Final Exam in Math1

WARNING: Calculators and electronic devices are strictly prohibited.

Exercise 1 **(4.5 Points)**

Let $\alpha \in [-1, +\infty[$. Prove by induction that for all $n \in \mathbb{N}$,

$$(1+\alpha)^n \geq 1 + n\alpha.$$

Exercise 2 **(4 Points)**

Consider the function:

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R}, \\ x & \longmapsto & \dfrac{2x}{1+x^2}. \end{array}$$

1. Compute $f(2)$ and $f\left(\frac{1}{2}\right)$.
2. Find $f^{-1}(\{2\})$.
3. Is f injective? Is it surjective?
4. Is f bijective?

Exercise 3 **(4 Points)**

Let f be the function defined by:

$$f(x) = \begin{cases} \ln(1+x), & \text{if } x \geq 0, \\ \sin(x), & \text{if } x < 0. \end{cases}$$

Study the continuity and differentiability of f on its domain of definition D_f.

Exercise 4 **(6 Points)**

1. Find the second-order Taylor expansion at $x_0 = 2$ for the functions:

$$f(x) = \ln x \quad \text{and} \quad g(x) = x^3 - x^2 - x - 2.$$

2. Using these Taylor expansions, compute:

$$\lim_{x \to 2} \frac{\ln x - \ln 2}{x^3 - x^2 - x - 2}.$$

Exercise 5 **(1.5 Points)**

Let the set E be defined as:

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + 5z = 0\}$$

Is the set E a vector subspace?

Good Luck!

Model Answer for Final Exam (MATH1)

Exercise 1 (4,5 pts)

For $\alpha \in [-1, +\infty[$, $\forall n \in \mathbb{N}$ $(1+\alpha)^n \geq 1+n\alpha$??

- Base Case:

For $n=0$ $(1+\alpha)^0 = 1$ and $1+0.\alpha = 1$

Thus $(1+\alpha)^0 = 1 \geq 1 = 1+0.\alpha$ 0.5pt

- Inductive Hypothesis:

Assume that for $n \in \mathbb{N}$, $(1+\alpha)^n \geq 1+n\alpha$ 0.5pt

- Inductive step:

We need to show that: $(1+\alpha)^{n+1} \geq 1+(n+1)\alpha$?

$(1+\alpha)^{n+1} = (1+\alpha).(1+\alpha)^n$ 0.5pt

$\geq (1+\alpha).(1+n\alpha)$ (By Inductive Hypothesis) 0.5pt

or

$(1+\alpha)(1+n\alpha) = 1+n\alpha+\alpha+n\alpha^2$ 0.5pt

$= 1+(n+1)\alpha + n\alpha^2$ 0.5pt

Since $n\alpha^2 \geq 0$ for $\alpha \geq -1$, we have

$1+(n+1)\alpha + n\alpha^2 \geq 1+(n+1)\alpha$ 0.5pt

Therefore:

$(1+\alpha)^{n+1} \geq 1+(n+1)\alpha$ 0.5pt

Conclusion:

For $\alpha \in [-1, +\infty[$, $\forall n \in \mathbb{N}$ $(1+\alpha)^n \geq 1+n\alpha$ 0.5pt

Exercise 2 : 4 pts

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{2x}{1+x^2}$$

1- $f(2) = \frac{2 \cdot 2}{1+2^2} = \frac{4}{5}$ — 0.5 pt

$f\left(\frac{1}{2}\right) = \frac{2 \cdot \frac{1}{2}}{1+\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$ — 0.5 pt

2- $f^{-1}(\{2\}) = \{x \in \mathbb{R} \,/\, f(x) = 2\}$ — 0.5 pt

We need to solve the equation

$$f(x) = \frac{2x}{1+x^2} = 2.$$

$$2x = 2(1+x^2) \Longrightarrow 2x^2 - 2x + 2 = 0$$

$$\Delta = 4 - 16 = -12$$

Since $\Delta < 0$, the quadratic equation has no solut — 0.5 pt

3- • Injectivity of f:

We need to show if $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$ — 0.25 pt

However, as we have already seen:

$$f(2) = f\left(\frac{1}{2}\right) = \frac{4}{5}$$ — 0.25 pt

This shows that two distinct element $x_1 = 2$ and $x_2 = \frac{1}{2}$, have the same image. Hence f is not injective — 0.25 pt

• Surjectivity of f:

$\forall y \in \mathbb{R} \quad \exists x \in \mathbb{R} \,/\, f(x) = y$ — 0.25 pt

As we showed earlier $f^{-1}(\{2\}) = \emptyset$ — 0.25 pt

This means $\nexists x \in \mathbb{R} \,/\, f(x) = 2$.

Consequently, f is not surjective. — 0.25 pt

4- Bijectivity of f:

Since f is neither injective nor surjective, is not bijective — 0.5 pt

Exercise 3 (4 pts)

$$f(x) = \begin{cases} \ln(1+x), & \text{if } x \geq 0 \\ \sin(x) & \text{if } x < 0 \end{cases}$$

1. Continuity of f:

- For $x \geq 0$: The function $f(x) = \ln(1+x)$ is a composit of continuous functions, so it is continuous on $]0, +\infty[$ 0,25

- For $x < 0$: The function $f(x) = \sin(x)$ is continuous on $]-\infty, 0[$, since it is continuous everywhere on $\mathbb{R}$. 0,25

- At $x = 0$:

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sin(x) = \sin(0) = 0.$ 0,5 p

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \ln(1+x) = \ln(1) = 0$ 0,5 p

Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$ f is continuous at $x = 0$ 0,25

Conclusion: f is continuous on $\mathbb{R}$. 0,25

2. Differentiability of f:

We wish to study the differentiability at $x = 0$.

- Case $x \to 0^+$

$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{\ln(1+x) - \ln(1)}{x} = \lim \frac{\ln(1+x)}{x}$ 0,5 p

Or $\lim_{x \to 0^+} \frac{\ln(1+x)}{x} \underset{\text{(Hopital)}}{=} \lim_{x \to 0^+} \frac{\frac{1}{1+x}}{1} = 1$

Thus $f'(0^+) = 1$ 0,5 p

- Case $x \to 0^-$

$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^-} \frac{\sin(x) - \sin(0)}{x} = \lim_{x \to 0^-} \frac{\sin(x)}{x}$ 0,5

$\underset{\text{Hopital}}{=} \lim_{x \to 0^-} \frac{\cos(x)}{1} = 1$

Thus $f'(0^-) = 1$ 0,5 p

Conclusion:

Since $f'(0^+) = f'(0^-) = 1$, f is differentiable at x

and $f'(0) = 1$

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Exercise 4: (6 pts)

1- • The second-order Taylor expansion of $f(x)$ at $x_0 = 2$:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + O((x - x_0)^3)$$

$f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$ 0,5pt + 0,5pt

$f(2) = \ln(2)$, $f'(2) = \frac{1}{2}$, $f''(2) = -\frac{1}{4}$

Therefore, the second-order Taylor expansion of $f(x)$ at $x_0 = 2$

$$f(x) = \ln(2) + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + O((x - 2)^3)$$ 1pt

• The second-order Taylor expansion of $g(x)$ at $x_0 = 2$:

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \frac{g''(x_0)}{2}(x - x_0)^2 + O((x - x_0)^3)$$

$g'(x) = 3x^2 - 2x - 1$, $g''(x) = 6x - 2$ 0,5pt + 0,5pt

$g(2) = 0$, $g'(2) = 7$, $g''(2) = 10$

Therefore, the second-order Taylor expansion of $g(x)$ at $x_0 = 2$ is

$$g(x) = 7(x - 2) + 5(x - 2)^2 + O((x - 2)^3).$$ 1pt

2- Limit calculation:

Using the Taylor expansions we have:

$$\ln x - \ln 2 = \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + O((x - 2)^3)$$ 0,25pt

$$x^3 - x^2 - x - 2 = 7(x - 2) + 5(x - 2)^2 + O((x - 2)^3).$$ 0,25pt

Simplify the fraction:

$$\frac{\ln x - \ln 2}{x^3 - x^2 - x - 2} = \frac{\frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + O((x - 2)^3)}{7(x - 2) + 5(x - 2)^2 + O((x - 2)^3)}$$ 0,25pt

$$= \frac{\frac{1}{2} - \frac{1}{8}(x - 2) + O((x - 2)^2)}{7 + 5(x - 2) + O((x - 2)^2)}$$ 0,25pt

Taking the limit as $x \to 2$, we get:

$$\lim_{x \to 2} \frac{\ln x - \ln 2}{x^3 - x^2 - x - 2} = \frac{\frac{1}{2}}{7} = \frac{1}{14}$$ 1pt

Exercise 5 (1,5 pts)

Let's show that: $E = \{(x,y,z) \in \mathbb{R}^3 / x + y + 5z = 0\}$ is vector subspac of $\mathbb{R}^3$

1. $E \neq \emptyset$, since $(0,0,0) \in E$. 0,5 pt

2. $\forall u = (x, y, z) \in E$, $\forall v = (x', y', z') \in E$, $\forall \lambda, \mu \in \mathbb{R}$:

$\lambda u + \mu v = (\lambda x + \mu x', \lambda y + \mu y', \lambda z + \mu z')$

Now,

$$(\lambda x + \mu x') + (\lambda y + \mu y') + 5(\lambda z + \mu z')$$

$$\parallel$$

$$\lambda(\underbrace{x + y + 5z}_{=0}) + \mu(\underbrace{x' + y' + z'}_{=0})$$

(because $u \in E$) (because $v' \in E$)

$$\parallel$$

$$\lambda \cdot 0 + \mu \cdot 0$$

$$\parallel$$

then: 0

$\lambda u + \mu v \in E$. 1 pt

Conclusion; E is a vector subspace.