1 \\ c=-1 \\ d=-1 \\ e=-1 \end{cases}$$

Donc,

$$\frac{1}{x(x^2+x+1)^2} = \frac{1}{x} - \frac{x+1}{x^2+x+1} - \frac{x+1}{(x^2+x+1)^2}$$

$$= \frac{1}{x} - \frac{1}{2}\,\frac{2x+2}{x^2+x+1} - \frac{1}{2}\,\frac{2x+2}{(x^2+x+1)^2}$$

$$= \frac{1}{x} - \frac{1}{2}\,\frac{2x+1}{x^2+x+1} - \frac{1}{2}\,\frac{1}{x^2+x+1} - \frac{1}{2}\,\frac{2x+1}{(x^2+x+1)^2} - \frac{1}{2}\,\frac{1}{(x^2+x+1)^2}$$

Alors

$$\int \frac{dx}{x} = \ln|x| + c, \qquad c \in \mathbb{R}$$

$$\int \frac{2x+1}{x^2+x+1}\,dx = \ln(x^2+x+1) + c,$$

$$\int \frac{dx}{x^2+x+1} = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$ (d'après l'exercice 3, 1/)

$$\int \frac{2x+1}{(x^2+x+1)^2}\,dx = -\frac{1}{x^2+x+1} + c$$

Il reste l'intégrale $I_1 = \int \frac{dx}{(x^2+x+1)^2}$.

On a : $x^2+x+1 = \frac{3}{4}\left(\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right)$.

Donc $I_1 = \frac{16}{9} \int \frac{dx}{\left(\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right)^2}$

Posons $t = \frac{2x+1}{\sqrt{3}} \Rightarrow dt = \frac{2}{\sqrt{3}} dx$

$$I_1 = \frac{8\sqrt{3}}{9} \int \frac{dt}{(t^2+1)^2} = \frac{8\sqrt{3}}{9} \int \frac{t^2+1-t^2}{(t^2+1)^2} dt$$

$$= \frac{8\sqrt{3}}{9} \int \frac{dt}{t^2+1} - \frac{8\sqrt{3}}{9} \int \frac{t^2\,dt}{(t^2+1)^2}$$

$$= \frac{8\sqrt{3}}{9} \operatorname{arctg} t - \frac{8\sqrt{3}}{9} \int \frac{t^2\,dt}{(t^2+1)^2}.$$

Calculons l'intégrale

$$I_2 = \int \frac{t^2\,dt}{(t^2+1)^2} = \int \frac{t}{(t^2+1)^2} \cdot t\,dt$$

par une intégration par partie,

$u = t \Rightarrow u' = 1$

$v' = \frac{t}{(t^2+1)^2} \Rightarrow v = -\frac{1}{2} \frac{1}{t^2+1}.$

Ainsi,

$$I_2 = -\frac{t}{2(t^2+1)} + \frac{1}{2} \int \frac{dt}{t^2+1} = \frac{-t}{2(t^2+1)} + \frac{1}{2} \operatorname{arctg} t + c, \quad c \in \mathbb{R}$$

Ceci implique que

$$I_1 = \frac{8\sqrt{3}}{9} \operatorname{arctg} t + \frac{4\sqrt{3}}{9} \frac{t}{t^2+1} - \frac{4\sqrt{3}}{9} \operatorname{arctg} t + C$$

$$= \frac{4\sqrt{3}}{9} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}}{4} \frac{2x+1}{x^2+x+1} + C.$$

Finalement,

$$\int \frac{dx}{x(x^2+x+1)^2} = \ln|x| - \frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$+ \frac{1}{2}\,\frac{1}{x^2+x+1} - \frac{2\sqrt{3}}{9} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{\sqrt{3}}{8}\,\frac{2x+1}{x^2+x+1} + c, \quad c \in \mathbb{R}.$$

Exercices:

(1) $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$

posons $t = \sin x \Rightarrow dt = \cos x\,dx$.

$$\int \cos^3 x\,dx = \int (1-t^2)\,dt = t - \frac{t^3}{3} + c, \quad c \in \mathbb{R}$$

$$= \sin x - \frac{1}{3}\sin^3 x + c.$$

(2) $\int \cos^2 x \sin^3 x\,dx = \int \cos^2 x \sin^2 x \sin x\,dx$

$$= \int \cos^2 x\,(1-\cos^2 x)\sin x\,dx$$

$$= \int (\cos^2 x - \cos^4 x)\sin x\,dx.$$

posons $t = \cos x \Rightarrow dt = -\sin x\,dx$. Ainsi,

$$\int \cos^2 x \sin^3 x\,dx = -\int (t^2 - t^4)\,dt = \int (t^4 - t^2)\,dt$$

$$= \frac{t^5}{5} - \frac{t^3}{3} + c, \quad c \in \mathbb{R}.$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c.$$

(3) $\int \sin^2\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right)dx = ?$

$$\int \sin^2\left(\frac{x}{2}\right) \cos\left(\frac{3x}{2}\right) dx = \int \left(\frac{1-\cos x}{2}\right) \cos\left(\frac{3x}{2}\right) dx$$

$$= \frac{1}{2} \int \cos\frac{3x}{2}\, dx - \frac{1}{2} \int \cos x \cos\left(\frac{3x}{2}\right) dx$$

$$= \frac{1}{2} \cdot \frac{2}{3} \sin\frac{3x}{2} - \frac{1}{2} \int \cos x \cos\left(\frac{3x}{2}\right) dx$$

Sachant que

$$\cos a \cos b = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right), \quad \text{alors}$$

$$\cos x \cos\left(\frac{3x}{2}\right) = \frac{1}{2}\left(\cos\left(\frac{5x}{2}\right) + \cos\left(-\frac{x}{2}\right)\right)$$

$$= \frac{1}{2}\left(\cos\left(\frac{5x}{2}\right) + \cos\left(\frac{x}{2}\right)\right).$$

$$\int \cos x \cos\left(\frac{3x}{2}\right) dx = \frac{1}{2} \int \left(\cos\left(\frac{5x}{2}\right) + \cos\left(\frac{x}{2}\right)\right) dx$$

$$= \frac{1}{2}\left(\frac{2}{5} \sin\left(\frac{5x}{2}\right) + 2 \sin\frac{x}{2}\right) + C \,;\; C \in \mathbb{R}.$$

$$= \frac{1}{5} \sin\left(\frac{5x}{2}\right) + \sin\left(\frac{x}{2}\right) + C.$$

Finalement,

$$\int \sin^2\left(\frac{x}{2}\right) \cos\left(\frac{3x}{2}\right) dx = \frac{1}{3} \sin\left(\frac{3x}{2}\right) - \frac{1}{10} \sin\left(\frac{5x}{2}\right) - \frac{1}{2} \sin\frac{x}{2} + C.$$

Exercice 06 :

① $\displaystyle\int \frac{dx}{\sin x} = ?$

posons $u = tg\left(\frac{x}{2}\right)$

$$\Rightarrow \begin{cases} dx = \dfrac{2\,du}{1+u^2} \\ \cos x = \dfrac{1-u^2}{1+u^2} \\ \sin x = \dfrac{2u}{1+u^2} \end{cases}$$

Ainsi,

$$\int \frac{dx}{\sin x} = \int \frac{\frac{2\,du}{1+u^2}}{\frac{2u}{1+u^2}} = \int \frac{du}{u} = \ln|u| + C, \; C \in \mathbb{R}$$

$$= \ln\left|tg\left(\frac{x}{2}\right)\right| + C$$

② $$\int \frac{dx}{2+\cos x} = \int \frac{\frac{2\,du}{1+u^2}}{2+\frac{1-u^2}{1+u^2}} = \int \frac{2\,du}{u^2+3}$$

$$= \frac{2}{3}\int \frac{du}{\left(\frac{u}{\sqrt{3}}\right)^2+1} = \frac{2\sqrt{3}}{3}\int \frac{\frac{du}{\sqrt{3}}}{\left(\frac{u}{\sqrt{3}}\right)^2+1}$$

$$= 2\frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{u}{\sqrt{3}}\right)+c, \quad c\in\mathbb{R}.$$

$$= 2\frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{1}{\sqrt{3}}\operatorname{tg}\left(\frac{x}{2}\right)\right)+c.$$

③ $$\int \frac{dx}{1+\sin^2 x} = ?$$

pour simplifier, on peut utiliser le changement de variable $u = \operatorname{tg} x$.

$$\int \frac{dx}{1+\sin^2 x} = \int \frac{dx}{1+\left(\frac{1-\cos(2x)}{2}\right)} = \int \frac{2\,dx}{3-\cos(2x)}$$

On pose $u = \operatorname{tg} x \Rightarrow dx = \frac{du}{1+u^2}$

$\cos 2x = \frac{1-u^2}{1+u^2}$. Ainsi,

$$\int \frac{2\,dx}{3-\cos(2x)} = 2\int \frac{\frac{du}{1+u^2}}{3-\left(\frac{1-u^2}{1+u^2}\right)} = \int \frac{du}{1+2u^2}$$

$$= \int \frac{du}{1+(\sqrt{2}u)^2}.$$ On pose $t = \sqrt{2}u \Rightarrow dt = \sqrt{2}\,du$

$$= \frac{1}{\sqrt{2}}\int \frac{dt}{1+t^2} = \frac{1}{\sqrt{2}}\operatorname{arctg} t + c = \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}u)+c, \quad c\in\mathbb{R}$$

$$= \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\operatorname{tg} x)+c.$$