University of Tlemcen
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1st Year LMD-M and MI

Correction of supervised work N° 1 of Mecanics

Dimensional analysis and uncertainty calculation

Exercice 1

- Surface :

We have [l]=L, [t]=T and [m]=M.

[Physical quantities]= $M^x L^y T^z$

$S = l \times l \implies$ [S]=L.L=L^2 $\implies$ [S]=L^2 (m^2)

- Volume :

V=l×l×l $\implies$ [S]=L.L.L=L^3 $\implies$ [V]=L^3 (m^3)

- Density :

$\rho = \frac{m}{V}$ so $[\rho] = \frac{[m]}{[V]} = \frac{M}{L^3} = ML^{-3} \implies$ [ρ]=ML^{-3} (kg/m^3)

- Frequency:

$f = \frac{1}{T} \implies [f] = \frac{1}{[T]} = \frac{1}{T} = T^{-1} \implies$ [f]=T^{-1} (s^{-1} or Hertz)

(Period $[T] = T$; unit is « s »)

- Linear velocity:

$v = \frac{dx}{dt} \implies [v] = \frac{[x]}{[t]} = \frac{L}{T} = LT^{-1} \implies [v] = LT^{-1}$ (m./s)

- Angulaire velocity :

$\omega = \theta^{\cdot} = \frac{d\theta}{dt} = \frac{v}{R} \implies [\omega] = \frac{[\theta]}{[t]} = \frac{1}{T} = T^{-1} \implies$ [ω]=T^{-1} (Rd/s)

$[angle] = 1\ i\ and\ its\ unit\ is\ the\ radian\ (rad).$

- Lineair acceleration:

$a = \frac{dv}{dt} \implies [a] = \frac{[dv]}{[dt]} = \frac{LT^{-1}}{T} = LT^{-2} \implies [a] = LT^{-2}$ ($m./s^2$)

- Angulaire acceleration :

$\omega^{\cdot} = \theta^{\cdot\cdot} = \frac{d\theta^{\cdot}}{dt} \Rightarrow [\omega^{\cdot}] = \frac{[d\theta^{\cdot}]}{[dt]} = \frac{T^{-1}}{T} = T^{-2} \Rightarrow [\omega^{\cdot}]=T^{-2}$ (Rd./s^2)

- Force :

$F = m\times a \Rightarrow [F]=[m]\times[a]=M.L.T^{-2} \Rightarrow [F]=MLT^{-2}$ ($kg.m.s^{-2}$ or Newton)

- Work :

$W = F\times d\times \cos\alpha \Rightarrow [W]=[F]\times[d]\times[\cos\alpha]=MLT^{-2}.L.1=ML^2T^{-2}$ ($kg.m^2.s^{-2}$ or Joule)

- Energy :

$E_C = (½).m.v^2 \Rightarrow [E]=[1/2].[m].[v]^2 = ML^2T^{-2}$ (Joule)

$E_P = m.g.h \Rightarrow [E]=[m].[g].[h]= M.LT^{-2}.L= ML^2T^{-2}$ (Joule)

- Power:

$P = W/t \Rightarrow [P]=[W]/[t]=(ML^2T^{-2})/T=ML^2T^{-3}$ ($kg.m^2.s^{-3}$ or Watt)

- Pressure:

$P = F/S \Rightarrow [P] = [F]/[S]=(MLT^{-2})/L^2=ML^{-1}T^{-2}$ ($kg.m^{-1}.s^{-2}$ or Pascal).

Summary :

Physical Quantities	Symbol	Formula used	Dimension	Unit (SI)
Surface	S	l×l	L^2	m^2
Volume	V	l×l×l	L^3	m^3
Density	ρ	m/V	ML^{-3}	$Kg./m^3$
Frequency	F	1/T	T^{-1}	s^{-1} or hertz
Linear vilocity	V	dx/dt	LT^{-1}	m/s^1
Angular Vilocity	Ω	dθ/dt	T^{-1}	$Rd./s^1$
Linear Acceleration	γ	dv/dt	LT^{-2}	$m./s^2$
Angular Acceleration	$\omega^{\cdot}$	$d\theta^{\cdot}/dt$	T^{-2}	$Rd./s^2$
Force	F	m.a	MLT^{-2}	Newton
Work	W	F.d	ML^2T^{-2}	Joule
Energy	E	$(½)mv^2$	ML^2T^{-2}	Joule
Power	P	W/t	ML^2T^{-3}	Watt
Pressure	$\wp$	F/S	$ML^{-1}T^{-2}$	Pascal

Exercice 2

We have $\left(P + \frac{a}{V^2}\right) \times (V - b) = C$

G=A+B or G=A-B then [G]=[A]=[B]

[b] = [V] = L^3

$[\frac{a}{V^2}] = [P] = \frac{[a]}{[V]^2} \Longrightarrow [a] = [P]\times[V]^2 = M.L^{-1}T^{-2}.L^6 = M.L^5T^{-2}$

$$[C] = \left[P + \frac{a}{V^2}\right] \times [V - b]$$

On the other hand : $\left[P + \frac{a}{V^2}\right] = [p] = \left[\frac{a}{V^2}\right]$ et $[V - b] = [V] = [b]$

Et $[C]=[P]\times[V]=ML^{-1}T^{-2}.L^3=ML^2T^{-2}$

Exercice 3

Check the homogeneity of this formula: $p = \rho g h_1 + h_2 F$

Such as: P a pressure, g an acceleration of gravity, h_1 and h_2 are heights and F a force.

We have $\begin{cases} [\mathrm{P}] = \mathrm{M}L^{-1}T^{-2} \\ [\mathrm{g}] = LT^{-2} \\ [h_1] = [h_2] = L \\ [F] = MLT^{-2} \\ [\rho] = ML^{-3} \end{cases}$

This expression is homogenious if: $[p] = [\rho g h_1] = [h_2 F]$

$$[\rho g h_1] = ML^{-3}.L.LT^{-2} = \mathrm{M}L^{-1}T^{-2} = [\mathrm{P}]$$

$$et\ [h_2 F] = \mathrm{M}L^2T^{-2} \neq \mathrm{M}L^{-1}T^{-2}$$

So the equation is heterogeneous (not homogeneous).

Exercice 4

We have : $F = -6\pi\eta r v$

1- $[\eta]$ =?

$$F = -6\pi\mu r v \implies \eta = -\frac{F}{6\pi r v}$$

$$[\eta] = \frac{[F]}{[r][v]} \quad \text{with} \begin{cases} [r] = L \\ [F] = MLT^{-2} \\ [v] = LT^{-1} \\ [-6\pi] = 1 \end{cases}$$

Where

$$[\eta] = \frac{MLT^{-2}}{L.LT^{-1}} = ML^{-1}T^{-1}$$

2- We have $v = a\left(1 - exp\left(-\frac{t}{b}\right)\right)$

we're looking for the dimension of [a] et [b]:

The argument of the exponential is therefore dimensionless:

so [v] = LT^{-1} = [a] $\Longrightarrow$ [a]=LT^{-1}

$$\left[exp\left(-\frac{t}{b}\right)\right] = 1 \Rightarrow \left[-\frac{t}{b}\right] = \left[-1.\frac{t}{b}\right] = [-1]\left[\frac{t}{b}\right] = \left[\frac{t}{b}\right] = 1$$

$$\Rightarrow \left[\frac{t}{b}\right] = \frac{[t]}{[b]} = 1$$

[b] =[t]= T

Exercice 5:

f=K $F^a L^b \rho^c$; This function is therefore homogeneous $[f] = [k][F]^a[L]^b[\rho]^c$

$$\text{with} \begin{cases} [F] = [m.a] = [m][a] = M.\ LT^{-2} \\ [L] = L \quad et \ \ [k] = 1 \\ [\rho] = \left[\frac{m}{V}\right] = ML^{-3} \\ [f] = T^{-1} \end{cases}$$

so $[f] = (MLT^{-2})^a(L)^b(ML^{-3})^c = T^{-1}$

$\Longrightarrow M^0L^0T^{-1} = M^{a+c}L^{a+b-3c}T^{-2a}$

$$By\ identification: \begin{cases} a + c = 0 \\ a + b - 3c = 0 \\ -2a = -1 \end{cases}$$

$$\Longrightarrow \begin{cases} a = 1/2 \\ b = -a + 3c = -\frac{4}{2} = -2 \\ c = -a = -1/2 \end{cases}$$

$$F = K\, F^{1/2} L^{-2} \rho^{-1/2} = K\sqrt{F}\frac{1}{L^2}\frac{1}{\sqrt{\rho}}$$

So $$f = k\frac{\sqrt{F}}{L^2\sqrt{\rho}}$$

Exercice 6

The period of a pendulum is written :

$T = K\eta^x R^y \rho^z$ avec $[\eta] = ML^{-1}T^{-1}$

Suppose the relationship is homogeneous so $[T] = [k][\eta]^x[R]^y[\rho]^z$

with $\begin{cases} [\eta] = ML^{-1}T^{-1} \\ [R] = L \quad et \;\; [k] = 1 \\ [\rho] = \left[\frac{m}{V}\right] = \frac{M}{L^3} = ML^{-3} \\ [T] = T \end{cases}$

so $[T] = (ML^{-1}T^{-1})^x L^y (ML^{-3})^z = T$

$$(A^X.A^Y = A^{X+Y})$$

$$\Longrightarrow T = M^x L^{-x} T^{-x} \;\; L^y \quad M^z\, L^{-3z}$$

$$\Longrightarrow M^0 L^0 T^1 = M^{x+z} \quad L^{-x+y-3z} \quad T^{-x}$$

$$by\ identification: \begin{cases} x + z = 0 \\ -x + y - 3z = 0 \\ -x = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} x = -1 \\ y = x + 3z = 2 \\ z = -x = 1 \end{cases}$$

$$\Longrightarrow \; T = K\eta^{-1}R^2\rho^1$$

$$\text{So } \; T = k\frac{\rho R^2}{\eta}$$

2- The relative uncertainty on T= f($\Delta\eta, \Delta R, \Delta m$) ?

$$T = \frac{K\rho R^2}{\mu} \quad \text{with} \quad \rho = \frac{m}{V} = \frac{m}{\frac{4}{3}\pi R^3} = \frac{3m}{4\pi R^3} \quad \text{so } T = \frac{3Km}{4\pi R\mu}$$

$$\Longrightarrow \log T = \log\left(\frac{3mK}{4\pi R\mu}\right) = log3K + \log(m) - \log(4\pi) - \log(R) - \log(\mu)$$

$$\Longrightarrow \frac{dT}{T} = \frac{dm}{m} - \frac{dR}{R} - \frac{d\mu}{\mu}$$

$$\Longrightarrow \frac{\Delta T}{T} = \left|\frac{\Delta m}{m}\right| + \left|-\frac{\Delta R}{R}\right| + \left|-\frac{\Delta \mu}{\mu}\right|$$

m, R, et μ are positive quantities, hence:

$$\Longrightarrow \frac{\Delta T}{T} = \frac{\Delta m}{m} + \frac{\Delta R}{R} + \frac{\Delta \mu}{\mu}$$

Exercice 7

1- The dimension of k :

We have $\begin{cases} [p] = M.L.T^{-2} \\ [S] = L^2 \\ [k] = 1 \\ [v] = L.T^{-1} \end{cases}$ $and\ \ k = \frac{p}{v^2.s} \Rightarrow [k] = \frac{[p]}{[v]^2.[s]}$

$\Rightarrow [k] = [p].[v]^{-2}.[s]^{-1} \quad \Rightarrow [k] = M.L^{-3}$

2- N.A : $v = \sqrt{\frac{P}{K.S}} = 3.097 m/s$

3- $\frac{\Delta P}{P} = 2\% = 0.02$ and $\frac{\Delta S}{S} = 3\% = 0.03$

The logarithmic method is used to calculate the relative uncertainty on v :

$v = \sqrt{\frac{P}{K.S}} \Rightarrow \log v = \log\sqrt{\frac{P}{K.S}} = \frac{1}{2} log\, P - \frac{1}{2} logk - \frac{1}{2}\log S$

$$\Rightarrow d\log v = \frac{1}{2} dlog\, P - \frac{1}{2} dlogk - \frac{1}{2}\text{dlog}\, S$$

$\frac{dv}{v} = \frac{1}{2}\frac{dp}{p} - \frac{1}{2}\frac{dS}{S} \Rightarrow \frac{\Delta v}{v} = \frac{1}{2}\left|\frac{\Delta p}{p}\right| + \frac{1}{2}\left|-\frac{\Delta S}{S}\right| \Rightarrow \frac{\Delta v}{v} = \frac{1}{2}\frac{\Delta p}{p} + \frac{1}{2}\frac{\Delta S}{S}$ A.N : $\frac{\Delta v}{v} = 0.025$

Absolute uncertainty on v is given by:

$\Delta v = v.\frac{\Delta v}{v} = v * \left(\frac{1}{2}\frac{\Delta p}{p} + \frac{1}{2}\frac{\Delta S}{S}\right) = 0.077 m/s$

hence the condensed writing of v is given by : v=(3.097±0.077)m/s