Chapter two: Mathematical formulation of the Linear Programming Model

2-1- Definition:

Linear programming is a mathematical optimization technique used to find the best possible outcome (such as maximum profit or minimum cost) under a given set of constraints and requirements. It is widely used in operations research, economics, engineering, and management to solve real-world problems involving limited resources.

The term "linear programming" consists of two words, linear and programming. the word linear tells the relation between various types of variables of degree one used in a problem, which can be represented by straight lines, i. e . , the relationships are of the form $y = ax + b$, and the word "programming" means "taking decisions systematically". (Step-by-step procedure to solve these problems)

2-2- Key Components of Linear Programming

- Decision Variables:

The variables that represent the choices to be made.

Example: $x1$ is the number of units of product A to produce,

$x2$ is the number of units of product B to produce.

- Objective Function:

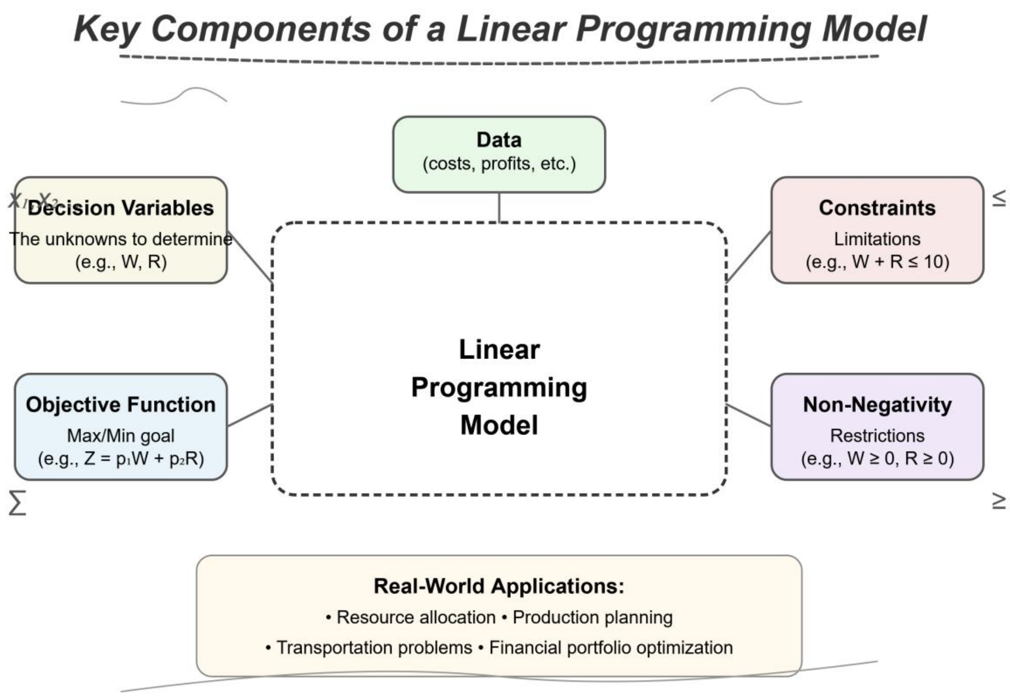
A mathematical representation of the goal, such as maximizing profit or minimizing cost.

- Constraints:

Linear inequalities or equations that represent limitations on resources (e.g., budget, time, labor, materials).

- Non-Negativity Restriction:

Decision variables must be non-negative ($X1,X2 \geq 0$) because negative quantities typically don't make sense in real-world problems.



2-3- Mathematical Formulation of a Linear Programming Problem

A standard LP problem is written as:

Objective Function:

Maximize or Minimize $Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

Subject to Constraints:

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2 \\ \quad \cdots\cdots\cdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m \end{cases}$$

Non-Negativity: $x_1, x_2, \ldots, x_n \geq 0$

Where:

Z: Objective function to be maximized or minimized.

$x_1, x_2, \ldots, x_n$: Decision variables.

$c_1, c_2, \ldots, c_n$: Coefficients of the objective function.

a_{ij} : Coefficients of the constraints.

$b_1, b_2, \ldots, b_m$: Right-hand side constants in the constraints.

2-4- Steps to Build a Linear Programming (LP) Model

Building a linear programming model involves systematically translating a real-world problem into a mathematical representation that can be solved using optimization techniques.

Below are the detailed steps to create a linear programming model:

Step 1: Identify the Decision Variables

After understanding the Problem, we define the decision variables that represent the quantities to be determined (e.g., number of products to produce, amount of resources to allocate, etc.).

Decision variables are the unknowns you want to determine, representing the quantities to be determined (e.g., number of products to produce, amount of resources to allocate, etc.).

Assign meaningful symbols (e.g., $x1,x2,x3$, ……………..)

Example:

X1: Number of product A to produce.

X2: Number of product B to produce.

Step 2: Formulate the Objective Function

The objective function is the mathematical expression of the goal you want to achieve, it is a linear combination of decision variables, weighted by their contribution to the goal;

If maximizing (e.g., profit, output): Example Max

$Z=c1 \ x1 \ +c2 \ x2$

(where : $c1,c2$ are coefficients like profit per unit of $x1$ and $x2$).

If minimizing (e.g., cost, time): The objective function is also a linear combination, but it aims to minimize the total cost or time.

Example: Min $Z=c1x1+c2x2$

Step 3: Write down all the constraints of the linear problems.

Constraints represent the limitations or restrictions in the problem. These are usually inequalities or equations

Resource constraints: Based on available resources (labor, material, budget, etc.).

Example: If producing product A and B requires raw materials,

$$a1x1+a2x2 \leq b$$

b is the resource limit, and a1,a2 are resource requirements per unit of x1,x2)

Demand constraints: If there is a minimum or maximum demand for a product.

Example: $x1 \geq 50$ (at least 50 units must be produced).

Technical constraints: Based on relationships between variables (e.g., production ratios, machine capacity, etc.).

Example: $x1 \leq 2x2$ (product A must not exceed twice the production of product B).

Types of Constraints:

- Less Than or Equal To: Representing limited resources or capacities, like production time or materials.
- Greater Than or Equal To: Enforcing minimum requirements, such as production quotas or quality standards.
- Equality Constraints: Representing fixed relationships, like balancing supply and demand or using a specific blend of ingredients.

Step 4: Ensure non-negative restrictions of the decision variables.

Decision variables must be non-negative since negative quantities don't make sense. Example: $x1, x2 \geq 0$

Step 5: Write the Complete Linear Programming Model :

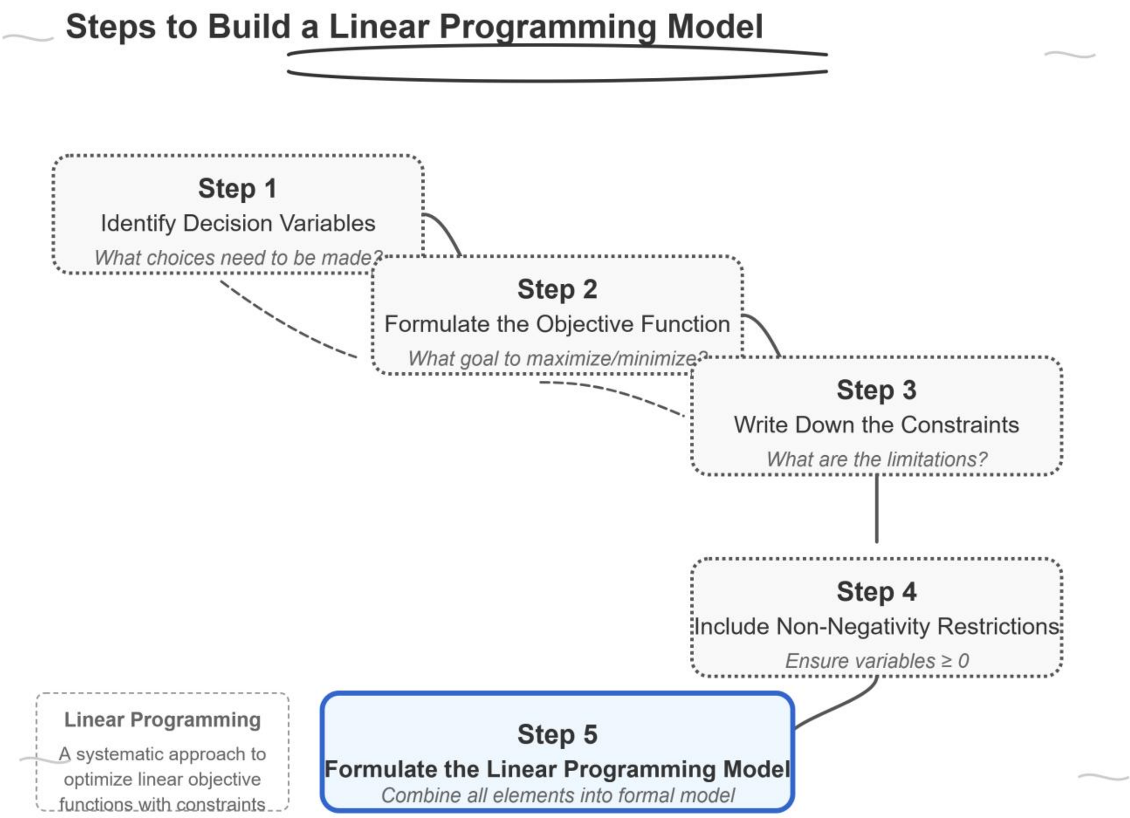
Combine all the components into a formal model. This typically includes the objective function, constraints, and non-negativity restrictions. For example:

Maximize $Z = 5x1 + 3x2$

Subject to:

$$\begin{cases} 2x1 + x2 \leq 100 \\ x1 + 3\,x2 \leq 90 \end{cases}$$

$x1 \geq 0, \ x2 \geq 0$



Example: (Formulating linear programming model)

A manufacturer produces two types of models *M* and *N*. Each *M* model requires 4 hours of grinding and 2 hours of polishing, whereas each *N* model requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on model *M* is 3€/Unit and model *N* is 4€/Unit. Whatever is produced in a week is sold in the market.

- ❖ Write the Linear Programming Model?

Solution:

-------------------- --	Product M	Product N	Available hours per week
Grinding	4 hours	2 hours	2 × 40 = 80 hours/ week
Polishing	2 hours	5 hours	3 × 60 = 180 hours/week
Profit	3 €/ unit	4 €/ unit	-----------------------------

Steps for building the Linear Programming Model:

Step N°1: Identify decision variables

Let: X_1 is the number of units of the product M produced per week;

X_2 is the number of units of the product N produced per week.

Step N°2: Formulate the objective function (the total profit)

The total profit required per week = profit required by the product M (per week) + profit required by the product N (per week)

$$Z = 3\ X_1 + 4\ X_2$$

Step N°3 Formulate the constraints (grinding & polishing constraints)

Grinding constraint: the number of grinding hours required by the product M + the number of grinding hours required by the product N should be at most 80 hours/week

$$4\ X_1 + 2\ X_2 \leq 80 \text{ hours}$$

Polishing constraint: the number of polishing hours required by the product M + the number of polishing hours required by the product N should be at most 180 hours / week

$$2X_1 + 5\ X_2 \leq 180 \text{ hours}$$

Step N°4: non-negativity condition of the decision variables

$$X_1 \geq 0$$

$$X_2 \geq 0$$

Step N° 5: The linear programming model is:

$$\text{MAX } Z = Z = 3\ X_1 + 4\ X_2$$

$$\text{St: } \begin{cases} 4\ X_1 + 2\ X_2 \leq 80 \\ 2X_1 + 5\ X_2 \leq 180 \end{cases}$$

$$X_1 \geq 0 \quad X_2 \geq 0$$