# Chapter Six: Transportation Problems

## 6-1- Introduction :

The origin of transportation models dates back to 1941 when F.L. Hitchcock presented a study entitled "The Distribution of a product from several sources to numerous localities". The presentation is regarded as the first important contribution to solving transportation problems. In 1947, T.C. Koopmans presented a study called "Optimum utilization of the transportation system". These two contributions are mainly responsible for the development of transportation models, which involve a number of shipping sources and a number of destinations

## 6-2- General Mathematical model of transportation problem:

Let there be $m$ sources of supply, $S1$, $S2$,..., $Sm$ having $ai$ ($i$=1,2,...,$m$) units of supply (or capacity), respectively to be transported to $n$ destinations, $D1$, $D2$,..., $Dn$ with $bj$ ($j$=1,2,...,$n$) units of demand (or requirement), respectively.

Let Cij be the cost of shipping one unit of the commodity from source i to destination j.

If $xij$ represents the number of units shipped from source i to destination j, the problem is to determine the transportation schedules so as to minimize the total transportation cost while satisfying the supply and demand conditions.

The general mathematical linear programming model for a transportation problem can be formulated as follows:

Minimize the total transportation cost:

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} X_{ij}$

Subjec To: $\sum_{j=1}^{n} X_{ij} \leq a_i$ **Supply Constraints**:

$\sum_{i=1}^{m} X_{ij} \leq b_j$ **Demand Constraints**

$x_{ij} \geq 0 \quad \forall i,j$

## 6-3- The Transportation algorithm:

The algorithm for solving a transportation problem may be summarized in to the following steps:

**Step 1: Formulate the problem and arrange the data in the matrix form**. The formulation of the transportation problem is similar to the LP problem formulation. In the transportation problem, the objective function is the total transportation cost, and the constraints are the amount of supply and demand available at each source and destination, respectively.

## Step2: Obtain an initial basic feasible solution.

In this chapter, the following two different methods are discussed to obtain an initial solution: North-West Corner Method, and Least Cost Method,

The initial solution obtained by any of the two methods must satisfy the following conditions:

(i) The solution must be feasible, i.e. it must satisfy all the supply and demand constraints (also called the rim conditions).

(ii) The number of positive allocations must be equal to m+n-1, where m is the number of rows and n is the number of columns.

Any solution that satisfies the above conditions is called a non-degenerate basic feasible solution; otherwise, degenerate solution.

Methods for finding an initial basic feasible solution:

### 1- North-West Corner Method:

**Step 1:**

Start with the cell at the upper left (north-west) corner of the transportation table (or matrix) and allocate a commodity equal to the minimum of the rim values for the first row and first column, i.e., min (a1, b1).

**Step2**:

(a) If the allocation made in Step 1 is equal to the supply available at first source (a1, in first row), then move vertically down to the cell (2,1),i.e., second row and first column. Apply Step 1 again for the next allocation.

(b) If the allocation made in Step 1 is equal to the demand of the first destination (b1 in the first column), then move horizontally to the cell (1,2),i.e., first row and second column. Apply Step 1 again for the next allocation.

(c)If $a1=b1$, allocate $x11=a1$ or $b1$and move diagonally to the cell (2,2).

**Step3:** Continue the procedure step by step till an allocation is made in the south- east corner cell of the transportation table.

Once the procedure is over, count the number of positive allocations. These allocations (occupied cells) should be equal (m+n)-1. If yes, then solution is non- degenerate feasible solution, Otherwise degenerate solution.

2- **Least Cost Method (LCM)**

The main objective is to minimize the total transportation cost, transport as much as possible through those routes (cells) where the unit transportation cost is lowest. This method takes into account the minimum unit cost of transportation for obtaining the initial solution and can be summarized as follows:

**Step 1:** Select the cell with the lowest unit cost in the entire transportation table and allocate as much as possible to this cell. Then eliminate (line out) that row or column in which either the supply or demand is fulfilled. If a row and a column are both satisfied simultaneously, then crossed off either a row or a column.

In case the smallest unit cost cell is not unique, then select the cell where the maximum allocation can be made.

**Step2:** After adjusting the supply and demand for all uncrossed rows and columns repeat the procedure to select a cell with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate as much as possible to this cell. Then crossed off that row and column in which either supply or demand is exhausted.

**Step 3:** Repeat the procedure until the available supply at various sources and demand at various destinations is satisfied. The solutions obtained need not be non-degenerate.

**Step3: Test the initial solution for optimality.**

In this chapter, the stepping stone method is discussed to test the optimality of the solution obtained in Step 2. The Stepping Stone

Method starts with an initial basic feasible solution (obtained using methods like the Northwest Corner Rule or Least Cost Method,) and systematically evaluates unoccupied cells to see if a better allocation can reduce the total transportation cost.

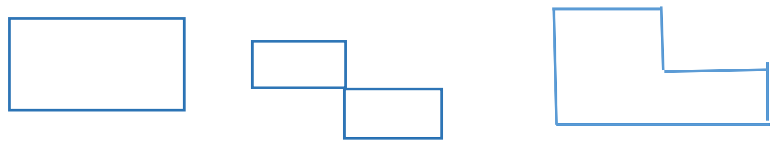
Here's a breakdown of the steps involved:

1. Evaluate Each Unoccupied Cell:

- For every unoccupied cell in the transportation table, you need to determine the change in the total transportation cost if one unit is allocated to that cell. This is done by following these substeps:
  - Trace a Closed Path (Loop): Starting from the unoccupied cell, trace a closed path using only *occupied* cells. The path should consist of alternating horizontal and vertical movements, and you must return to the starting unoccupied cell. You can "step" on occupied cells (the "stepping stones") to change direction.
  - Assign Signs: Assign a plus (+) sign to the unoccupied cell you are evaluating. Then, move along the closed path, assigning alternating minus (-) and plus (+) signs to the occupied cells at each corner of the path.

- Calculate the Net Change in Cost: Sum the transportation costs of the cells in the closed path, considering the signs you assigned. For example, if the costs in the loop are c1,c2,c3,c4 with signs +,−,+,− respectively, the net change in cost (Δc) for the unoccupied cell is: Δc=+c1−c2+c3−c4

There are three types of loops:



2. Identify the Cell with the Most Negative Net Change:

- After evaluating all unoccupied cells, identify the one with the most negative net change in cost. A negative value indicates that allocating one unit to this cell will decrease the total transportation cost.

3. Reallocate Units:

- If there is an unoccupied cell with a negative net change, you can improve the current solution by allocating as many units as possible to this cell.
- Look at the occupied cells in the closed path that have a minus (-) sign. Determine the smallest quantity allocated to these cells.

- Allocate this smallest quantity to the unoccupied cell you are evaluating (the one with the most negative net change).
- Adjust the allocations in the other cells of the closed path:
    - Subtract this smallest quantity from the occupied cells with a minus (-) sign.
    - Add this smallest quantity to the occupied cells with a plus (+) sign.
- This reallocation maintains the supply and demand constraints and results in a new basic feasible solution with a lower total transportation cost.

4. Repeat Steps 2-4:

- Continue evaluating all unoccupied cells in the new transportation table.
- If there are still unoccupied cells with negative net changes, repeat the reallocation process.

5. Check for Optimality:

- The process stops when all unoccupied cells have a net change in cost that is greater than or equal to zero ($\Delta c \geq 0$). At this point, no further reduction in transportation cost is possible, and the current allocation represents the optimal solution.

Step 4: Dating the solution.

If the current solution is optimal, then stop. Otherwise, determine a new improved solution. Repeat Step 3 until an optimal solution is reached.

## 6-4- Numerical example :

- **Balanced Transportation Problem Exercise: The Brick Delivery**

A construction company has three brick factories (Factory A, Factory B, and Factory C) with the following weekly production capacities:

- Factory A: 80 units
- Factory B: 100 units
- Factory C: 220 units

They need to supply bricks to three major construction sites (Site 1, Site 2, and Site 3) with the following weekly demand:

- Site 1: 150 units
- Site 2: 70 units
- Site 3: 180 units

The cost of transporting one unit from each factory to each site is given in the following table:

| From/To | Site 1 (€/unit) | Site 2 (€/unit) | Site 3 (€/unit) |
|---|---|---|---|
| Factory A | 4 | 2 | 3 |
| Factory B | 3 | 5 | 4 |
| Factory C | 6 | 1 | 2 |

Now, check if the problem is balanced:

- Total Supply: 80 + 100 + 220 = 400 units
- Total Demand: 150 + 70 + 180 = 400 units

The total supply = the total demand, so this is a balanced transportation problem.

**Solution:**

STEP 1: Write the linear programming model

Let $X_{ij}$ : The number of units shipped from the factory i to the site j

Min CT= 4X11 + 2 X12 +3 X13 +3 X21 +5 X22 +4 X23 +6X31 +1X32+ 2 X33

$$\text{ST:} \begin{cases} X11 + X12 + X13 = 80 \\ X21 + X22 + X23 = 100 \\ X31 + X32 + X33 = 220 \end{cases}$$

(Supply constraints)

$$\begin{cases} X11 + X21 + X31 = 150 \\ X12 + X22 + X32 = 70 \\ X13 + X23 + X33 = 180 \end{cases}$$

(Demand constraints)

$$X11 , X12 , X13 , X21 , X22, X23 , X31 , X32 , X33 \geq 0$$

Formulate the transportation table.

We can represent the problem in a transportation table:

| From/To | Site 1 | | Site 2 | | Site 3 | | Ɛ Supply |
|---|---|---|---|---|---|---|---|
| Factory A (Supply:) | 4 | $X_{11}$ | 2 | $X_{12}$ | 3 | $X_{13}$ | 80 |
| Factory B (Supply:) | 3 | $X_{21}$ | 5 | $X_{22}$ | 4 | $X_{23}$ | 100 |
| Factory C (Supply:) | 6 | $X_{31}$ | 1 | $X_{32}$ | 2 | $X_{33}$ | 220 |
| Ɛ Demand | 150 | | 70 | | 180 | | 400=400 |

**Step 2: Find an initial basic feasible solution using:**

1- The North-West Corner Rule.

The North-West Corner Rule starts by allocating as much as possible to the cell in the top-left corner of the transportation table and then proceeds systematically.

The initial basic feasible solution using the North-West Corner Rule is:

| From/To | Site 1 (Demand:) | Site 2 (Demand:) | Site 3 (Demand:) | Ɛ Supply |
|---|---|---|---|---|
| Factory A (Supply:) | 4 80 | 2 ---- | 3 ---- | 80 |
| Factory B (Supply:) | 3 70 | 5 30 | 4 ---- | 100 |
| Factory C (Supply:) | 6 ---- | 1 40 | 2 180 | 220 |
| Ɛ Demand | 150 | 70 | 180 | 400=400 |

All supply and demand constraints are satisfied. The number of allocated cells is (3+3-1 = 5), which satisfies the condition for a basic feasible solution in a transportation problem with (m) sources and (n) destinations. Therefore, this initial solution is non-degenerate. Thus, an optimal solution can be obtained.

The total cost according to this method is:

CT= (4× 80) + (3× 70)+(5 ×30) +(1× 40)+ (2× 180)= 1080 €

## 2- The Least Cost Method (LCM):

| From/To | Site 1 (Demand:) | | Site 2 (Demand:) | | Site 3 (Demand:) | | Ɛ Supply |
|---|---|---|---|---|---|---|---|
| Factory A (Supply:) | 4 | 50 | 2 | ---- | 3 | 30 | 80 |
| Factory B (Supply:) | 3 | 100 | 5 | ---- | 4 | ---- | 100 |
| Factory C (Supply:) | 6 | ---- | 1 | 70 | 2 | 150 | 220 |
| Ɛ Demand | 150 | | 70 | | 180 | | 400=400 |

ll supply and demand constraints are satisfied. The number of allocated ls is (3 + 3 - 1 = 5), which satisfies the condition for a basic feasible solution in a transportation problem with (m) sources and (n) destinations. Therefore, this initial solution is non-degenerate. Thus, an optimal solution can be obtained.

The total cost according to this method is:

**CT= (4× 50) + (3× 30) + (3 ×100) +(1× 70) + (2× 150) = 960 €**

**NOTE**: The North-West corner method ignores the cost information. However, the Least Cost method incorporates the cost factor directly into the allocation process.

**Step 3: Test the initial solution for optimality using the stepping stone method.**

| Unoccupied cells | The Net Change in Cost | Closed Path |
|---|---|---|
| $S_1D_2$ | $\Delta C= +2-3+2-1=0$ | |
| $S_2D_2$ | $\Delta C=+5-1+2-3+4-3=+4$ | |
| $S_2D_3$ | $\Delta C=+4-3+4-3= +2$ | |
| $S_3D_1$ | $\Delta C=+6-2+3-4=+3$ | |

all unoccupied cells have a net change in cost that is greater than or equal to zero ($\Delta c \geq 0$). At this point, no further reduction in transportation cost is possible, and the current allocation represents the optimal solution.

## Conclusion: The optimal solution is:

$X_{11}$= 50 units shipped from factory A to site 1, $X_{21}$=100 units shipped from factory B to site1, $X_{13}$= 30 units shipped from factory A to site 3, $X_{32}$=70 units shipped from factory C to site 2, $X_{33}$= 150 units shipped from factory C to sit3.

Min total cost = 960 €