Chapter four: Theory of the Simplex Method

4-1- Introduction:

Simplex method also called simplex technique or simplex algorithm was developed in 1947 by G.Dantzig an American mathematician. It has the advantage of being universal.

In principle, it consists of starting with a certain solution of which all that we know is that it is basic feasible i.e, it satisfies the constraints as well as non-negativity conditions ($x_j \geq 0$, j=1, 2, 3, ………). Then, we improve upon this solution at consecutive stages, we arrive at the optimal solution. The method also helps the decision maker to identify the redundant constraints, an unbounded solution, multiple solutions and an infeasible solution.

The simplex method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another in such a manner that the value of the objective function t at the succeeding vertex is less in a minimization problem (or more in a max problem) than at the proceeding vertex. this procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, this method leads to an optimal vertex in a finite number of steps.

4-2- Steps for solving LPM using the SIMPLEX method

The simplex method consists of:

(i) having a trial basic feasible solution to the constraint equations.

(ii) testing whether it is an optimal solution or not.

(iii) improving, if required, the first trial solution by a set of rules and repeating the process till an optimal solution is obtained.

The simplex technique will be explained by considering this example:

Example:

Max $Z= 3X_1 +4X_2$

$$\text{ST:} \quad \begin{cases} X_1+2X_2 \leq 450 \text{ mn (time constraint of machine M1)} \\ 2X_1+ X_2 \leq 600 \text{ mn (time constraint of machine M2)} \end{cases}$$

$X_1, X_2 \geq 0$

SOLUTION

To solve this model by the simplex method, we have to follow these steps:

Step N°1: Express the problem in standard form:

The given problem is said to be expressed in standard form if the decision variables are non-negative, right-hand side of the constraints are non-negative and the constraints are expressed as equations.

Since the first two conditions are not with in the problem, non-negative slack variables Y_1, Y_2, are added to the left hand side of the two constraints respectively to convert them into equations.

Accordingly, the problem in standard form can be written as follows:

Max $Z = 3X_1 + 4X_2 + 0Y_1 + 0Y_2$

ST: $\begin{cases} X_1 + 2X_2 + Y_1 = 450 \\ 2X_1 + X_2 + Y_2 = 600 \end{cases}$ $\longrightarrow$ STANDARD FORM

$X_1, X_2, Y_1, Y_2 \geq 0$

Slack variables Y_1, Y_2 represent unutilized capacity or resources

In current problem Y_1 denotes the time (in mn) for which machine M1 remains unutilized, similarly Y_2 denotes the unutilized time for machine M2.

<u>Step N°2: find initial basic feasible solution:</u>

In the simplex method a start is made with a feasible which we shall get by assuming that the profit earned is zero.

The simplex tabular representation of the linear programming problem, organizing the coefficients of the variables, the right- hand side values, and the objective function coefficients.

Note that: Slack variables added to (≤) constraints to convert them into equalities, form columns with a single ‘1’ and the rest ‘Os’ creating part of the identity matrix (unit matrix where n×n = number of constraints

Note that every simplex table will have identity matrix under the basic variables. The identity matrix is also called unit matrix or basis matrix. It is always a square matrix and its size is equal to the number of constraints.

When setting up the initial simplex tableau, the variables that construct the unit matrix are typically the:

- **Slack variables:**
    - These are added to "≤" constraints to convert them into equalities. In the initial tableau, they form columns with a single "1" and the rest "0s," creating part of the identity matrix.
- **Artificial variables:**
    - These are added to "≥" and "=" constraints when there's no readily available basic feasible solution. Similar to slack variables, they also form columns that contribute to the initial unit matrix.

Here's a breakdown of why and how:

- **Creating an Initial Basic Feasible Solution:**
    - The simplex method requires an initial basic feasible solution to start its iterative process. The unit matrix provides this. Each column of the unit matrix represents a

basic variable, and the corresponding right-hand side values give their initial values.

- **How They Form the Unit Matrix:**
  - When you add a slack variable to a "≤" constraint, its coefficient is "1" in that constraint's row and "0" in all other constraint rows. This creates a column with a "1" and zeros else where.
  - Similarly, artificial variables are introduced to create those necessary "1" and zero columns.
  - So, in the initial tableau, the columns corresponding to these slack and/or artificial variables will have the form of a unit matrix.

In essence, these variables provide the initial "handles" for the simplex method to manipulate as it searches for the optimal solution.

The initial basic tableau for the example is:

Basic variables	X_1	X_2	Y_1	Y_2	B
Y_1	1	2	1	0	450
Y_2	2	1	0	1	600
Z	3	4	0	0	0

In the simplex method a start is made with a feasible solution.

Step N°3: select the entering variable from the row of objective function

The entering variable is the non-basic variable. If there are more than one positive coefficient of z, then choose the most positive of them.

The entering variable column is known as the key column or pivot column which is shown marked with an arrow at the side.

Basic variables	X_1	X_2	Y_1	Y_2	B	$\frac{B}{X_2}$
Y_1	1	2	1	0	450	450/1=450
Y_2	2	1	0	1	600	600/1=600
Z	3	4	0	0	0	

Let the entering variable is X_2 where its coefficient = 4 (a positive number in the case of MAX Z)

Step N°4: select the leaving variable

Compute the ratios $\frac{B}{X_2}$ and choose the minimum of them. Let the minimum ratio be Y_1 (=450) Then, the vector Y_1 will leave the base YB.

The element lying at the intersection of key column and key row is called KEY or PIVOT element. The column to be entered is called *key column*

Pivot = 2

Basic variables	X_1	X_2	Y_1	Y_2	B	$\frac{B}{X_2}$
Y_1	1	$\boxed{2}$	1	0	450	450/1=450
Y_2	2	1	0	1	600	600/1=600
Z	3	4	0	0	0	

X_2 enters into the basis and *Y_1* leaves the basis

To find the leaving variable we have to compute the ratio $\frac{B}{X_2}$ and we choose the minimum of them which is 450, SO the leaving variable is Y_1 it means that Y_1 leaves the base to be non-basic element.

The smallest positive ratio of the two equations is 450/2 (row Y1). Row Y1 now becomes the *pivot row* and 1 at the intersection of the pivot row and pivot column, becomes *the pivot.*

The element at the intersection of key row and key column is called *key element*

Step N°5: Drop the leaving variable and introduce the entering variable along with its associated value. h

Convert the pivot element 2 to unity and all other element in its column to zero by the following transformations.

New pivot equation = Old pivot equation / Pivot element

The remaining rows are formed by using the formula:

$$\text{New number} = \text{Old number} - \frac{(\underset{\text{in pivot row}}{\text{corresponding nmber}}) \times (\underset{\text{in pivot column}}{\text{corresponding number}})}{\text{the pivot}}$$

X_2 enters into the basis and Y_1 leaves the basis. The iterative table is as follows:

Basic variables	X_1	X_2	Y_1	Y_2	B
X_2	1/2	1	1	0	225
Y_2	3/2	0	-1/2	1	375
Z	1	0	-2	0	900

Since all the coefficients of Z are not less than or equal to zero, the current basic feasible solution is not optimal.

STEP N°6: If it is not the optimal solution, repeat from step N°3

Basic variables	X_1	X_2	Y_1	Y_2	B
X_2	1/2	1	1/2	0	225
Y_2	3/2	0	-1/2	1	375
Z	1	0	-2	0	900

Key column

In this table only the coefficient of X_1 is positive, so this is the entering variable

To find the leaving variable we calculate $\frac{B}{X_1}$ AS FOLLOWS:

Basic variables	X_1	X_2	Y_1	Y_2	B	$\frac{B}{X_1}$
X_2	1/2	1	1/2	0	225	225×2/1=450
Y_2	3/2	0	-1/2	1	375	375×2/3= 250
Z	1	0	-2	0	900	

PIVOT ELEMENT= 3/2

X_1 enters into the basis and Y_2 leaves the basis. The iterative table is as follows:

Basic variables	X_1	X_2	Y_1	Y_2	B
X_2	0	1	2/3	-1/3	100
X_1	1	0	-3	2/3	250
Z	0	0	-5/3	-2/3	1150

Since all the coefficients of Z $cj \leq 0$, the current basic feasible solution is optimal.

The optimal solution is : $MAX\ Z = 1150$ $X_1 = 250$ $X_2 = 100$.