The Matrices :

2- Determinant of Matrix

Let's consider A a square matrix of order n

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

Definition :

The determinant of a square matrix A is a number (a value) denoted by $det\ (A)\ \ or\ \ |A|$.

1) Determinant of a matrix order 2

when $n = 2$, the matrix A is structured as follows

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

In this case, the determinant of A is a cross product such that:

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{12}.a_{21}$$

Examples :

Let

$$A = \begin{pmatrix} -3 & 2 \\ -1 & 4 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}$$

Then,

- $det(A) = \begin{vmatrix} -3 & 2 \\ -1 & 4 \end{vmatrix} = (-3).4 - (-1).2 = -10$
- $det(B) = \begin{vmatrix} 2 & 3 \\ -4 & -6 \end{vmatrix} = (2).(-6) - (-4).(3) = 0.$

Now, let's see how to calculate the determinant of a matrix of dimension greater than 2.

2) Determinant matrix order n, $(n \geq 3)$

Definition ***: (the minor)***

We call M_{ij} a minor of A is the determinant of the matrix formed by removing the $i-th$ row and the $j-th$ column from A. It is also referred to as the $(i,j)-th$ minor of A.

Exemple :

Let A be a matrix *:*

$$A=\begin{pmatrix} 4 & 1 & 2 \\ 3 & 1 & 4 \\ -2 & -2 & 0 \end{pmatrix},$$

- The minor M_{12} is the determinant obtained by eliminating the 1st row and the 2nd column of matrix A, indeed:

$$M_{12}=\begin{vmatrix} 3 & 4 \\ -2 & 0 \end{vmatrix}=8$$

- The minor M_{22} is the determinant obtained by eliminating the 2nd row and the 2nd column of the matrix :

$$M_{22}=\begin{vmatrix} 4 & 2 \\ -2 & 0 \end{vmatrix}=4$$

- The minor M_{13} is the determinant obtained by eliminating the 1st row and the 3rd column of matrix A, it means:

$$M_{13}=\begin{vmatrix} 3 & 1 \\ -2 & -2 \end{vmatrix}=-4.$$

Remark :

A matrix has several minor determinants, as demonstrated in the following example.

Definition :

We call C_{ij} the cofactor of the element a_{ij}, given by the formula

$$\boldsymbol{C_{ij}=(-1)^{i+j}.M_{ij}}$$

The determinant of A (at the i-th row) is calculated using cofactor expansion as follows:

$$\det(A)=\sum_{j=1}^{n} a_{ij}.C_{ij}$$

Example

Let

$$A=\begin{pmatrix} 4 & 1 & 2 \\ 3 & 1 & 4 \\ -2 & -2 & 0 \end{pmatrix},$$

We expand along the first row:

$$\det(A) = 4M_{11} - M_{12} + 2M_{13}$$

With

$$M_{11} = 8 \qquad M_{12} = 8 \quad et \quad M_{13} = -4$$

Therefore

$$\det(A) = 4 \times 8 - 8 + 2 \times (-4) = 16$$

Properties :

Let $A \, and \, B$ be tow matrices order n, satisfying:

- $\det(A.B) = \det(A).\det(B)$
- $\det(A^T) = \det(A)$
- $\det(k.A) = k.\det(A)$
- The determinant of a triangular or diagonal matrix is equal to the product of its diagonal elements.
- The determinant of a matrix containing a null row (or column) is 0.