# Elementary functions

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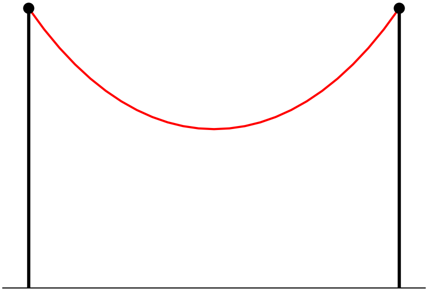
# Chapter 4

## **Elementary Functions**

# Outline

# Motivation

We already know some classic functions : exp, ln, cos, sin, tan. In this chapter, we are going to add some new functions to our catalog : cosh, sinh, tanh, arccos, arcsin, arctan, arg cosh, arg sinh, arg tanh.



For example when a necklace is held between two hands then the drawn curve is a **chain** of which the equation involves the hyperbolic cosine and a parameter $a$ (that depends on the length of the wire and the spacing of the posts)

$$y = a \cosh\left(\frac{x}{a}\right).$$

# Logarithm and exponential

# Logarithm

## Definition 1

There exists a unique fucntion, written $\ln : ]0,+\infty[ \to \mathbb{R}$ such that :

$$\ln'(x) = \frac{1}{x} \quad (\text{for all } x > 0) \qquad \text{and} \qquad \ln(1) = 0.$$

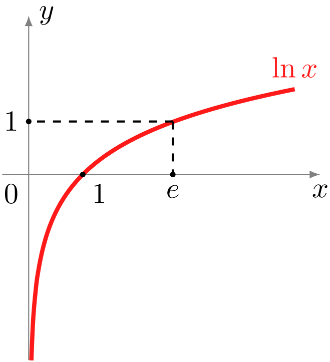
In addition, this function verifies (for all $a, b > 0$) :

1. $\ln(a \times b) = \ln a + \ln b$,
2. $\ln(\frac{1}{a}) = -\ln a$,
3. $\ln(a^n) = n \ln a$, (for all $n \in \mathbb{N}$)
4. $\ln$ is a continuous function, strictly increasing and defines a bijection of $]0,+\infty[$ on $\mathbb{R}$,
5. $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$,
6. We have $\ln x \leq x - 1$ (for all $x > 0$).

$\ln x$ is called the **Natural Logarithm**. It is caracterized by $\ln(e) = 1$.
We define the **base $a$'s logarithm** by

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

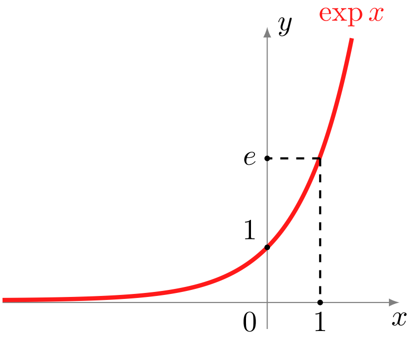
So that $\log_a(a) = 1$.
For $a = 10$ we get the **decimal logarithm** $\log_{10}$ such that $\log_{10}(10) = 1$ (and so $\log_{10}(10^n) = n$).



# Exponential

## Definition 2

The reciprocal bijection of $\ln : ]0, +\infty[ \to \mathbb{R}$ is called the **exponential** function, written $\exp : \mathbb{R} \to ]0, +\infty[$.



# Properties

For $x \in \mathbb{R}$ we also write down $e^x$ for $\exp(x)$.

## Proposition 1

The exponential function veries the following properties :

- $\boxed{\exp(\ln x) = x \text{ for all } x > 0}$ and $\boxed{\ln(\exp x) = x \text{ for all } x \in \mathbb{R}}$
- $\exp(a + b) = \exp(a) \times \exp(b)$
- $\exp(nx) = (\exp x)^n$
- $\exp : \mathbb{R} \to ]0, +\infty[$ is a continuous function, strictly increasing verifying $\lim_{x \to -\infty} \exp x = 0$ and $\lim_{x \to +\infty} \exp = +\infty$.
- The exponential function is a a differentiable function and $\exp' x = \exp x$, for all $x \in \mathbb{R}$. In addition $\exp x \geq 1 + x$.

# Power and Comparison

# Definitions

By definition, for $a > 0$ and $b \in \mathbb{R}$,

$$\boxed{a^b = \exp\left(b \ln a\right)}$$

## Notes 1

- $\sqrt{a} = a^{\frac{1}{2}} = \exp\left(\frac{1}{2}\ln a\right)$.
- $\sqrt[n]{a} = a^{\frac{1}{n}} = \exp\left(\frac{1}{n}\ln a\right)$ ($n^{th}$ **root**).
- We also write down $\exp x$ by $e^x$ which is justified by the following calculus : $e^x = \exp\left(x \ln e\right) = \exp(x)$.
- The functions $x \mapsto a^x$ are also called exponential functions and systematically reduce to the classical exponential function by the equality $a^x = \exp(x \ln a)$. Do not confuse this functions with the power functions $x \mapsto x^a$.

**Proposition 2**

Let $x, y > 0$ and $a, b \in \mathbb{R}$.

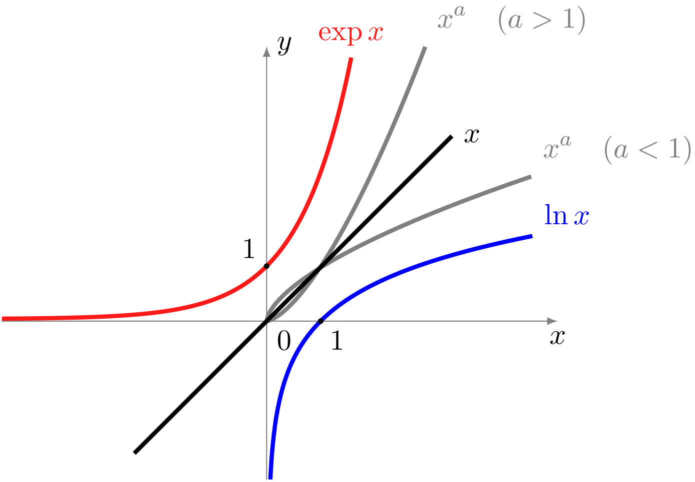
- $\boxed{x^{a+b} = x^a x^b}$
- $\boxed{x^{-a} = \dfrac{1}{x^a}}$
- $\boxed{(xy)^a = x^a y^a}$
- $\boxed{(x^a)^b = x^{ab}}$
- $\boxed{\ln(x^a) = a \ln x}$

Let us compare the functions $\ln x$, $\exp x$ with $x$ :

**Proposition 3**

$$\lim_{x \to +\infty} \frac{\ln x}{x} = 0 \qquad \text{et} \qquad \lim_{x \to +\infty} \frac{\exp x}{x} = +\infty.$$

# Graphically



# Inverse Circular Functions

# Arccosinus

Let us consider the cosinus function $\cos : \mathbb{R} \to [-1, 1]$, $x \mapsto \cos x$. To obtain a bijection from this function, we must consider restricting the cosinus to the interval $[0, \pi]$. In this interval, the cosinus function is continuous and stricly decreasing, so the restriction

$$\cos_| : [0, \pi] \to [-1, 1]$$

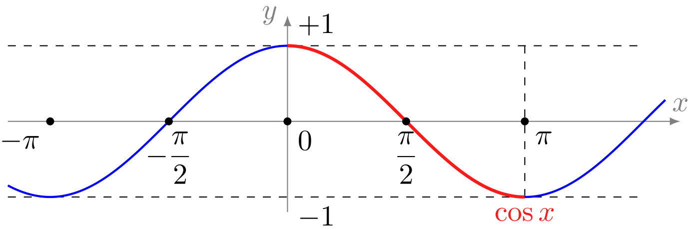
is a bijection.

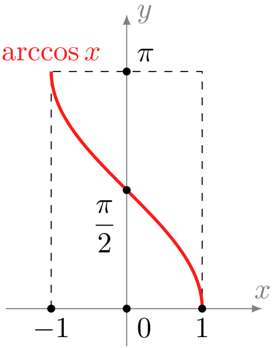
## Definition 3

The reciprocal bijection of the function cos is the **arccosinus** function :

$$\arccos : [-1, 1] \to [0, \pi]$$

# Graphically



# Properties

## Proposition 4

We have by definition the reciprocal bijection :

$$\boxed{\begin{array}{ll} \cos\big(\arccos(x)\big) = x & \forall x \in [-1,1] \\ \arccos\big(\cos(x)\big) = x & \forall x \in [0,\pi] \end{array}}$$

In other words :

$$\boxed{\text{If} \quad x \in [0,\pi] \qquad \cos(x) = y \iff x = \arccos y}$$

Let us finish with the derivative function of arccos:

$$\boxed{\arccos'(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \forall x \in ]-1,1[}$$

# Arcsinus

The restriction

$$\sin_{|} : [-\tfrac{\pi}{2}, +\tfrac{\pi}{2}] \to [-1, 1]$$

is a bijection. Its reciprocal bijection is the function **arcsinus** :

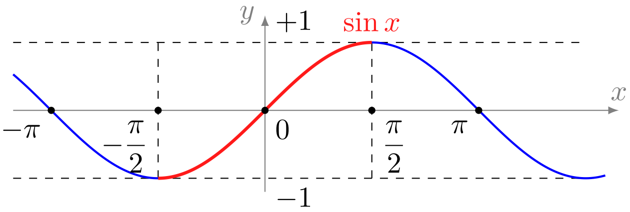
$$\arcsin : [-1, 1] \to [-\tfrac{\pi}{2}, +\tfrac{\pi}{2}].$$

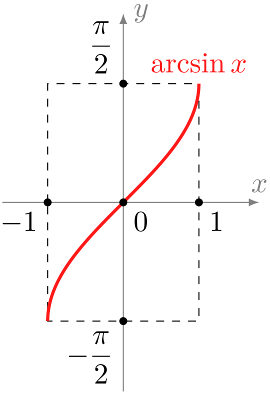
$$\boxed{\begin{array}{ll} \sin\big(\arcsin(x)\big) = x & \forall x \in [-1, 1] \\ \arcsin\big(\sin(x)\big) = x & \forall x \in [-\dfrac{\pi}{2}, +\dfrac{\pi}{2}] \end{array}}$$

$$\boxed{\text{If} \quad x \in [-\frac{\pi}{2}, +\frac{\pi}{2}] \qquad \sin(x) = y \iff x = \arcsin y}$$

$$\boxed{\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}} \qquad \forall x \in ]-1, 1[}$$

# Geometrically



# Arctangent

The restriction

$$\tan_{|} : ]-\tfrac{\pi}{2}, +\tfrac{\pi}{2}[ \to \mathbb{R}$$

is a bijection. Its reciprocal bijection is the function **arctangente** :

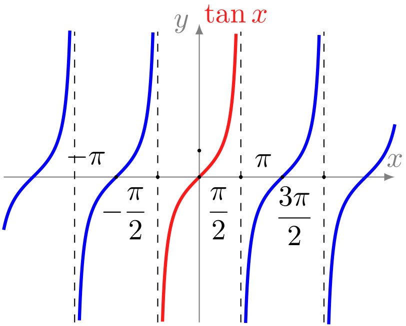
$$\arctan : \mathbb{R} \to ]-\tfrac{\pi}{2}, +\tfrac{\pi}{2}[.$$

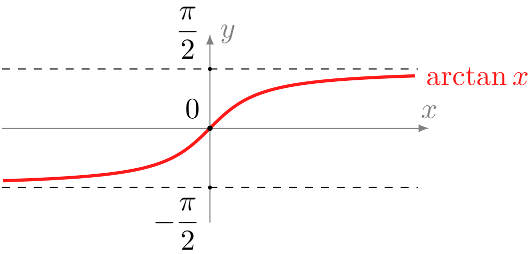
$$\boxed{\begin{aligned} \tan\left(\arctan(x)\right) &= x \quad \forall x \in \mathbb{R} \\ \arctan\left(\tan(x)\right) &= x \quad \forall x \in ]-\frac{\pi}{2}, +\frac{\pi}{2}[ \end{aligned}}$$

$$\boxed{\text{If} \quad x \in ]-\frac{\pi}{2}, +\frac{\pi}{2}[ \qquad \tan(x) = y \iff x = \arctan y}$$

$$\boxed{\arctan'(x) = \frac{1}{1+x^2} \qquad \forall x \in \mathbb{R}}$$

# Geometrically



# hyperbolic Functions and inverses

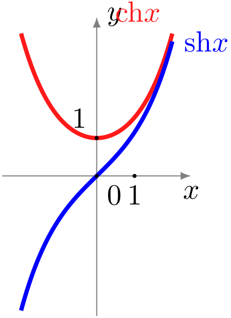
# Hyperbolic Cosinus

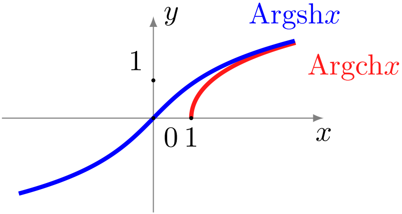
For $x \in \mathbb{R}$, The **hyperbolic cosinus** is :

$$\boxed{\cosh x = \frac{e^x + e^{-x}}{2}}$$

The restriction $\cosh_|$ : $[0, +\infty[\to [1, +\infty[$ is a bijection. Its reciprocal bijection is the function $\arg\cosh : [1, +\infty[\to [0, +\infty[$.

# Geometrically



# Hyperbolic Sinus and its inverse

For $x \in \mathbb{R}$, the **hyperbolic sinus** is :

$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$$

$\sinh : \mathbb{R} \to \mathbb{R}$ Is a continuous function, differentiable, strictly increasing and verifying $\lim_{x \to -\infty} \sinh x = -\infty$ and $\lim_{x \to +\infty} \sinh x = +\infty$, so it's a bijection. Its reciprocal bijection is the function $\arg\sinh : \mathbb{R} \to \mathbb{R}$.

## Proposition 5

- $\cosh^2 x - \sinh^2 x = 1$
- $\cosh' x = \sinh x$, $\sinh' x = \cosh x$
- $\arg\sinh : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous.
- $\arg\sinh$ is a differentiable function and $\arg\sinh' x = \dfrac{1}{\sqrt{x^2+1}}$.
- $\arg\sinh x = \ln\left(x + \sqrt{x^2+1}\right)$.

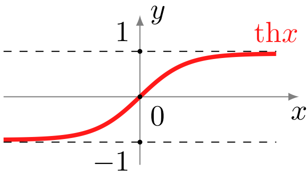
# Hyperbolic Tangent

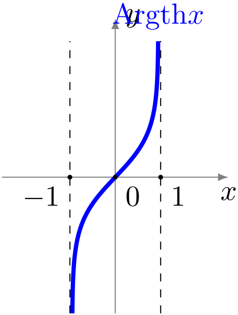
By definition the **hyperbolic tangent** is :

$$\boxed{\tanh x = \frac{\sinh x}{\cosh x}}$$

The function $\tanh : \mathbb{R} \to ]-1, 1[$ is a bijection, we write $\arg\tanh : ]-1, 1[ \to \mathbb{R}$ its reciprocal bijection.

# Geometrically



# Hyperbolic Trigonometry

$$1 = \cosh^2 x - \sinh^2 x$$

$$\cosh(a+b) = \cosh a \cdot \cosh b + \sinh a \cdot \sinh b$$

$$\sinh(a+b) = \sinh a \cdot \cosh b + \sinh b \cdot \cosh a$$

$$\cosh' x = \sinh x \qquad \sinh' x = \cosh x$$

$$\tanh' x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}$$

$$\arg\cosh' x = \frac{1}{\sqrt{x^2-1}} \quad (x > 1) \qquad \arg\sinh' x = \frac{1}{\sqrt{x^2+1}}$$

$$\arg\tanh' x = \frac{1}{1-x^2} \quad (|x| < 1)$$

$$\arg\cosh x = \ln\left(x + \sqrt{x^2-1}\right) \quad (x \geq 1)$$

$$\arg\sinh x = \ln\left(x + \sqrt{x^2+1}\right) \quad (x \in \mathbb{R})$$

$$\arg\tanh x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (-1 < x < 1)$$

GOOD LUCK !!