4- invertible matrix

Definition :

Let A be a square matrix of order n. The matrix A is invertible if and only if there exists a square matrix B of order n, such that

$$A \times B = I_n$$

And $$B \times A = I_n$$

With I_n is the Identity matrix of order n.
Additionally, the inverse of A is denoted by A^{-1} and

$$A^{-1} = B.$$

Remark :
The concept of matrix inverses only applies to square matrices. This means that:

$$A \times A^{-1} = A^{-1} \times A = I_n$$

Example 01 : Let A be a square matrix of order 3 which satisfies the following relation

$$A^3 = 3A - 2I_3$$

Let's show that A is invertible. Indeed:

$$A^3 = 3A - 2I_3 \Leftrightarrow A^3 - 3A = -2I_3$$

$$\Leftrightarrow -\frac{1}{2}(A^3 - 3A) = I_3$$

$$\Leftrightarrow A\left[-\frac{1}{2}(A^2 - 3I_3)\right] = I_3$$

Thus, we have

$$\left[-\frac{1}{2}(A^2 - 3I_3)\right]A = I_3$$

Therefore, A is invertible and its inverse is

$$A^{-1} = -\frac{1}{2}(A^2 - 3I_3)$$

Example 02 :
Let A and B be the two matrices such that:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad and \quad B = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Since,

$$AB = BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then , , A is invertible, and

$$A^{-1} = B.$$

How to calculate the inverse of an n-order matrix?

It is recalled that the cofactor of the element a_{ij} is denoted by C_{ij} and defined by

$$\boldsymbol{C_{ij} = (-1)^{i+j}.M_{ij}}$$

Where $\boldsymbol{M_{ij}}$ is the minor determinant of the matrix A .

Definition :(cofactor matrix) :

We call the cofactor matrix (or adjoint matrix) of A, the square matrix of order n, denoted as $cof(A)$ and defined by:

$$cof(A) = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}$$

Proposition : if $det\,(A) \neq 0,\ then\ A\ is\ invertible\ .$

Définition :

The inverse matrix of A is also found using the following equation:

$$A^{-1} = \frac{1}{\det(A)}.\ cof(A)^t$$

Propriétés

Let $A, B\ and\ C$ three invertible matrices of order n

- $if\ A\ is$ invertible, then the matrix A^{-1} is unic.
- $det(A^{-1}) = \frac{1}{\det(A)}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^t)^{-1} = (A^{-1})^t$

Application : let the matrix A

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{pmatrix}$$

Show that A is invertible and calculate its inverseA^{-1}.