Chapter 01 :

1st order Différential Equations

Definition 01:

An Ordinary Differential Equation (ODE) of order n is an equation where the unknown is a function $y(t)$. It is of the form

$$F\left(t, y, y', y'', \ldots \ , y^{(n)}\right) = 0\,,$$

With :

- F:is a continues function.
- y : The unknown function of the variable t (to be determined)
- t : is the real variable (in physics, representing time).
- n : It is the highest order of the derivative of y, which is also the order of the ODE.

Definition 02 : A first-order ordinary differential equation (ODE) is of the form:

$$y' = F(t, y) \qquad \ldots .(1)$$

Remark : To find the solution of an ODE, one must search for a function y(t) that satisfies this ODE, according to each type. Integration is the process that allows us to do this.

1. Separable Variable Equation:

Definition 03: The ODE (1) is said to be in separated variables form if it is of the form:

$$f(y)dy = g(t)dt$$

Where f and g are two real (continuous) functions of the real variable.

Resolution method: To find the solution from (1), you need to follow the following steps:

1. **Use the relation** $\boldsymbol{y' = \frac{dy}{dt}}$.
2. **Separate the $\boldsymbol{y}$ terms from the $\boldsymbol{t}$ variable:** Put the y terms on one side and what depends on t on the other side.
3. **Integrate both sides of the equation with respect to both $\boldsymbol{t}$ and $\boldsymbol{y}$.**

Example: Solve the following ODE:

$$\boldsymbol{y^2 - (1 + 3t)y' = 0} \qquad \ldots \ldots (\boldsymbol{E})$$

- **From the relation** $\boldsymbol{y'} = \frac{dy}{dt}$, we obtain :

$$y^2 - (1 + 3t)y' = 0 \;\Leftrightarrow\; y^2 - (1 + 3t)\frac{\boldsymbol{dy}}{dt} = 0$$

- To begin, we separate the variables, we have

$$y^2 - (1+3t)y' = 0 \quad \Leftrightarrow y^2 = (1+3t)\frac{d\boldsymbol{y}}{dt}$$

$$\Leftrightarrow \frac{1}{(1+3t)} = \frac{1}{y^2}y'$$

$$\Leftrightarrow \frac{1.dt}{(1+3t)} = \frac{1}{y^2}dy$$

- By integrating each side of the equation:

$$\int \frac{1.dt}{(1+3t)} = \int \frac{1}{y^2}dy \quad \Leftrightarrow \frac{1}{3}ln|1+3t| + c = -\frac{1}{y}$$

- Finally, the solution is:

$$\boldsymbol{y} = -\frac{\mathbf{1}}{\frac{1}{3}ln|1+3t| + c}$$

2. Homogeneous Differential Equation:

Definition 04 : A homogeneous differential equation is written as:

$$y' = f\left(\frac{y}{t}\right) \ldots..(2)$$

Resolution method: Let's consider the differential equation $y' = f\left(\frac{y}{t}\right)$,

1. Introduce the change of variable : $\boldsymbol{u(t)} = \frac{\boldsymbol{y(t)}}{\boldsymbol{t}} \quad \Rightarrow \quad y(t) = t.u(t)$
2. Use the derivative: $\boldsymbol{y' = t.u'(t) + u(t)}$
3. Replace the values of u and y' in equation (2).
4. Use the method of separated variables to solve the resulting equation.
5. Find the final solution $y(t)$.

Example: Solve the following ODE:

$$ty' + y = t$$

Solution :

We have,

$$ty' + y = t$$

In order to express$(\boldsymbol{E})$ in the form $\boldsymbol{y}' = \boldsymbol{f}\left(\frac{\boldsymbol{y}}{\boldsymbol{t}}\right)$, We need to divide everything by t.

$$ty' + y = t \quad \Leftrightarrow \quad \frac{t.y'}{t} + \frac{y}{t} = \frac{t}{t}$$

$$\Rightarrow \quad y' = 1 - \frac{y}{t} \ldots \ (\boldsymbol{E})$$

Let's introduce a variable transformation

$$u(t) = \frac{y(t)}{t} \quad \Rightarrow \quad y(t) = t.u(t)$$

The derivative is $y' = t.u'(t) + u(t)$

By substituting into (E)

$$y' = 1 - \frac{y}{t}$$

We get ,

$$t.u'(t) + u(t) = 1 - u$$

$$\Rightarrow \quad t.u'(t) = 1 - 2u$$

- **Using the method of separated variables, we solve the equation:**

$$t.u'(t) = \frac{1}{2} - 2u$$

Remark that $u'(t) = \frac{du}{dt}$, so ,

$$t.u'(t) = 1 - 2u \qquad \Leftrightarrow \; t.\frac{du}{dt}. = \quad 1 - 2u$$

$$\Leftrightarrow \frac{du}{1-2u}. = \frac{dt}{t}$$

We integrate both sides.

$$\int \frac{du}{1-2u}. = \int \frac{dt}{t}$$

$$\frac{-1}{2} ln|1 - 2u| = ln|t| + c_1$$

Then ,

$$ln|1 - 2u| = -2ln|t| + c \qquad \Leftrightarrow \quad e^{ln|1-2u|} = e^{-2ln|t|+c}$$

$$\Leftrightarrow \quad |1 - 2u| = e^{-ln|t|^2}.e^c$$

$$\Leftrightarrow \quad 1 - 2u = \pm e^{ln\frac{1}{t^2}}.e^c$$

$$\Leftrightarrow u = \frac{-1}{2}\left(\pm\frac{1}{t^2}.e^c - 1\right)$$

Then the solution is

$$u(t) = \frac{1}{2}\left(\frac{k}{t^2} + 1\right) \qquad where \;\; k = \mp e^c$$

- **In order to find $y(t)$, we simply substitute the value of u into the solution, such that,**

$$u(t) = \frac{y(t)}{t} = \frac{1}{2}\left(\frac{k}{t^2} + 1\right)$$

Finally, the solution is :

$$y(t) = \frac{t}{2}\left(\frac{k}{t^2} + 1\right) = \frac{k}{2t} + \frac{t}{2}, \qquad k \in \mathbb{R}$$