Series expansion

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Chapter 5

Series expansion

Summary

Introduction

In physics and mathematics, a series expansion (denoted DL) of a function at a point is a polynomial approximation of this function at the neighborhood of this point, i.e. the writing of this function in the form of the sum of :

- a polynomial function denoted by $P_n(x)$ and
- a negligible remainder noted $R_n(x)$ in the neighborhood of the considered point.

In physics, the expansion series make it possible to approach the functions to simplify the calculations.
In mathematics, they make it easier to find limits of functions, to calculate derivatives, to prove that a function is integrable or not, or to study the positions of curves in relation to tangents.

Prerequisites

Before studying the expansion series strictly speaking, it is necessary to make some reminders on the small o of x^n, noted $o(x^n)$.

We consider two functions f and g, and a real number a, such that : $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$.

Then we can say that f is negligible compared to g in the neighborhood of a (it is important to specify in the neighborhood of which point !).
Indeed, in the neighborhood of 0 for example, x is negligible compared to $1/x$, but in the neighborhood of $+\infty$, it is $1/x$ that is negligible compared to x. So just saying $1/x$ is negligible compared to ... doesn't make sense if you don't specify near which point..
When f is negligible compared to g in the neighborhood of a, we then write : $f(x) = o(g(x))$.

Examples : in the neighborhood of 0 : $x^6 = o(x^3)$, $x^2 = o(x)$.

$$\forall p > n, \ x^p = o(x^n).$$

Taylor's theorem

Definition 1 (Taylor's formula with Peano form of the remainder.)

Let I be an interval of $\mathbb{R}$, a an element of I and $f : I \to \mathbb{R}$ a differentiable function at a up to a certain order $n \geqslant 1$.
Then for any real number x belonging to I, we have the Taylo's formula with Peano form of the remainder

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + o\left((x-a)^n\right)$$

or equivalently

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$

where the remainder $R_n(x)$ is a negligible function with respect to $(x-a)^n$ in the neighborhood of a.

Remarks :

- By setting $h = x - a$, this formula can also be expressed as :

$$f(a+h) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} h^k + R_n(h)$$

where the remainder $R_n(h)$ is a negligible function compared to h^n in the neighborhood of 0.

- According to the hypotheses on the function f, we can give expressions and estimates of the remainder $R_n(x)$ to obtain other more precise formulas such as : the Taylor-Lagrange formula, Taylor-Cauchy formula and Taylor with integral remainder.

Example : Consider the function $f : x \to \mathrm{e}^x$.
f is of class $\mathcal{C}^\infty$ on $\mathbb{R}$ with $f^{(k)} = \mathrm{e}^x$ and therefore $f^{(k)}(0) = 1$, $\forall k \in \mathbb{N}$.
By the Taylor-Young formula in the neighborhood of 0, we get

$$\begin{aligned} \mathrm{e}^x &= \sum_{k=0}^{n} \frac{1}{k!} x^k + o(x^n) \\ &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + o(x^n) \end{aligned}$$

in particular

$$\mathrm{e}^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + o(x^4)$$

Remark : The Taylor polynomial of order 4 at the point 0 is very close to e^x in neighborhood of 0.

To verify, let $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$ and evaluate at $x = 0.1$ (close to 0), we get

$$\begin{aligned} e^{0.1} &= 1.10517091 \\ P_4(0.1) &= 1.10517083 \end{aligned}$$

Series expansion

I denotes a non-singular interval and n a natural number.
The functions considered here are real-valued.

Definition 2 (Series expansion)

Let a be a point from I or a finite end of I and $D = I$ or $D = I \setminus \{a\}$.
We say that $f : D \to \mathbb{R}$ admits a series expansion to the order n at a (abbreviated $SE_n(a)$) if there exist $a_0, a_1, \cdots, a_n$ such that when $x \to a$

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + o\left((x-a)^n\right)$$

The polynomial function $x \to a_0 + a_1(x-a) + \cdots + a_n(x-a)^n$ is then called **regular part** of $SE_n(a)$ of f.

Remarks : In the regular part of SE, each term of the sum is negligible compared to the one before it.
A $SE_n(a)$ gives information on the behavior of f at a and only at a.

Example : $SE_n(0)$ of $x \to \frac{1}{1-x}$

We have

$$1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

then

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \frac{x^{n+1}}{1-x}$$

but

$$\frac{x^{n+1}}{1-x} = x^n \frac{x}{1-x} = o\left(x^n\right)$$

since $\frac{x}{1-x} \xrightarrow[x \to 0]{} 0.$

so

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + o\left(x^n\right)$$

which is a series expansion to the order n at 0.

Proposition 1

If $f : D \to \mathbb{R}$ admits a $SE_n(a)$ of the form :

$$f(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n + o\left((x-a)^n\right)$$

then, for every $m \leqslant n$, f admits a $SE_m(a)$ being obtained by truncation :

$$f(x) = a_0 + a_1(x-a) + \cdots + a_m(x-a)^m + o\left((x-a)^m\right)$$

Example : let $P(x) = a_0 + a_1 x + \cdots + a_p x^p$ a polynomial function.
its $SE_n(0)$ is given by
for $n \leqslant p$, we have $P(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n)$.
for $n > p$, we have $P(x) = a_0 + a_1 x + \cdots + a_p x^p + o(x^p)$.

Theorem 1 (Uniqueness)

If $f : D \to \mathbb{R}$ admits a $SE_n(a)$ then this one is unique.

Existence of SE

Proposition 2

f admits a SE of order 0 at a if and only if f converges at a.
Moreover, if this is the case i.e. $\lim_{x\to a} f(x) = a_0$, then the SE of order 0 of f is given by $f(x) = a_0 + o(1)$.

Proposition 3

f is continuous at a and admits a SE of order 1 at a if and only if f is derivable at a.
Moreover, $f'(a) = a_1$, so the SE of order 1 of f is given by

$$f(x) = f(a) + f'(a)(x-a) + o(x-a)$$

Examples :

- $f(x) = \ln(x-2), \quad a = 2.$
  we have $\lim_{x\to 2^+} \ln(x-2) = -\infty$ so f does not admit a SE at $a = 2$.

- $f(x) = |x|$ converges at 0 so it admits a SE of order 0 at 0.
  On the other hand f is not differentiable at 0, so it does not admit a SE of order 1 at 0.
- $f(x) = \sqrt{1 + \sin x}$, $a = 0$.
  Since f is of class C^∞ in neighborhood of 0, it admits a Taylor expansion at 0 of order n, $\forall n \in \mathbb{N}$, therefore it admits a SE at 0 of order n, $\forall n \in \mathbb{N}$.
- $x \to \sin\left(\frac{1}{x}\right)$ does not converge at 0, so there is no SE at 0.

Theorem 2 (Relationship between SE and Taylor's expansion)

Let $f : I \to \mathbb{R}$ and $a \in I$.
If f is of class $\mathcal{C}^n$ then f admits a series expansion of order n at a of the form :

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + o\left((x-a)^n\right)$$

Remark : This theorem provides a sufficient condition for the existence of a $SE_n(a)$.
In other words : If f is of class $\mathcal{C}^n$ on I then it admits a $SE_n(a)$ and its polynomial part coincides with the Taylor polynomial of f of order n at a.
The converse implication is false; there exist functions admitting a $SE_n(a)$ which are not even of class $\mathcal{C}^1$ and which therefore do not admit a Taylor expansion of order n at a.
For example the function f defined by

$$f(x) = \begin{cases} 1 + x + x^2 + x^3 \sin(1/x^2), & \text{if } x \neq 0 \\ 1, \quad \text{if } x = 0 \end{cases}$$

This function admits a $SE_2(0)$ (just notice that $x^3 \sin(1/x^2) = o(x^2)$), on the other hand it is not even twice differentiable at 0 (because its first derivative is not continuous at 0).

SE at 0 of usual functions

- $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n) = \displaystyle\sum_{k=0}^{n} x^k + o(x^n)$
- $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + o(x^n) = \displaystyle\sum_{k=0}^{n} (-1)^k x^k + o(x^n)$
- $e^x = 1 + x + \dfrac{1}{2!}x^2 + \dfrac{1}{3!}x^3 + \cdots + \dfrac{1}{n!}x^n + o(x^n) = \displaystyle\sum_{k=0}^{n} \dfrac{1}{k!}x^k + o(x^n)$
- $\cos x = 1 - \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 + \cdots + \dfrac{(-1)^n}{(2n)!}x^{2n} + o(x^{2n}) = \displaystyle\sum_{k=0}^{n} \dfrac{(-1)^k}{(2k)!}x^{2k} + o(x^{2n})$
- $\sin x = x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 + \cdots + \dfrac{(-1)^n}{(2n+1)!}x^{2n+1} + o(x^{2n+1})$
  $= \displaystyle\sum_{k=0}^{n} \dfrac{(-1)^k}{(2k+1)!}x^{2k+1} + o(x^{2n+1})$

- Since $\operatorname{ch}(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$, we get by summing the expansions of e^{x} and of e^{-x} :
  $$\operatorname{ch}(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{1}{(2n)!}x^{2n} + o(x^{2n}) = \sum_{k=0}^{n} \frac{1}{(2k)!}x^{2k} + o(x^{2n})$$
- $$\operatorname{sh}(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{1}{(2n+1)!}x^{2n+1} + o(x^{2n+1})$$
  $$= \sum_{k=0}^{n} \frac{1}{(2k+1)!}x^{2k+1} + o(x^{2n+1})$$
- $$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^{n-1}}{n}x^n + o(x^n) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}x^k + o(x^n)$$
- For $\alpha \in \mathbb{R}$ fixed
  $$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n)$$
- For $\alpha = p \in \mathbb{N}$
  $$(1+x)^{\alpha} = \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots + \binom{p}{n}x^n + o(x^n) = \sum_{k=0}^{n} \binom{p}{k}x^k + o(x^n)$$

Determination of series expansion

Transfering the problem to 0

Method

To determine a series expansion at a of a function $x \to f(x)$, we relocate the problem to 0 using the change of variable $x = a + h$.
We then determine a series expansion at 0 of the function $h \to f(a+h)$ then we transpose this expansion at a by replacing h by $x - a$.

Examples :

- $SE_2(1)$ of $x \to \mathrm{e}^x$.
  When $x \to 1$, we put $x = 1 + h$, $h = x - 1$ with $h \to 0$

  $$\mathrm{e}^x = \mathrm{e}^{1+h} = \mathrm{e}.\mathrm{e}^h = \mathrm{e}\left(1 + h + \frac{1}{2}h^2 + o(h^2)\right) = \mathrm{e} + \mathrm{e}.h + \frac{e}{2}h^2 + o(h^2)$$

  then

  $$\mathrm{e}^x = \mathrm{e} + \mathrm{e}(x-1) + \frac{\mathrm{e}}{2}(x-1)^2 + o\left((x-1)^2\right)$$

  Remark : Do not develop this expression because otherwise we lose the visualization of the orders of magnitude in the neighborhood of 1.

- $SE_3(\pi/3)$ of $x \to \cos x$.
When $x \to \pi/3$, we put $x = \pi/3 + h$, $h = x - \pi/3$ with $h \to 0$.

$$\cos x = \cos\left(\frac{\pi}{3} + h\right) = \frac{1}{2}\cos h - \frac{\sqrt{3}}{2}\sin h$$

But $\cos h = 1 - \frac{1}{2}h^2 + o(h^3)$ and $\sin h = h - \frac{1}{6}h^3 + o(h^3)$ so

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}h - \frac{1}{4}h^2 + \frac{\sqrt{3}}{12}h^3 + o(h^3)$$

then

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + o\left(\left(x - \frac{\pi}{3}\right)^3\right)$$

- $SE_2(1)$ of $x \to \ln x$. When $x \to 1$, let's put $x = 1 + h$ with $h \to 0$

$$\ln x = \ln(1+h) = h - \frac{1}{2}h^2 + o(h^2) = (x-1) - \frac{1}{2}(x-1)^2 + o\left((x-1)^2\right)$$

- $DL_2(2)$ of $\sqrt{x}$
  When $x \to 2$, let's put $x = 2 + h$ with $h \to 0$

$$\sqrt{x} = \sqrt{2+h} = \sqrt{2}\sqrt{1+h/2}$$

Or $\sqrt{1+u} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + o(u^2)$, when $u \to 0$ then for $u = \frac{h}{2} \to 0$ we have
$\sqrt{1+h/2} = 1 + \frac{1}{4}h - \frac{1}{32}h^2 + o(h^2)$
then

$$\sqrt{x} = \sqrt{2} + \frac{\sqrt{2}}{4}(x-2) - \frac{\sqrt{2}}{32}(x-2)^2 + o\left((x-2)^2\right)$$

Remark : Here the change of variable $x = 1 + h$ would have been unsuitable, since when $x \to 2$, we have $h \to 1$ and not $h \to 0$.

SE of a product

Method

Assuming that in neighborhood of 0 we have
$f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n)$ and $g(x) = b_0 + b_1 x + \cdots + b_n x^n + o(x^n)$.
The series expansion of a product is the product of the series expansions of the factors i.e.

$$f(x)g(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)\, x + \cdots + (a_0 b_n + \cdots + a_n b_0) x^n + o(x^n)$$

which determines the $SE_n(0)$ of $x \to f(x)g(x)$.

Examples :

- $SE_3(0)$ of $x \to \dfrac{\mathrm{e}^x}{1-x}$, when $x \to 0$

$$\mathrm{e}^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3) \quad \text{and} \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + o(x^3)$$

then

$$\frac{\mathrm{e}^x}{1-x} = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + o(x^3)$$

- $SE_4(0)$ of $x \to \cos x.\text{ch}x$, when $x \to 0$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4) \quad \text{and} \quad \text{ch}x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

then

$$\begin{aligned}\cos x.\text{ch}x &= 1 + \left(\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{24} + \frac{1}{24} - \frac{1}{4}\right)x^4 + o(x^4) \\ &= 1 - \frac{1}{6}x^4 + o(x^4)\end{aligned}$$

- $SE_3(0)$ of $x \to \ln(1+x)\mathrm{e}^x$.
  Since the expansion of $\ln(1+x)$ starts by the term x, an expansion of order 2 of e^x is enough to carry out the calculations.
  Indeed, by multiplying by $\ln(1+x)$, the term $o(x^2)$ of the expansion of e^x becomes $o(x^3)$.
  When $x \to 0$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3) \quad \text{and} \quad \mathrm{e}^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

  then

$$\begin{aligned} \ln(1+x)\mathrm{e}^x &= x + \left(1 - \frac{1}{2}\right)x^2 + \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{3}\right)x^3 + o(x^3) \\ &= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3) \end{aligned}$$

- $SE_4(0)$ of $x \to \ln(1+x)\,(1-\cos x)$.
  The expansion of $\ln(1+x)$ starting by x, an expansion of order 3 of $(1-\cos x)$ is sufficient to carry out the calculations.

$$1 - \cos x = \frac{1}{2}x^2 + o(x^3)$$

  Also, the expansion of $1-\cos x$ starting by a term x^2, an expansion to the order 2 of $\ln(1+x)$ is enough.

$$\ln(1+x) = x - \frac{1}{2}x^2 + o(x^2)$$

  then

$$\ln(1+x)\,(1-\cos x) = \frac{1}{2}x^3 - \frac{1}{4}x^4 + o(x^4)$$

SE of composition of functions

Method

Suppose $f(x) \xrightarrow[x\to 0]{} 0$ and $g(u) = a_0 + a_1 u + \cdots + a_n u^n + o(u^n)$ when $u \to 0$.
Since we can write $o(u^n) = u^n \varepsilon(u)$ with $\varepsilon \underset{0}{\to} 0$, we have if the composition is allowed

$$g(f(x)) = a_0 + a_1 f(x) + \cdots + a_n \left(f(x)\right)^n + \left(f(x)\right)^n \varepsilon\left(f(x)\right)$$

with $\varepsilon\left(f(x)\right) \xrightarrow[x\to 0]{} 0$
This can be written as :

$$g\left(f(x)\right) = a_0 + a_1 f(x) + \cdots + a_n \left(f(x)\right)^n + o\left(\left(f(x)\right)^n\right)$$

We can substitute u by $f(x)$ in the $SE_n(0)$ of $g(u)$ since $f(x) \xrightarrow[x\to 0]{} 0$.

So by knowing a series expansion of f, we can deduce a series expansion of $g\left(f(x)\right)$.

Examples :

- $SE_6(0)$ of $x \to \ln(1+x^2+x^3)$ when $x \to 0$
  $\ln(1+x^2+x^3) = \ln(1+u)$ with

$$u = x^2 + x^3 \to 0$$
$$u^2 = x^4 + 2x^5 + x^6$$
$$u^3 = x^6 + o(x^6)$$
$$\text{and } o(u^3) = o(x^6)$$

A series expansion to order 3 of $\ln(1+u)$ is enough.

$$\ln(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 + o(u^3)$$

then

$$\ln(1+x^2+x^3) = x^2 + x^3 - \frac{1}{2}x^4 - x^5 - \frac{1}{6}x^6 + o(x^6)$$

- $SE_3(0)$ of $\mathrm{e}^{\frac{1}{1+x}}$, when $x \to 0$

$$\mathrm{e}^{\frac{1}{1+x}} = \mathrm{e}^{1-x+x^2-x^3+o(x^3)} = \mathrm{e}.\mathrm{e}^u$$

with

$$u = -x + x^2 - x^3 + o(x^3) \to 0$$
$$u^2 = x^2 - 2x^3 + o(x^3)$$
$$u^3 = -x^3 + o(x^3)$$
$$\text{and } o(u^3) = o(x^3).$$

$$\mathrm{e}^u = 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3 + o(u^3)$$

then

$$\mathrm{e}^{\frac{1}{1+x}} = \mathrm{e} - \mathrm{e}.x + \frac{3\mathrm{e}}{2}x^2 - \frac{13\mathrm{e}}{6}x^3 + o(x^3)$$

SE of a Quotient

Method

The regular part of SE of the quotient $\dfrac{f}{g}$ is the quotient in the division according to the increasing powers of the regular part of f by the regular part of g.

Examples :

- $SE_3(0)$ of $x \to \dfrac{\mathrm{e}^{x}}{1-x}$

$$
\begin{array}{ccccc|l}
1 & +x & +\dfrac{x^2}{2} & +\dfrac{x^3}{6} & & 1-x \\ \hline
-1 & +x & & & & \\ \cline{1-3}
 & 2x & +\dfrac{x^2}{2} & & & 1 \quad +2x \quad +\dfrac{5}{2}x^2+\dfrac{8}{3}x^3 \\
 & -2x & +2x^2 & & & \\ \cline{2-5}
 & & \dfrac{5}{2}x^2 & +\dfrac{x^3}{6} & & \\
 & & -\dfrac{5}{2}x^2 & +\dfrac{5}{2}x^3 & & \\ \cline{3-5}
 & & & \dfrac{8}{3}x^3 & &
\end{array}
$$

$$\frac{\mathrm{e}^{x}}{\ldots} = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + o(x^3)$$

- $SE_5(0)$ of $x \to \tan x$

$$
\begin{array}{rrr|lll}
x & -\dfrac{1}{6}x^3 & +\dfrac{1}{120}x^5 & 1- \quad \dfrac{1}{2}x^2 & +\dfrac{1}{24}x^4 & \\ \cline{4-6}
-x & +\dfrac{x^3}{2} & -\dfrac{x^5}{24} & & & \\ \cline{1-3}
 & \dfrac{x^3}{3} & -\dfrac{x^5}{30} & x & +\dfrac{1}{3}x^3 & +\dfrac{2}{15}x^5 \\
 & -\dfrac{x^3}{3} & \dfrac{x^5}{6} & & & \\ \cline{2-3}
 & & \dfrac{2}{15}x^5 & & &
\end{array}
$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^5)$$

SE by derivation

Theorem 3

Let f be differentiable on I and admits a SE of order n at 0.
If f' admits a SE of order $n-1$ at 0 then the regular part of expansion of f' is the derivative of the regular part of the expansion of f.

Example : Since $x \to \dfrac{1}{1-x}$ and its derivative $x \to \dfrac{1}{(1-x)^2}$ admit SE of order n and $n-1$ respectively at 0 (as they are of class $\mathcal{C}^\infty$ on $\mathbb{R}^*$), so by differentiating the SE of $x \to \dfrac{1}{1-x}$ we obtain that of its derivative.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n)$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + o(x^{n-1})$$

Remark : it is possible that a differentiable function admits a SE of order n at 0 without its derivative admitting a SE of order $n-1$ at 0. It is necessary to check the conditions of the theorem before using it.

Example : Let $n \in \mathbb{N}$ and

$$f(x) = \begin{cases} 1 + x + x^2 \cos(1/x), & \text{si } x \neq 0 \\ 1, \quad \text{if } x = 0 \end{cases}$$

When $x \to 0$, we have $f(x) = 1 + x + o(x)$ so then f admits a series expansion of order 1 at 0.
On the other hand, its derivative

$$f'(x) = \begin{cases} 1 + 2x\cos(1/x) + \sin(1/x), & \text{if } x \neq 0 \\ 1, \quad \text{if } x = 0 \end{cases}$$

does not admit a limit at 0 and therefore neither a series expansion at 0.

Applications

I Determination of equivalents

Definition 3 (Equivalents of functions)

We say that f is equivalent to g at a if we can write at neighborhood of a

$$f(x) = g(x)\theta(x)$$

with $\theta \underset{a}{\to} 1$ We note then $f \sim g$ or $f(x) \sim g(x)$ when $x \to a$.
If the function g does not vanish in the neighborhood of a then we have

$$f(x) \sim g(x) \Leftrightarrow \frac{f(x)}{g(x)} \xrightarrow[x \to a]{} 1$$

Example :

$$\text{when } x \to +\infty, \quad x^2 + x + 2\ln(x) \sim x^2$$
$$\text{when } x \to 0, \quad x^2 + x + 2\ln(x) \sim 2\ln x$$

SE for Determination of simple equivalents

The first non-zero term of a series expansion provides a simple equivalent of the function studied at the considered point.

Examples :

- From the series expansions of the usual functions, we obtain these famous equivalents when $x \to 0$:

$$\sin x \sim x,\ \tan x \sim x,\ \ln(1+x) \sim x,\ \mathrm{e}^x - 1 \sim x,\ 1-\cos x \sim \frac{x^2}{2},\ (1+x)^\alpha \sim 1+\alpha x.$$

- Let us determine a simple equivalent of $x^x - x$ when $x \to 1$.
  It is important to begin by changing variable to get near 0.
  Put $x = 1 + h$, with $h = x - 1 \to 0$

$$x^x - x = (1+h)^{1+h} - (1+h) = \mathrm{e}^{(1+h)\ln(1+h)} - (1+h)$$

but

$$\mathrm{e}^{(1+h)\ln(1+h)} = \mathrm{e}^{(1+h)\left(h - \frac{h^2}{2} + o(h^2)\right)} = \mathrm{e}^{h + \frac{h^2}{2} + o(h^2)} = 1 + h + h^2 + o(h^2)$$

so

$$x^x - x = h^2 + o(h^2) \sim h^2 \sim (x-1)^2$$

II Limit determination

SE to determine a limit

Obtaining an equivalent makes it possible to obtain the limit of the considered function.

Examples :

- Let's find

$$\lim_{x\to 0} \frac{\tan(2x) - 2\tan x}{\sin(2x) - 2\sin x}$$

when $x \to 0$

$$\tan x = x + \frac{1}{3}x^3 + o(x^3) \quad \text{then} \quad \tan(2x) - 2\tan x \sim 2x^3$$

$$\sin x = x - \frac{1}{6}x^3 + o(x^3) \quad \text{then} \quad \sin(2x) - 2\sin x \sim -x^3$$

Therefore

$$\frac{\tan(2x) - 2\tan x}{\sin(2x) - 2\sin x} \sim \frac{2x^3}{-x^3} \to -2$$

- Let us find

$$\lim_{x\to+\infty}\left(\cos\frac{1}{x}\right)^{x^2}$$

When $x \to +\infty$

$$\left(\cos\frac{1}{x}\right)^{x^2} = \exp\left(x^2\ln\left(\cos\frac{1}{x}\right)\right)$$

but

$$\cos\frac{1}{x} = 1 - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)$$

then $\quad \ln\left(\cos\frac{1}{x}\right) \sim -\frac{1}{2x^2} \quad$ so $\quad x^2\ln\left(\cos\frac{1}{x}\right) \to -\frac{1}{2}$

and finally

$$\lim_{x\to+\infty}\left(\cos\frac{1}{x}\right)^{x^2} = \frac{1}{\sqrt{\mathrm{e}}}$$

III Local positioning of a curve and its tangent

SE to position a curve in relation to its tangent

Let $f : I \to \mathbb{R}$ and $a \in I$.
We assume that f admits a series expansion of order n at a of the form :

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + o\left((x-a)^n\right)$$

The function being defined at a, we have necessarily $a_0 = f(a)$ and $a_1 = f'(a)$.
The equation of the tangent T to f at a is then given by $y = a_0 + a_1(x-a)$.
It suffices then to study the sign of

$$f(x) - y = a_2(x-a)^2 + \cdots + a_n(x-a)^n + o\left((x-a)^n\right)$$

to deduce the position of the curve of f with respect to T at a.

Example : the curve of $x \to \mathrm{e}^x$ is above its tangent at 0 because

$$\mathrm{e}^x - (1+x) \sim \frac{x^2}{2} \geqslant 0, \quad \text{when} \quad x \to 0.$$