University of Tlemcen
Faculty of Sciences
Mathematics Department

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1[ST] YEAR LMD-M AND MI

COURSE OF MECHANICS

OF THE MATERIAL POINT

Chapter II:

Vector analysis

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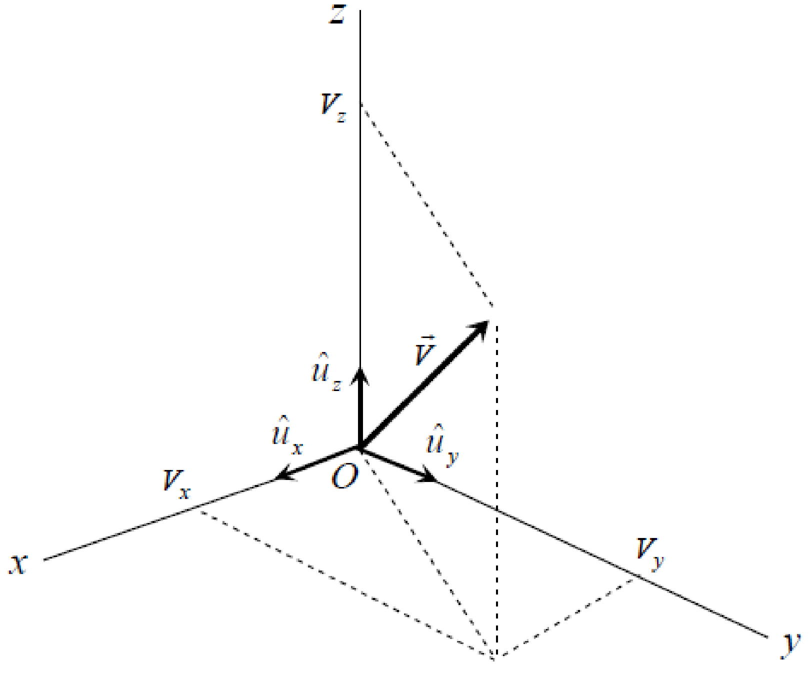


Summary

1. Introduction

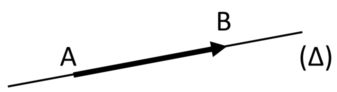
Vectors are fundamental mathematical entities used to represent quantities that have both magnitude and direction. Unlike scalars, which only have magnitude (e.g., distance, time, temperature), vectors provide a more comprehensive description of physical quantities by including information about their orientation or direction.

In other words, in physics, two types of quantities are used: scalar quantities and vector quantities:

- Scalar quantity المقدار السلمي : defined by a number (a scalar) and an appropriate unit such as: volume, mass, temperature, time ...
- Vector quantity المقدار الشعاعي: this is a quantity defined by a scalar, a unit and a direction such as : Displacement vector, velocity $\vec{v}$, weight $\vec{p}$, electric field ...

2. Definition

Vectors are physical or mathematical quantities carrying two properties: magnitude and direction. It is an oriented segment. Symbolically, a vector is usually represented by an arrow.

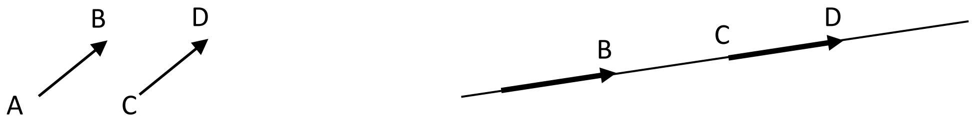


- Origin (المبدأ): presents the point of application "A".
- Support (الحامل): the straight line that carries the vector (Δ).
- Direction (الاتجاه): Vectors have a specific direction or orientation in space, often indicated by angles or coordinate systems (from A to B).
- Modulus (الطويلة): The size or length of a vector represents its magnitude. This is typically represented by a positive numerical value gives the algebraic value of the vector $\overrightarrow{AB}$ noted.

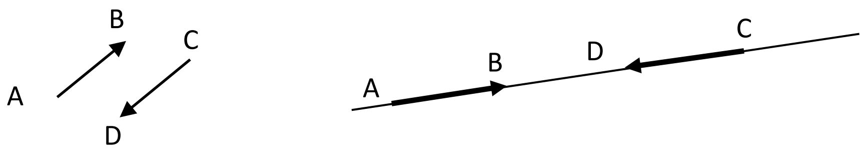
3. Vector types

- **Free vector:** the origin is not fixed.
- **Sliding vector:** the support is fixed, but the origin is not.
- **Linked vectors:** the origin is fixed.

- **Equal vectors**: if they have the same direction, the same support or parallel supports and the same modulus.



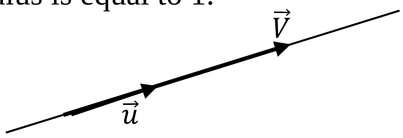
- **Opposite vector:** if they have the same support or parallel supports, the same modulus but the direction is opposite.



4. Unit Vector شعاع الوحدة

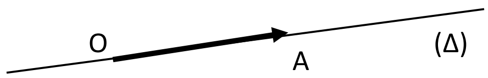
A vector is said to be unitary if its modulus is equal to 1.

We write: $|\vec{u}|$=1 and $\vec{V} = |\vec{V}|\,\vec{u}$



5. Algebraic measurement

Consider an axis (Δ) bearing points O and A. O is the origin, and the abscissa of point A is the algebraic measure of the vector $\overline{OA}$.

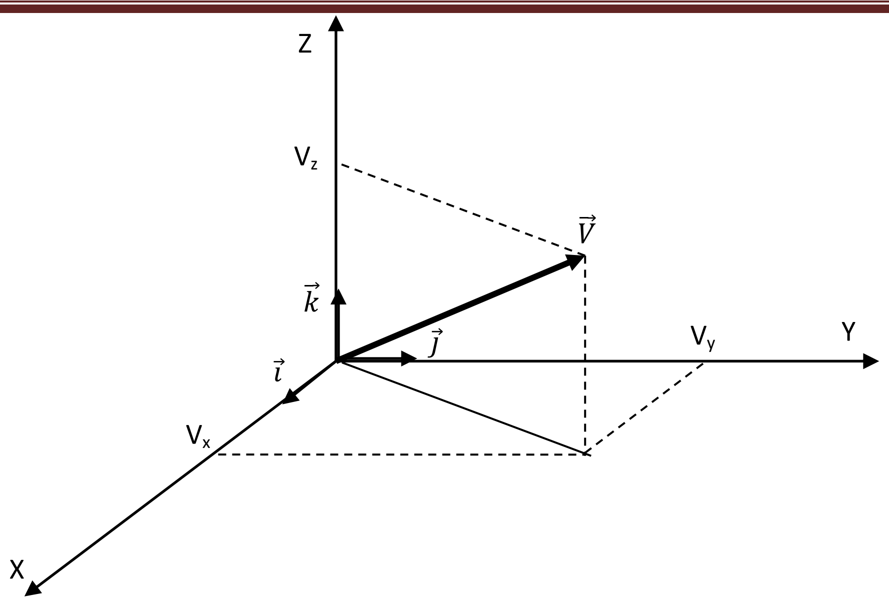


6. Components of a vector مركبات شعاع

The coordinates of a vector in space, represented in an orthonormal base frame R(O, $\vec{\imath}, \vec{\jmath}, \vec{k}$) are : V_x, V_y et V_z such that:

$$\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$$

Where a **position vector** $\vec{V} = \overrightarrow{OM}$ is a vector used to determine the position of a point M in space, relative to a fixed reference point O which, typically, is chosen to be the origin of our coordinate system.



The modulus of the vector $\vec{V}$ is : $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$

In cartesian coordinates, a vector is written as:

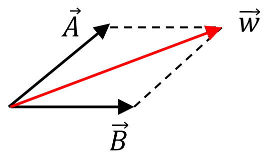
$$\vec{V} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} \Rightarrow V = \|\vec{V}\| = \sqrt{x^2 + y^2 + z^2}$$

7. Elementary operations on vectors

7.1. Vector addition

The sum of two vectors $\vec{A}$ and $\vec{B}$ is $\vec{w}$, obtained using the parallelogram:

$$\vec{A} + \vec{B} = \vec{w}$$

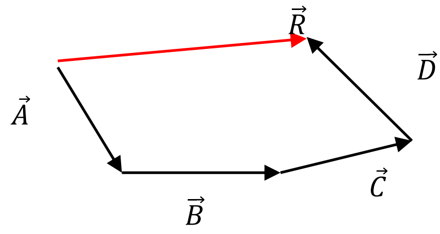


Let two vectors $\vec{A}$ *and* $\vec{B}$: $\vec{A} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ *and* $\vec{B} = x'\vec{\imath} + y'\vec{\jmath} + z'\vec{k}$

$$\vec{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{and } \vec{B}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \text{so } \vec{A} + \vec{B} = \vec{w} = (x + x')\vec{\imath} + (y + y')\vec{\jmath} + (z + z')\vec{k}$$

<u>Note</u> :

1. For several vectors: $\vec{A} + \vec{B} + \vec{C} + \vec{D} = \vec{R}$



2. Properties :

$\vec{A}+\vec{B}=\vec{B}+\vec{A}$, $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$, $\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$

3. Charles relationship:

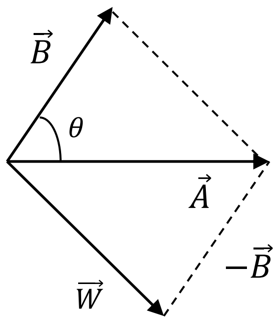
Or the three points: A, B and C, we have: $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

7.2. Subtracting two vectors

This is an anticommutative operation such that: $\vec{W}=\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$

Let two vectors: $\vec{A}\,and\,\vec{B}$, $\vec{A}=x\vec{\imath}+y\vec{\jmath}+z\vec{k}\ et\ \vec{B}=x'\vec{\imath}+y'\vec{\jmath}+z'\vec{k}$

$\vec{A}\begin{pmatrix}x\\y\\z\end{pmatrix}and\ \vec{B}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$ so $\vec{A}-\vec{B}=\vec{w}=(x-x')\vec{\imath}+(y-y')\vec{\jmath}+(z-z')\vec{k}$



7.3. Product of a vector and a scalar

The product of a vector $\vec{v}$ by a scalar α is the vector $\alpha\vec{v}$, this vector has the same support as $\vec{v}$.

The two vectors ($\vec{v}$ and $\alpha\vec{v}$) have the same direction if α>0 and they are opposite supports if α<0 .

$$\alpha\vec{v}=\alpha\begin{pmatrix}x\\y\\z\end{pmatrix}=\alpha x\,\vec{\imath}+\alpha y\vec{\jmath}+\alpha z\vec{k}$$

Notes: $\lceil\alpha\vec{v}\rceil=|\alpha||\vec{v}|,\ \alpha(\vec{u}+\vec{v})=\alpha\vec{u}+\alpha\vec{v}$ and $(\alpha+\beta)\vec{u}=\alpha\vec{u}+\beta\vec{u}$

8. Products

8.1. Scalar product الجداء السلمي

Given two vectors $\vec{A}\ and\ \vec{B}$ making an angle θ between them, the scalar product $\vec{\boldsymbol{A}}.\vec{\boldsymbol{B}}=\boldsymbol{m}$ with **m** is a scalar such that:

$$\vec{\boldsymbol{A}}.\vec{\boldsymbol{B}}=\boldsymbol{m}=|\vec{\boldsymbol{A}}|.|\vec{\boldsymbol{B}}|\cos(\vec{\boldsymbol{A}},\vec{\boldsymbol{B}})$$

With : $\widehat{(\vec{A},\vec{B})}=\theta$

Note : The properties of the scalar product are:

- The scalar product is commutative $\vec{A}.\vec{B} = \vec{B}.\vec{A}$
- The scalar product isn't associative $\overrightarrow{V_1}.(\overrightarrow{V_2}.\overrightarrow{V_3})$, doesn't exist, because the result would be a vector.
- $\vec{A}.\vec{B} = 0$ when both vectors are perpondicular $(\vec{A} \perp \vec{B})$.
- If $\vec{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\vec{B}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ so $\boldsymbol{\vec{A}.\vec{B} = x.x' + y.y' + z.z'}$

8.2. Vector product الجداء الشعاعي

The vector product of two vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C}$ and is written as:

$$\boldsymbol{\vec{C} = \vec{A} \wedge \vec{B}}$$

To calculate the vector product of two vectors $\vec{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\vec{B}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ we have :

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x & y & z \\ x' & y' & z' \end{vmatrix} = \vec{\imath}\begin{vmatrix} y & z \\ y' & z' \end{vmatrix} - \vec{\jmath}\begin{vmatrix} x & z \\ x' & z' \end{vmatrix} + \vec{k}\begin{vmatrix} x & y \\ x' & y' \end{vmatrix} = \boldsymbol{\vec{C}}$$

$$\vec{A} \wedge \vec{B} = \vec{\imath}(yz' - zy') - \vec{\jmath}(xz' - zx') + \vec{k}(xy' - yx') = \boldsymbol{\vec{C}}$$

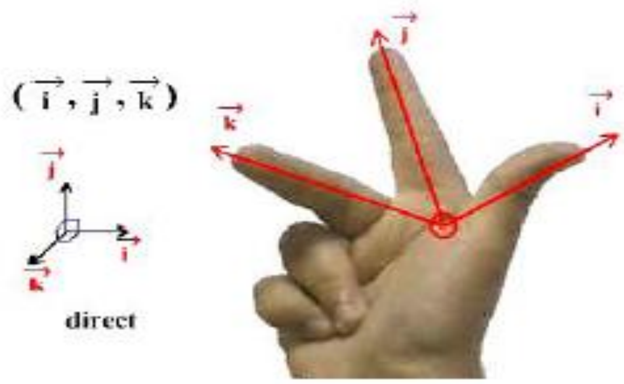
So the modulus of the vector product can be given by another method such as:

$$W = \sqrt{(yz' - zy')^2 + (xz' - zx')^2 + (xy' - yx')^2}$$

Characteristics of vector $\vec{C}$:

The support : $\vec{C}$ is perpondicular to the plane formed by the two vectors $\vec{A}$ and $\vec{B}$.

The direction: the three vectors $\vec{A}, \vec{B}$ and $\vec{C}$ form a direct trihedron. The direction is given by the rule of the three fingers of the right hand.



The modulus :

$$|\vec{C}| = |\vec{A}|.|\vec{B}|\sin(\vec{A},\vec{B})$$

The modulus of the vector product corresponds to the area (the surface مساحة) of the parallelogram (متوازي الاضلاع) formed by the two vectors $\vec{A}$ and $\vec{B}$.

<u>Example</u>:

In an orthonormal Cartesian coordinate base $(\vec{\imath},\vec{\jmath},\vec{k})$:

$\vec{\imath}\wedge\vec{\jmath}=\vec{k}$, $\vec{\jmath}\wedge\vec{k}=\vec{\imath}$ et $\vec{k}\wedge\vec{\imath}=\vec{\jmath}$. On the other hand $\vec{\imath}\wedge\vec{k}=-\vec{\jmath}$

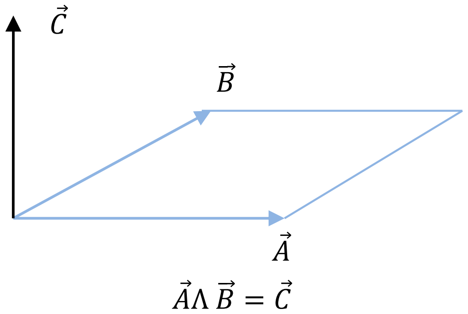
<u>Notes</u> : The properties of the vector product are:

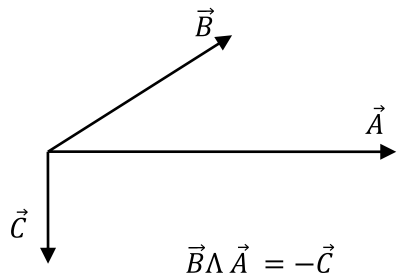
- The vector product is not commutative (Anticommutative).
- Not associative : $\overrightarrow{V_1}\wedge(\overrightarrow{V_2}\wedge\overrightarrow{V_3}) \neq (\overrightarrow{V_1}\wedge\overrightarrow{V_2})\wedge\overrightarrow{V_3}$.
- Distributive with respect to vector sum: $\vec{A}\wedge(\overrightarrow{B_1}+\overrightarrow{B_2}) = \vec{A}\wedge\overrightarrow{B_1} + \vec{A}\wedge\overrightarrow{B_2}$

But :

$\overrightarrow{V_1}\wedge(\overrightarrow{V_2}+\overrightarrow{V_3}) \neq (\overrightarrow{V_1}\wedge\overrightarrow{V_2}) + (\overrightarrow{V_1}\wedge\overrightarrow{V_3})$

- $\vec{A}\wedge\vec{B} = -\vec{B}\wedge\vec{A}$ car $sin(\vec{A},\vec{B}) = -sin(\vec{B},\vec{A})$





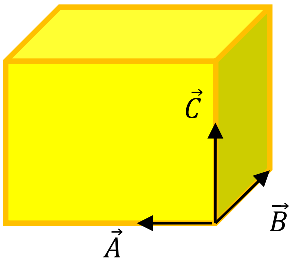
- $\vec{A}\wedge\vec{B} = \vec{0}$ *when the two vectors are parallel* $(\vec{A}\parallel\vec{B})$

8 .3. Mixed product

The mixed product of three vectors is $\vec{A},\vec{B}$ *and* $\vec{C}$ a scalar quantity m such that:

$$m = (\vec{A}\wedge\vec{B}).\vec{C}$$

Where **m** represents the volume of the parallelepiped (حجم متوازي المستطيلات) constructed by the three vectors :



Note: The mixed product is commutative, $(\vec{A}\wedge\vec{B}).\vec{C} = \vec{A}.(\vec{B}\wedge\vec{C}) = (\vec{C}\wedge\vec{A}).\vec{B}$

9. Derivative of a vector

Let the vector $\vec{A} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ which varies with time:

Its first derivative in relation to time is:

$$\overrightarrow{A'} = \frac{d\vec{A}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

The second derivative is:

$$\overrightarrow{A''} = \frac{d^2\vec{A}}{dt^2} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

Note :

- Derivative of a scalar product $(\vec{A}.\vec{B})' = \overrightarrow{A'}.\vec{B} + \vec{A}.\vec{B}$
- If $\vec{B}$ is constant $(\vec{A}.\vec{B})' = \overrightarrow{A'}.\vec{B}$
- $(\vec{A}^2)' = 0\ because\ (\vec{A}^2)' = 2\overrightarrow{A'}.\vec{A} = 0$
- The derivative vector is perpendicular to the vector.
- A vector is written as $\vec{A} = |\vec{A}|\vec{u} = A\vec{u}$, if $\vec{u}$ is a variable vector, then $\vec{A}' = A'\vec{u} + A\overrightarrow{u'}$.

Example: The position vector on Cartesian Coordinate is written as:

$$\vec{A} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

The velocity vector in Cartesian Coordinates is written as:

$$\vec{V} = \frac{d\overrightarrow{OM'}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

The acceleration vector in Cartesian Coordinates is written as:

$$\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

References

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3. M.D. Greenberg, *Advanced Engineering Mathematics*, 2nd Edition (Prentice-Hall, 1998).