Chapter five: Duality Theory:

5-1- Introduction:

Duality in linear programming is a powerful concept where every linear programming problem (the primal) has a corresponding linear programming problem (the dual). Introducing the canonical form of the primal makes the relationship and the conversion to the dual more systematic and easier to understand.

5-2- Canonical Forms:

There are two main types of canonical forms for a linear programming problem:

1. Maximization Canonical Form:

Maximize: $c_i X_i$

Subject to: $A_i X_i \leq b_i$

$X_i \geq 0$

2. Minimization Canonical Form:

Minimize $c_i X_i$

Subject to: $A_i X_i \geq b_i$

$x \geq 0$

Where:

- x is the vector of decision variables.
- c is the vector of objective function coefficients.
- A is the matrix of constraint coefficients.
- b is the vector of the right-hand side values of the constraints.

5-3- Duality with Canonical Form:

When the primal problem is expressed in one of these canonical forms, the rules for constructing its dual are straightforward and symmetric:

Case 1: Primal is in Maximization Canonical Form:

Primal:	**Dual**
Maximize: $c_i X_i$	Minimize: $b_j Y_j$
Subject to: $A_i X_i \leq b$	Subject to: $ATy \geq c$
$X_I \geq 0$	$y_J \geq 0$

- For each primal constraint (≤), there is a corresponding non-negative dual variable (yi ≥0).
- For each primal variable (xj ≥0), there is a corresponding dual constraint (≥).
- The objective function coefficients of the primal (c) become the right-hand side of the dual constraints.

- The right-hand side values of the primal constraints (b) become the objective function coefficients of the dual.
- The constraint matrix A is transposed (AT).
- The direction of optimization is reversed (maximization becomes minimization).

<u>Case 2: Primal is in Minimization Canonical Form</u>:

Primal:	**Dual**
Minimize: cTx	Maximize: bTy
Subject to: Ax≥b	Subject to: ATy≤c
x≥0	y≥0

- For each primal constraint (≥), there is a corresponding non-negative dual variable (yi ≥0).
- For each primal variable (xj ≥0), there is a corresponding dual constraint (≤).
- The objective function coefficients of the primal (c) become the right-hand side of the dual constraints.
- The right-hand side values of the primal constraints (b) become the objective function coefficients of the dual.

- The constraint matrix A is transposed (AT).
- The direction of optimization is reversed (minimization becomes maximization).

Why Canonical Form is Helpful for Duality:

1. **Symmetry:** The relationship between the primal and dual is clear and symmetric when both are in canonical form. The dual of the dual is the primal.
2. **Consistent Rules**: Using the canonical form provides a consistent set of rules for converting the primal to the dual, making the process less prone to errors.

In summary, while you can derive the dual from a primal problem in any form, converting the primal to its canonical form (either maximization with ≤ constraints or minimization with ≥ constraints, and all variables non-negative) simplifies the process of formulating the dual by establishing a clear and systematic relationship between the two problems. This makes it easier to apply the duality transformation rules and understand the structure of the resulting dual problem.

Let's illustrate duality in linear programming with an example, starting with the primal problem in canonical form and then deriving its dual.

5-4- Example 01; Primal Problem (Maximization Canonical Form):

Maximize: $Z=6x_1+8x_2$

Subject to:

$$\begin{cases} 5x_1+2x_2\leq 20 & \text{(Constraint 1)} \\ x_1+3x_2\leq 15 & \text{(Constraint 2)} \end{cases}$$

$x_1\geq 0,\quad x_2\geq 0$

Deriving the Dual Problem:

Following the rules for converting a maximization canonical primal to a minimization canonical dual:

1. Dual Variables: Introduce a dual variable for each primal constraint. Let $y_1\geq 0$ correspond to the first constraint and $y_2\geq 0$ correspond to the second constraint. So, $y=[y_1\ y_2]$.
2. Dual Objective Function: The dual objective function will be to minimize b^Ty:

Minimize: $W=20y_1+15y_2$

3. Dual Constraints: For each primal variable, create a dual constraint ($\geq$). The coefficients of the dual constraints are the columns of A^T, and the right-hand side values are the coefficients of the primal objective function (c).

$A^T=\begin{bmatrix}5 & 2 \\ 1 & 3\end{bmatrix}$

The dual constraints are:

$5y1 \quad +1y2 \quad \geq 6$ (Corresponding to x1)

$2y1 \quad +3y2 \quad \geq 8$ (Corresponding to x2)

4. Non-negativity of Dual Variables: Since the primal constraints were ≤, the dual variables are non-negative:

$y1 \quad \geq 0$

$y2 \quad \geq 0$

Dual Problem (Minimization Canonical Form):

$$\text{Minimize } W=20y1+15y2$$

Subject to:

$$\begin{cases} 5y1+y2\geq 6 \\ 2y1+3y2\geq 8 \end{cases}$$

$$y1\geq 0 \text{ , } y2\geq 0$$

Interpretation:

- The primal problem seeks to maximize profit (Z) given limited resources (represented by the constraints on x1 and x2).

- The dual problem seeks to minimize the total value of the resources used (W), where y1 and y2 can be interpreted as the shadow prices or the marginal value of one unit of the resource in the first and second primal constraints, respectively. The dual constraints ensure that the imputed value of the resources used to produce one unit of each primal variable is at least as much as the profit gained from that unit.

Key Duality Relationships Illustrated:

- Number of Variables and Constraints: The primal has 2 variables and 2 constraints, while the dual has 2 variables and 2 constraints.
- Objective Function: Maximization in the primal becomes minimization in the dual.
- Coefficients: The coefficients of the primal objective function (6, 8) become the right-hand side of the dual constraints. The right-hand side values of the primal constraints (20, 15) become the coefficients of the dual objective function.
- Constraint Matrix: The constraint matrix A in the primal is transposed (AT) in the dual.
- Inequality Direction: For a maximization primal with ≤ constraints and ≥0 variables, the dual is a minimization with ≥ constraints and ≥0 variables.

5-2- Example 2: Maximization Primal (non- canonical form)

$$\text{Maximize: } Z=2x1+3x2+5x3$$

subject to:

$$\begin{cases} x1+2x2+3x3\leq 10 & \text{(Constraint 1)} \\ 2x1-x2+x3\geq 8 & \text{(Constraint 2)} \\ x1+x2-x3\leq 5 & \text{(Constraint 3)} \\ -x1+3x2+2x3\geq 12 & \text{(Constraint 4)} \end{cases}$$

$$x1\geq 0,\ x2\geq 0,\ x3\geq 0$$

To convert this problem which is not in its canonical form we have to convert it in its canonical form then we convert it to dual form as follows;

Canonical form :

$$\text{Maximize: } Z=2x1 \quad +3x2 \quad +5x3$$

Subject to;

$$\begin{cases} x1+2x2+3x3 \leq 10 & \text{(Constraint 1)} \\ -2x1+x2-x3 \leq -8 & \text{(Constraint 2)} \\ x1+x2-x3 \leq 5 & \text{(Constraint 3)} \\ x1-3x2-2x3 \leq -12 & \text{(Constraint 4)} \end{cases}$$

$$x1\geq 0,\ x2\geq 0,\ x3\geq 0$$

Dual problem :

Minimize: $W=10y_1+8y_2+5y_3+12y_4$

Subject to:

$$\begin{cases} y_1-2y_2+2y_3+y_4 \geq 2 \\ 2y_1+y_2+y -3y_4 \geq 3 \\ 3y_1-1y_2-1y_3-2y_4 \geq 5 \end{cases}$$

$y_1\geq 0,\ y_2\geq 0,\ y_3\geq 0,\ y_4\geq 0$

Remember the key relationships: the number of dual variables equals the number of primal constraints, and the number of dual constraints equals the number of primal variables. The objective function is reversed, the coefficients are swapped and transposed, and the inequality directions are flipped according to the primal's form.