Chapter 05 :

Linear Equations Systems

A system of linear equations consists of several linear equations involving the same variables, which are called unknowns.

A system of n linear equations with p unknowns: $x_1, x_2, x_3, \dots, x_n$, is written in the following form:

$$(S)\quad \begin{cases} a_{11}x_1 + a_{12}.x_2 + a_{13} + \cdots + a_{1p}x_p = b_1 \\ a_{21}x_1 + a_{22}.x_2 + a_{23} + \cdots + a_{2p}x_p = b_2 \\ \vdots \\ a_{31}x_1 + a_{32}.x_2 + a_{33} + \cdots + a_{1p}x_p = b_n \end{cases}$$

Where :

- The a_{ij} are given real numbers, referred to as the coefficients of the system. (pour $1 \leq i \leq n \quad and \quad 1 \leq j \leq p$)
- The b_i are also real numbers representing the constants on the right-hand side of the system (S) ($for \ 1 \leq i \leq n$).
- The x_j are the unknowns of the system, where $1 \leq j \leq p$.

Solving the system (S) involves finding the values of x_j that satisfy all the equations of the system. The system (S) can be rewritten in matrix form as:

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{ip} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{np} \end{pmatrix}}_{A} \times . \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_p \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{pmatrix}}_{B}$$

In other words, (S) is equivalent to

$$A\,.X = B$$

With $X = (x_1, x_2, \dots, x_j, \dots, x_p)$ represents the unknown to be determined.

Remark :

If all $b_i = 0$ (for $1 \leq i \leq n$), then the system is called homogeneous; otherwise (i.e., if at least one $b_i = 0$ is not equal to zero), it is called non-homogeneous.

I. Cramer's system :

Definition :

A system (S) is said to be a Cramer's system if it satisfies these three conditions:

- A is a square matrix, meaning it contains the same number of equations as unknowns. In other words, the number of unknowns equals the number of equations.
- A is invertible, meaning $det(A) \neq 0$

II. Methods of Solution :

Cramer's systems of linear equations are solved using one of the following methods:

- Method of matrix inversion (or Matrix inversion)
- Cramer's method
- Gauss method (Elimination Method) .

For this purpose, consider the following system, which is non-homogeneous and Cramer's system written in its matrix form.

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{ip} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{np} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_p \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{pmatrix}}_{B}$$

$$A \times X = B$$

1) Method of matrix inversion :

It is evident that in Cramer's systems, the matrix A^{-1} exists (because $det(A) \neq 0$) . Therefore, to determine the vector X it suffices to multiply both sides of the system by the inverse matrix A^{-1}. Indeed:

$$A.\boldsymbol{X} = B \iff A^{-1}.A.\boldsymbol{X} = A^{-1}.B$$

$$\iff I_n.\boldsymbol{X} = A^{-1}.B \qquad (because \quad A^{-1}.A = I_n \;)$$

$$\iff \boldsymbol{X} = A^{-1}.B \qquad (because \quad I_n.X = X \;)$$

This means that the values of the unknown vector �X are calculated from the matrix product $A^{-1}.B$.

Example :

Let the system (S_1) :

$$(s_1)\quad \begin{cases} 3x_1 + 2x_2 + x_3 = 4 \\ x_1 + x_2 + x_3 = 1 \\ x_1 - 2x_3 = -1 \end{cases}$$

To solve(S_1) , we first need to write (S_1) in matrix form:

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$$

It is in the form:

$$A.X = B$$

With

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \quad , \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$$

Consequently, the solution X of (S_1) is given by:

$$X = A^{-1}.B.$$

Now, let us calculate A^{-1}, the inverse of A , using the formula:

$$A^{-1} = \frac{1}{\det(A)} . \, com(A)^t$$

We have

$$det(A) = -1$$

Furthermore, after calculation, the matrix of cofactors of A is given by:

$$Com(A) = \begin{pmatrix} -2 & 3 & -1 \\ 4 & -7 & 2 \\ 1 & -2 & 1 \end{pmatrix}$$

It follows that:

$$A^{-1} = \frac{1}{-1} \begin{pmatrix} -2 & 4 & 1 \\ 3 & -7 & -2 \\ -1 & 2 & 1 \end{pmatrix}$$

It is deduced that:

$$X = A^{-1}.B = \begin{pmatrix} 2 & -4 & -1 \\ -3 & 7 & 2 \\ 1 & -2 & -1 \end{pmatrix} . \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \quad \Longrightarrow \quad X = \begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix}$$

2) Cramer Method's :

It is also called the method of determinants. Indeed,

Let (S) be the matrix system to solve.

$$A.X = B$$

And let Δ denote the determinant of the matrix A, such that:

$$\Delta = \det(A)$$

Thus, Δi is the determinant of the matrix obtained by replacing the i-th column of matrix A with the vector B (column of constants).

Therefore, the unknown $\boldsymbol{x_i}$ is obtained by calculating the following ratio:

$$\boldsymbol{x_i} = \frac{\Delta_i}{\Delta} \qquad pour\ 1 \leq i \leq n$$

Example (Application) :

Let the system :

$$A.X = B$$

Where

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \ , \ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad and \qquad B = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$$

The determinant of this systeme is :

$$\Delta = |A| = -1$$

Let's calculate the value of x, indeed:

$$\Delta_x = \begin{vmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & -2 \end{vmatrix} = -5$$

So,

$$\boldsymbol{x} = \frac{\Delta_x}{\Delta}$$

$$\Longrightarrow \boldsymbol{x} = \frac{-5}{\boldsymbol{-1}}$$

$$\Longrightarrow \quad \boldsymbol{x} = \mathbf{5}$$

- Let's calculate the value of y:

$$\Delta_y = \begin{vmatrix} 3 & 4 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -2 \end{vmatrix} = 7$$

Then

$$\boldsymbol{y} = \frac{\Delta_y}{\Delta} = \frac{\mathbf{7}}{\mathbf{-1}}$$

$$\Longrightarrow \boldsymbol{y} = \mathbf{-7}$$

- Similarly, to find z, we have

$$\Delta_z = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = -3$$

Then,

$$\boldsymbol{z} = \frac{\Delta_z}{\Delta} = -\frac{\mathbf{3}}{\mathbf{-1}}$$

$$\Longrightarrow \boldsymbol{y} = \mathbf{3}$$

So, the solution of$(S_1)est$:

$$X = (x, y, z,) = (5, -7, 3).$$

3) Gauss Method's:

Given the following matrix system:

$$A.X = B \qquad (S)$$

To solve (S) using the Gauss method involves transforming the system matrix (S) into an upper triangular matrix using elementary row operations, and then solving the resulting system using the back-substitution method.

To successfully achieve this transformation, we first need to define the Gauss table, written as follows:

$$[A \mid B]$$

Definition : Elementary row operations on a matrix :

The Elementary row operations (on the rows of a linear system) include the following:

- Swapping two equations, which means interchanging two rows.
- Multiplying a row by a constant.
- Replacing an equation with a linear combination of two rows (or two equations).

Definition : (pivot)

A pivot is a value by which we must divide to solve the linear system. These are the diagonal elements of the square matrix (i.e., $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$). It is necessary for these pivots to be non-zero in order to determine the solution of the system.
Note: This method is also called the Gauss elimination method, or the Gauss pivot method.

Example (Application) :

Let's solve the system using the Gauss method:

$$A.X = B$$

Where

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \quad , \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad et \quad B = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$$

The Gauss table is defined by:

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & -1 \end{array}\right] \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

We denote L_1, L_2 *and* L_3 as the rows defined in the Gauss table.

We also denote L'_1, L'_2 *and* L'_3 as the new rows calculated from L_1, L_2 *and* L_3.To transform the system into an upper triangular system, we first fix the first row $L_1$and apply the following operations:

$$L'_2 = L_1 + (-3)L_2$$

And

$$L'_3 = L_1 + (-3)L_3$$

We obtain the new table:

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 0 & -1 & -2 & -2 \\ 0 & 2 & -5 & 7 \end{array}\right] \begin{array}{c} L_1 \\ L'_2 \\ L'_3 \end{array}$$

To obtain the upper triangular matrix in the first part of the table, we use this operation while fixing the row L'_2

$$L''_2 = (2)L'_2 + L'_3$$

This results in

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -9 - & 9 \end{array}\right] \begin{array}{c} L_1 \\ L'_2 \\ L'_3 \end{array}$$

The new system obtained is:

$$\begin{pmatrix} 3 & 2 & -1 \\ 0 & -1 & -4 \\ 0 & 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix}$$

The system is rewritten as linear equations, starting from the last row and moving upwards to the first. Indeed,

$$\begin{cases} -3z = 9 \\ -y - 4z = 1 \\ 3x + 2y - z = 4 \end{cases}$$

By substitution, we find:

$$\begin{cases} z = -3 \\ y = 11 \\ x = -7 \end{cases}$$

I. Non-cramerian systems :

Let (S) be a linear system with n equations and p unknowns. (S) is non-Cramerian if:

1) $\boldsymbol{n > p}$, , meaning there are more equations than unknowns. In this case, the system is termed “ *overdetermined* ”.

2) $\boldsymbol{n < p}$, , indicating fewer equations than unknowns. In this case, the system is termed “ *underdetermined* ”

3) If $\boldsymbol{n = p}$ et $\boldsymbol{det\,(A) = 0}$, the system is square but non-invertible.

Solving overdetermined systems $(n > p)$:

It suffices to follow these steps

a) Extract a subsystem with p equations and p unknowns such that the determinant associated with the subsystem is non-zero.

b) Solve the subsystem.
c) Check the solution obtained for the $n - p$ equations. There are two cases:
   - If the solution satisfies all the equations, we conclude that the global system has a unique solution.
   - If the solution does not satisfy all the equations, then it is clear that the global system has no solution.

Example: Consider the system:

$$(s)\quad \begin{cases} x + 2y = 1 \\ 3x - y = 2 \\ 5x - 4y = -2 \end{cases}$$

The system(s) contains 3 equations with 2 unknowns. Here, $n = 3\ and\ \ p = 2$. To solve (S), we choose the subsystem (S') with 2 equations:

$$(s')\quad \begin{cases} x + 2y = 1 \\ 3x - y = 2 \end{cases}$$

The resolution of (S') is very simple and yields $x = \frac{5}{7}\ \ et\ y = \frac{1}{7}$. In conclusion, we need to verify if this solution:

$$x = \frac{5}{7}\ \ et\ y = \frac{1}{7}$$

satisfies the last equation (the one that was not chosen).

$$5x - 4y = -2$$

We have

$$5x - y = 5\left(\frac{5}{7}\right) - 4\left(\frac{1}{7}\right) = \frac{21}{7} = 3\ \neq -2.$$

We deduce that the solution obtained from the subsystem (S') does not satisfy all the equations. Consequently, the system (S) has no solutions.

Solving underdetermined systems $(\boldsymbol{n} < \boldsymbol{p})$:

To determine the solution of this type of system, we need to:

1. Consider a subsystem that contains $\boldsymbol{n}$ equations with $\boldsymbol{n}$ unknowns and assume the remaining unknowns $\boldsymbol{n}-\boldsymbol{p}$, as constants.
2. The solution obtained demonstrates that underdetermined systems have infinitely many solutions.

Example: Consider the system:

$$(S)\quad \begin{cases} x + 2y - z = 1 \\ 3x - y + z = 2 \end{cases}$$

The system (S) contains 2 equations with 3 unknowns:: x, y and z (here, $n = 2$ and $p = 3$). To solve(S), we choose the subsystem (S') with 2 equations and 2 unknowns while assuming the third unknown as a constant:

Let's set:

$$z = \alpha \quad \text{where} \quad \alpha \in \mathbb{R}$$

The subsystem of (S) is:

$$\begin{cases} x + 2y - \alpha = 1 \\ 3x - y + \alpha = 2 \end{cases} \quad \Longrightarrow \quad \begin{cases} x + 2y = 1 + \alpha \\ 3x - y = 2 - \alpha \end{cases}$$

Let's find the value of x and y in terms of α which gives:

$$x = \frac{5-\alpha}{7} \qquad and \quad y = \frac{4\alpha + 1}{7} \qquad where \ \ \alpha \in \mathbb{R}$$

Hence, (S) has infinitely many solutions of the form:

$$X = (x; y; z) = \left(\frac{5-\alpha}{7} ; \quad \frac{4\alpha+1}{7} ; \ \alpha \right) \qquad avec \ \alpha \in \mathbb{R}$$