# I. 2nd Order Differential Equations

**Definition :** A second-order linear differential equation with constant coefficients has the form

$$a.y'' + by' + c.y = f(t) \qquad (E)$$

Where : $a, b$ and $c$ are a reals constants with $a \neq 0$, $\forall t \in \mathbb{R}$.

And $f(t)$ is the second member.

if $f(t) = 0$, then ( E) becomes an equation without a second member (EWSM), called a ***linear homogeneous equation,*** denoted by($E_h$) :

$$a.y'' + by' + c.y = 0 \qquad (E_h)$$

## Resolution method:

The general solution y of $(E)$ is the sum of the homogeneous solution $y_h$ of $(E_h)$ and a particular solution $(y_p)$ of (E): such that

$$\boldsymbol{y = y_h + y_p}$$

### 2. *How to find $y_h$?*

Let the homogeneous equation be

$$a.y'' + by' + c.y = 0$$

a) Write the caracteristic equation: $a.r^2 + b.r + c = 0$.

b) Findthe root $r$ according to the sign of $\Delta$ given in the following table where $\Delta = b^2 - 4ac$ : $hier\ a = 1$

| Sign of $\Delta$ | The roots : $r_i$ | The soltion $y_h$ |
|---|---|---|
| $\Delta > 0$ | There is $tow\ roots$ $r_1 = \frac{-b-\sqrt{\Delta}}{2a}$ $r_2 = \frac{-b+\sqrt{\Delta}}{2a}$ | $y_h = C_1 e^{r_1.t} + C_2 e^{r_2.t}$ |
| $\Delta = 0$ | $r_0 = \frac{-b}{2a}$ | $y_h = (C_1 t + C_2) e^{r_0.t}$ |

### 1. *How to find the particular solution $y_p$ ?*

We determine the particular solution $y_p$ of (E1), according to the form of the second member $f(t)$, and using the identification method of coefficient, the following table shows how to choose the form of $y_p$

| f(t) takes the form : | $y_p$ |
|---|---|
| $\boldsymbol{f(t) = P(t)e^{\alpha t}}$ $with\ P(t)$ is a polynomial of degree $n$, where $\alpha$ is a real number, and $m$ is not a root of $\boldsymbol{P}$ | $\boldsymbol{y_p = Q(t)e^{\alpha t}}$ <br> $Q(t)$ is a polynomial $deg(Q) = n$ |
| $\boldsymbol{f(t) = P(t)e^{\alpha t}}$ $with\ P(t)$ is a polynomial of degree 2, $\alpha$ is a real number, and $\boldsymbol{m}$ **Is a simple root** | $\boldsymbol{y_p = Q(t).t.e^{\alpha t}}$ |
| $\boldsymbol{f(t) = P(t)e^{\alpha t}}$ $with\ P(t)$ is a polynomial of degree 2, $\alpha$ is a real number, and $\boldsymbol{m}$ **Is a double root** | $\boldsymbol{y_p = Q(t).t^2.e^{\alpha t}}$ |

### 3. Donne la solution finale $\boldsymbol{y(t) = y_h + y_p}$.

**Exemple :** *Solve the equation (E) given by*

$$y" + 2y' = 2e^{-2x} \qquad (E)$$

**Solution :**

- **Find the homogeneous solution of $\boldsymbol{y" + 2y' = 0}$**

The characteristic equation associate to ( E ) is:

$$r^2 + 2r = 0$$

Wich has tow roots : $r_1 = 0\ and\ r_2 = -2$

Then , the solution $y_h$ is

$$y_h = k_1 + k_2e^{-2x} \quad where\ k_1 and\ k_2 \in \mathbb{R}$$

- **Find the particular solution of ( E )**

Hier $\alpha = -2$, then $y_p$ tak the form

$$y_p = kxe^{-2x} \ , \qquad k \in \mathbb{R}$$

Then the derivative of $y_p$ gives

$$y_p' = ke^{-2x} - 2kxe^{-2x} \text{ et } y_p" = -4ke^{-2x} + 4kxe^{-2x}$$

By substituting into ( E)

$$y_p" + 2y_p' = -4ke^{-2x} + 4kxe^{-2x} + 2ke^{-2x} - 4kxe^{-2x} = -2ke^{-2x} = 2e^{-2x}$$

Using the identification method, it follow :

$$-2k = 2 \Rightarrow k = -1$$

Then ,

$$y_p = -xe^{-2x}$$

Finally , the general solution of ( E ) is

$$y = y_h + y_p = k_1 + k_2e^{-2x} \ - xe^{-2x}$$

$$where\ k_1 and\ k_2 \in \mathbb{R}$$