People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

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Department of Accounting and Finance

Fundamentals of Operations Research

A Handout For L2 Students

International Finance & Applied English

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OUTLINE:FUNDAMENTALS OF OPERATIONS RESEARCH

Introduction:

The Fundamentals of Operations Research Module designed for second-year undergraduate students (International Finance & Applied English) represents a valuable opportunity for them to learn mathematical techniques that researchers use to make important decisions in organizations. This involves identifying the best use of available resources, whether they are financial or human. Moreover, operations research techniques are applied in various fields such as industry, commerce, and services.

Through these courses, we have simplified the module, supported by a detailed example for each chapter.

Chapter one: Introduction to Operations Research

1-1-Definition:

Operations Research (OR) is a multidisciplinary field that applies mathematical, analytical, and computational methods to solve complex decision-making problems. It provides a systematic and scientific approach to optimize processes, allocate resources, and improve organizational efficiency.

1-2 Origin and Background of Operations Research

Operations Research (OR) originated as a scientific discipline during the early 20th century. The term "Operational Research" was coined in 1940 by A.P. Rowe, a British Air Ministry scientist. It referred to the application of scientific methods to integrate and optimize new radar technologies for military use.

Inspired by British successes, the United States adopted OR during World War II. The first organized OR group in the U.S. was formed in 1942 at the Naval Ordnance Laboratory, focusing on mine warfare and submarine tactics.

After World War II, OR expanded into civilian industries, including manufacturing, transportation, and finance. It became a key tool for optimizing business processes and resource management.

Universities and professional societies began formalizing OR as an academic discipline, further advancing its methodologies.

1-3- Core Concepts of Operations Research

1. Mathematical Modeling:
   - OR relies heavily on mathematical models to represent problems. These models often include these key components:
     - Objective Function: Defines the goal, such as maximizing profit or minimizing cost.
     - Decision Variables: Represent the choices or actions available.
     - Constraints: Define the limitations or requirements of the system.
     - Feasible Region: The set of all possible solutions that satisfy the constraints.
     - Non-negativity of the decision variables.
2. Optimization:
   - Optimization is the backbone of OR, aiming to find the best possible solution from a set of feasible alternatives.
   - Techniques:

- Linear Programming (LP): Solves problems with linear relationships.
- Integer Programming (IP): Deals with discrete decision variables.
- Non-linear Programming (NLP): Handles non-linear relationships.
- Dynamic Programming (DP): Breaks problems into simpler subproblems with overlapping solutions.

3. Deterministic vs. Stochastic Models:
   - Deterministic Models: Assume certainty in all inputs and outcomes.
   - Stochastic Models: Incorporate randomness and uncertainty, often using probability distributions.
4. Simulation:
   - Simulation models replicate real-world systems to test different scenarios and predict outcomes without implementing changes in reality.
5. Queuing Theory:

   - Focuses on analyzing and optimizing waiting lines or queues, commonly applied in service industries and transportation systems.

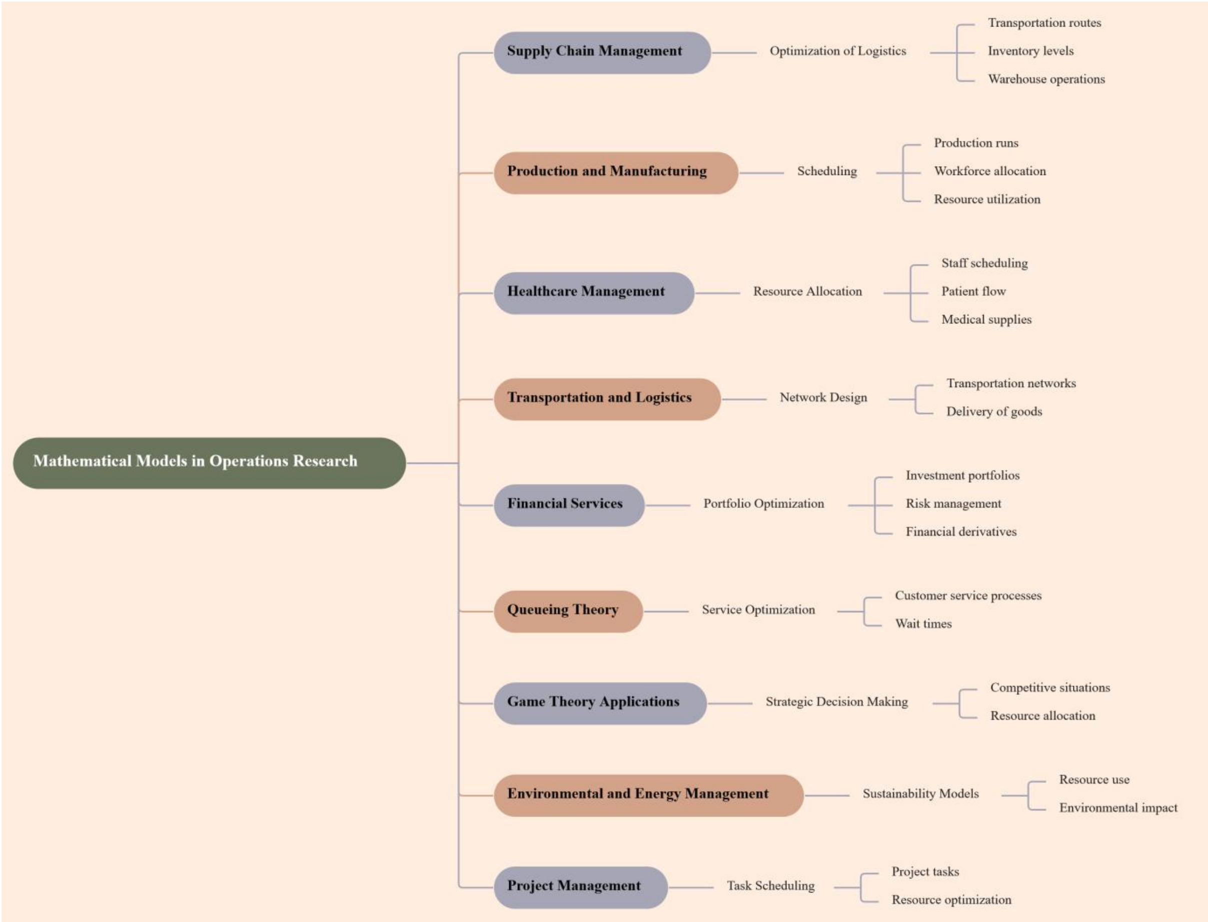
6. Network Analysis:
   - Models systems as networks of interconnected entities to optimize flows, such as transportation routes or supply chains.

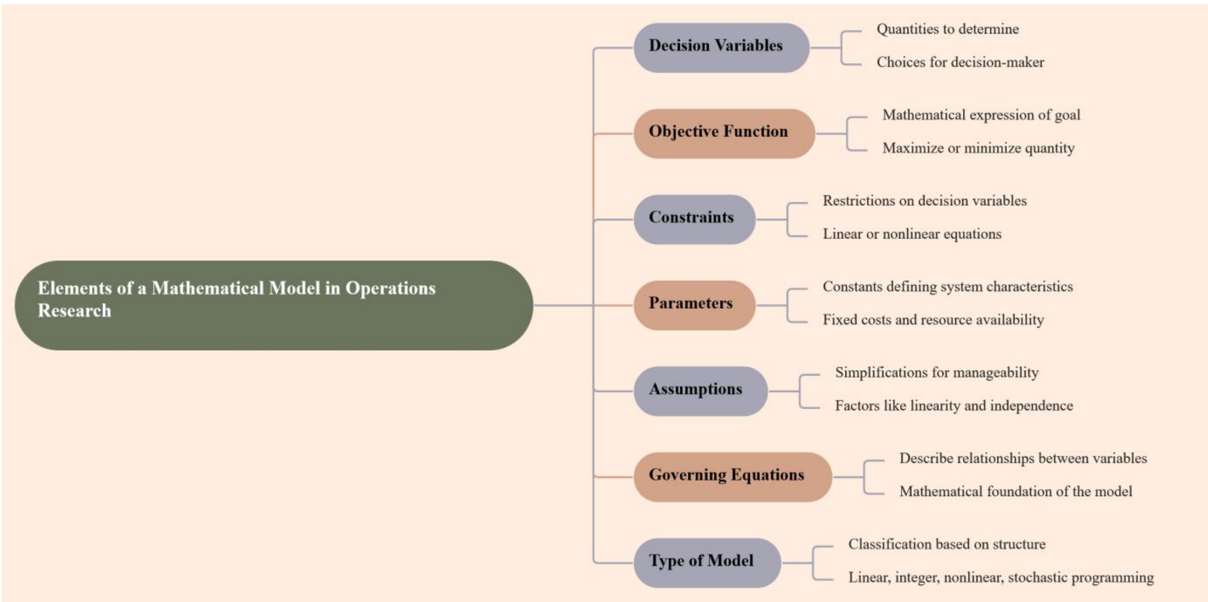
7. Inventory Control:
   - Ensures optimal inventory levels to minimize costs while meeting demand.

1-4- Applications of Operations Research

- Supply Chain Management: Optimizing logistics, inventory, and transportation.
- Scheduling: Allocating resources like employees or machinery efficiently.
- Risk Management: Identifying and mitigating uncertainties.
- Urban Planning: Designing efficient layouts for cities and infrastructure.
- Healthcare: Resource allocation and patient flow optimization.

- Finance: Portfolio Optimization OR techniques are used to balance risk and return in investment portfolios.





<u>Chapter two:</u> Mathematical formulation of the Linear Programming Model

2-1- Definition:

Linear programming is a mathematical optimization technique used to find the best possible outcome (such as maximum profit or minimum cost) under a given set of constraints and requirements. It is widely used in operations research, economics, engineering, and management to solve real-world problems involving limited resources.

The term "linear programming" consists of two words, linear and programming. the word linear tells the relation between various types of variables of degree one used in a problem, which can be represented by straight lines, i. e . , the relationships are of the form $y = ax + b$, and the word "programming" means "taking decisions systematically". (Step-by-step procedure to solve these problems)

2-2- <u>Key Components of Linear Programming</u>

- **Decision Variables**:

The variables that represent the choices to be made.

Example: $x1$ is the number of units of product A to produce,

$x2$ is the number of units of product B to produce.

- Objective Function:

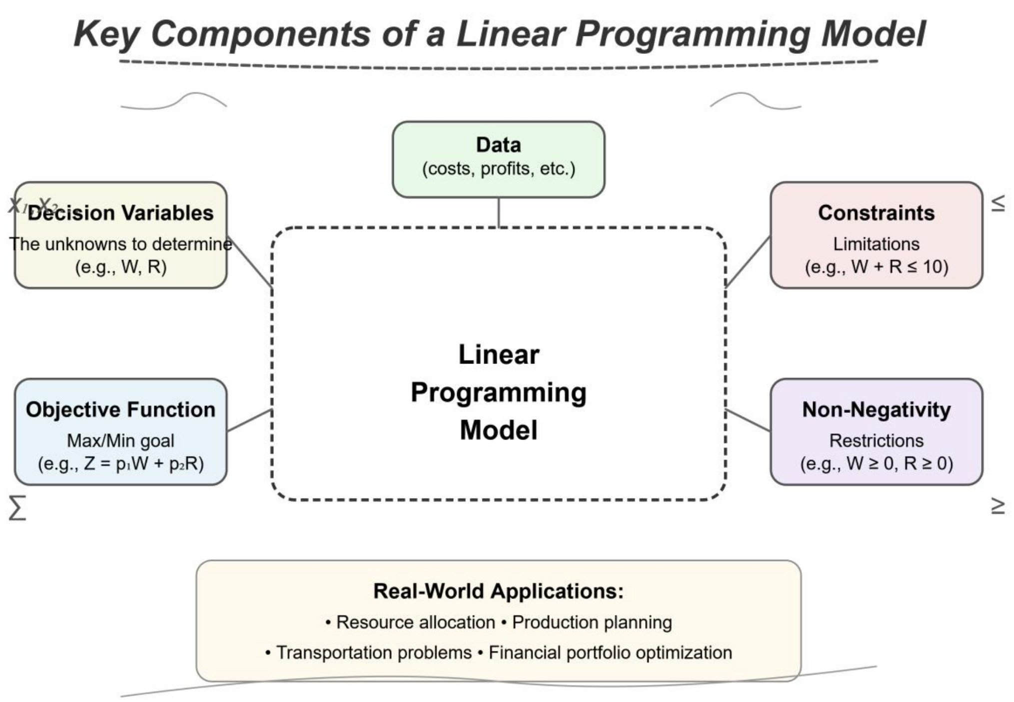
A mathematical representation of the goal, such as maximizing profit or minimizing cost.

- Constraints:

Linear inequalities or equations that represent limitations on resources (e.g., budget, time, labor, materials).

- Non-Negativity Restriction:

Decision variables must be non-negative ($X1,X2 \geq 0$) because negative quantities typically don't make sense in real-world problems.



2-3- Mathematical Formulation of a Linear Programming Problem

A standard LP problem is written as:

Objective Function:

Maximize or Minimize $Z=c_1x_1+c_2x_2+\cdots+c_nx_n$

Subject to Constraints:

$$\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n\leq b_1 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n\leq b_2 \\ \ldots\ldots\ldots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n\leq b_m \end{cases}$$

Non-Negativity: $x_1,x_2,\ldots,x_n\geq 0$

Where:

Z: Objective function to be maximized or minimized.

$x_1,x_2,\ldots,x_n$: Decision variables.

$c_1,c_2,\ldots,c_n$: Coefficients of the objective function.

a_{ij} : Coefficients of the constraints.

$b_1,b_2,\ldots,b_m$: Right-hand side constants in the constraints.

2-4- Steps to Build a Linear Programming (LP) Model

Building a linear programming model involves systematically translating a real-world problem into a mathematical representation that can be solved using optimization techniques.

Below are the detailed steps to create a linear programming model:

Step 1: Identify the Decision Variables

After understanding the Problem, we define the decision variables that represent the quantities to be determined (e.g., number of products to produce, amount of resources to allocate, etc.).

Decision variables are the unknowns you want to determine, representing the quantities to be determined (e.g., number of products to produce, amount of resources to allocate, etc.).

Assign meaningful symbols (e.g., x_1, x_2, x_3 , ……………..)

Example:

X_1: Number of product A to produce.

X_2: Number of product B to produce.

Step 2: Formulate the Objective Function

The objective function is the mathematical expression of the goal you want to achieve, it is a linear combination of decision variables, weighted by their contribution to the goal;

If maximizing (e.g., profit, output): Example Max

$Z = c_1 x_1 + c_2 x_2$

(where : c_1, c_2 are coefficients like profit per unit of x_1 and x_2).

If minimizing (e.g., cost, time): The objective function is also a linear combination, but it aims to minimize the total cost or time.

Example: Min $Z=c1x1+c2x2$

Step 3: Write down all the constraints of the linear problems.

Constraints represent the limitations or restrictions in the problem. These are usually inequalities or equations

Resource constraints: Based on available resources (labor, material, budget, etc.).

Example: If producing product A and B requires raw materials,

$$a1x1+a2x2 \leq b$$

b is the resource limit, and a1,a2 are resource requirements per unit of x1,x2)

Demand constraints: If there is a minimum or maximum demand for a product.

Example: $x1 \geq 50$ (at least 50 units must be produced).

Technical constraints: Based on relationships between variables (e.g., production ratios, machine capacity, etc.).

Example: $x1 \leq 2x2$ (product A must not exceed twice the production of product B).

Types of Constraints:

- Less Than or Equal To: Representing limited resources or capacities, like production time or materials.
- Greater Than or Equal To: Enforcing minimum requirements, such as production quotas or quality standards.
- Equality Constraints: Representing fixed relationships, like balancing supply and demand or using a specific blend of ingredients.

Step 4: Ensure non-negative restrictions of the decision variables.

Decision variables must be non-negative since negative quantities don't make sense. Example: $x1, x2 \geq 0$

Step 5: Write the Complete Linear Programming Model :

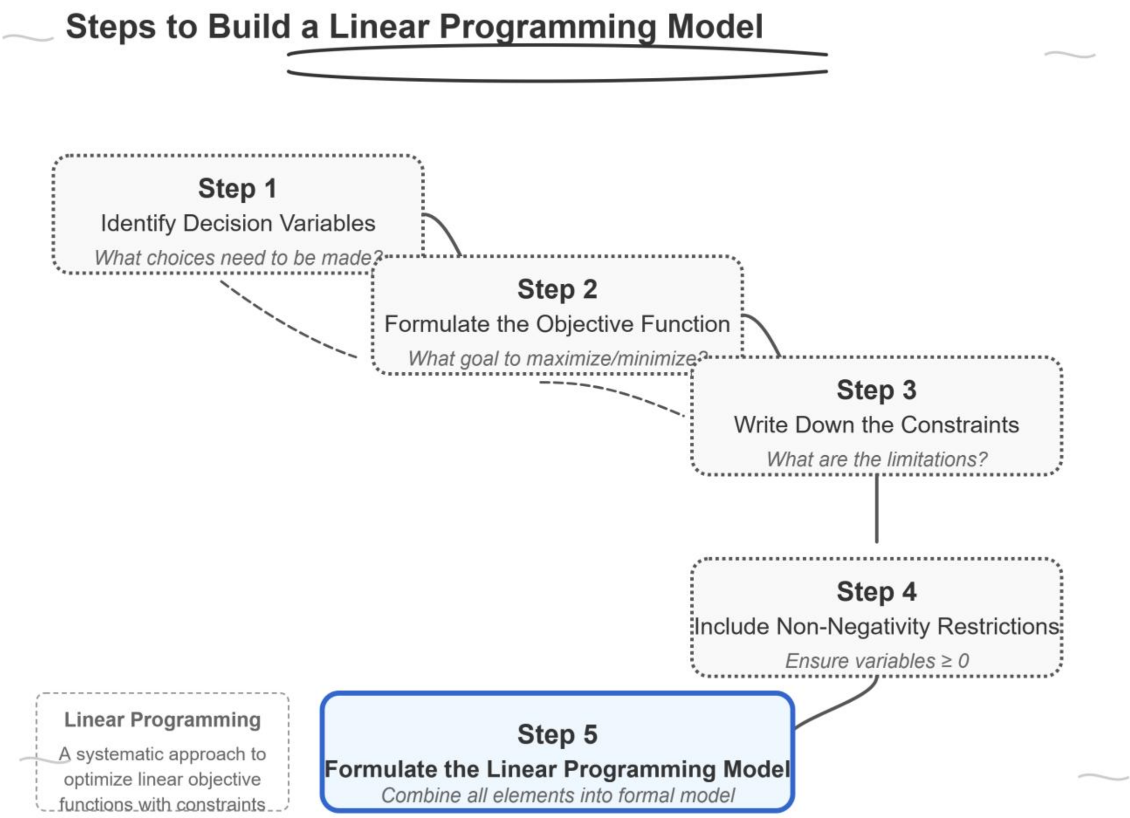
Combine all the components into a formal model. This typically includes the objective function, constraints, and non-negativity restrictions. For example:

Maximize $Z = 5x1 + 3x2$

Subject to:

$$\begin{cases} 2x1 + x2 \leq 100 \\ x1 + 3\,x2 \leq 90 \end{cases}$$

$x1 \geq 0, \; x2 \geq 0$



Example: (Formulating linear programming model)

A manufacturer produces two types of models *M* and *N*. Each *M* model requires 4 hours of grinding and 2 hours of polishing, whereas each *N* model requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on model *M* is 3€/Unit and model *N* is 4€/Unit. Whatever is produced in a week is sold in the market.

- ❖ Write the Linear Programming Model?

Solution:

--------------------	Product M	Product N	Available hours per week
Grinding	4 hours	2 hours	2 × 40 = 80 hours/ week
Polishing	2 hours	5 hours	3 × 60 = 180 hours/week
Profit	3 €/ unit	4 €/ unit	-----------------------------

Steps for building the Linear Programming Model:

Step N°1: Identify decision variables

Let: X_1 is the number of units of the product M produced per week;

X_2 is the number of units of the product N produced per week.

Step N°2: Formulate the objective function (the total profit)

The total profit required per week = profit required by the product M (per week) + profit required by the product N (per week)

$$Z = 3\,X_1 + 4\,X_2$$

Step N°3 Formulate the constraints (grinding & polishing constraints)

Grinding constraint: the number of grinding hours required by the product M + the number of grinding hours required by the product N should be at most 80 hours/week

$$4\,X_1 + 2\,X_2 \leq 80 \text{ hours}$$

Polishing constraint: the number of polishing hours required by the product M + the number of polishing hours required by the product N should be at most 180 hours / week

$$2X_1 + 5\,X_2 \leq 180 \text{ hours}$$

Step N°4: non-negativity condition of the decision variables

$$X_1 \geq 0$$

$$X_2 \geq 0$$

Step N° 5: The linear programming model is:

$$\text{MAX } Z = Z = 3\,X_1 + 4\,X_2$$

$$\text{St: } \begin{cases} 4\,X_1 + 2\,X_2 \leq 80 \\ 2X_1 + 5\,X_2 \leq 180 \end{cases}$$

$$X_1 \geq 0 \quad X_2 \geq 0$$

Chapter three : The Graphical Method for solving Linear Programming Models

3-1- Introduction

The graphical method is a visual approach for solving linear programming (LP) problems with **two decision variables**.

3-2- Steps for Solving Linear Programming Problems Graphically

Here are the steps for solving a linear programming model (LPM) graphically:

1. Formulate the Linear Programming Problem: Define the decision variables, objective function, and constraints clearly.

2. Construct a graph: Set up a coordinate system (typically the x_1-x_2 plane) to represent the decision variables.

3. Plot the Constraint Lines: For each constraint, convert the inequalities into equations and plot the corresponding lines on the graph.

4. Determine the Feasible Region: Identify the area that satisfies all the constraints. This region is where all the inequalities overlap.

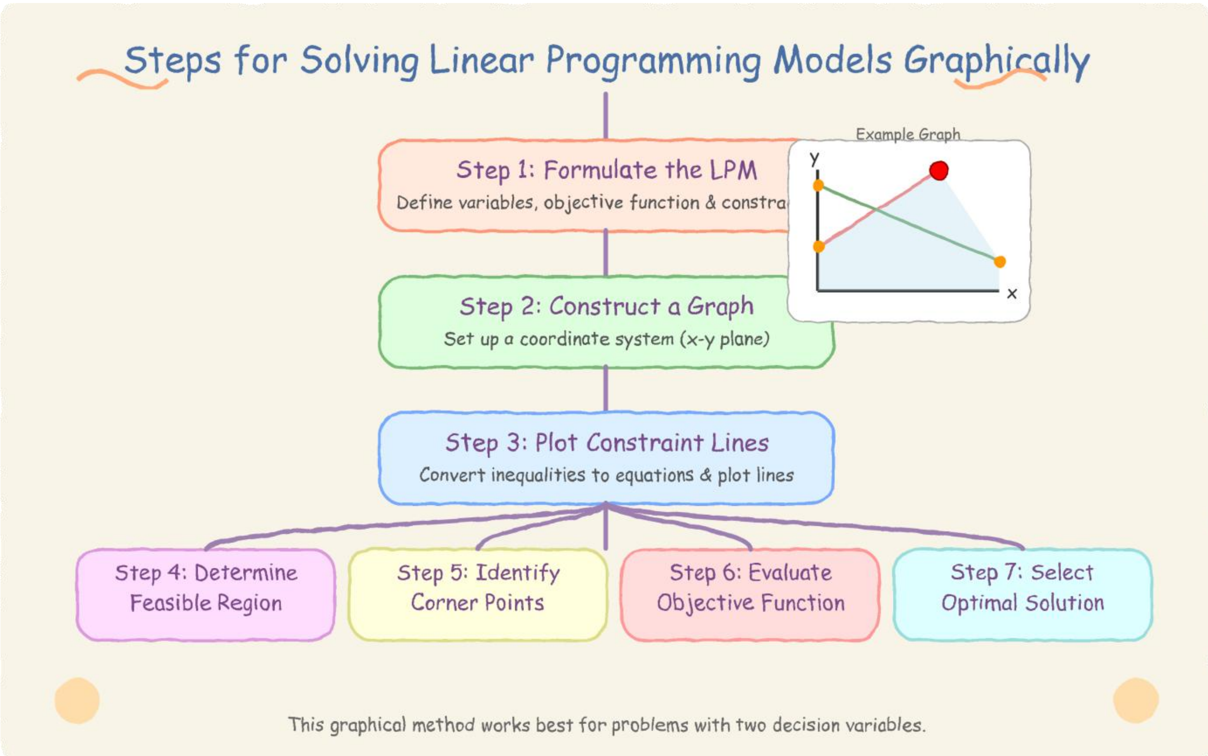
5. Identify the Corner Points: Locate the vertices (corner points) of the feasible region. These points are potential candidates for the optimal solution.

6. Evaluate the Objective Function: Calculate the value of the objective function at each corner point to determine which point yields the maximum or minimum value.

7. Select the Optimal Solution

The corner point that provides the best value for the objective function (maximum or minimum) is the optimal solution.

This graphical method is particularly effective for problems involving two decision variables, allowing for a visual representation of constraints and the feasible region.



Example:(The graphical method for solving LPM)

Solve this linear programming model using graphical method;

Max $Z= 6x_1 + 9x_2$

$$ST: \begin{cases} x_1 + 3x_2 \leq 9 \\ x_1 + x_2 \leq 5 \end{cases}$$

$X_1 \geq 0 , X_2 \geq 0$

Solution:

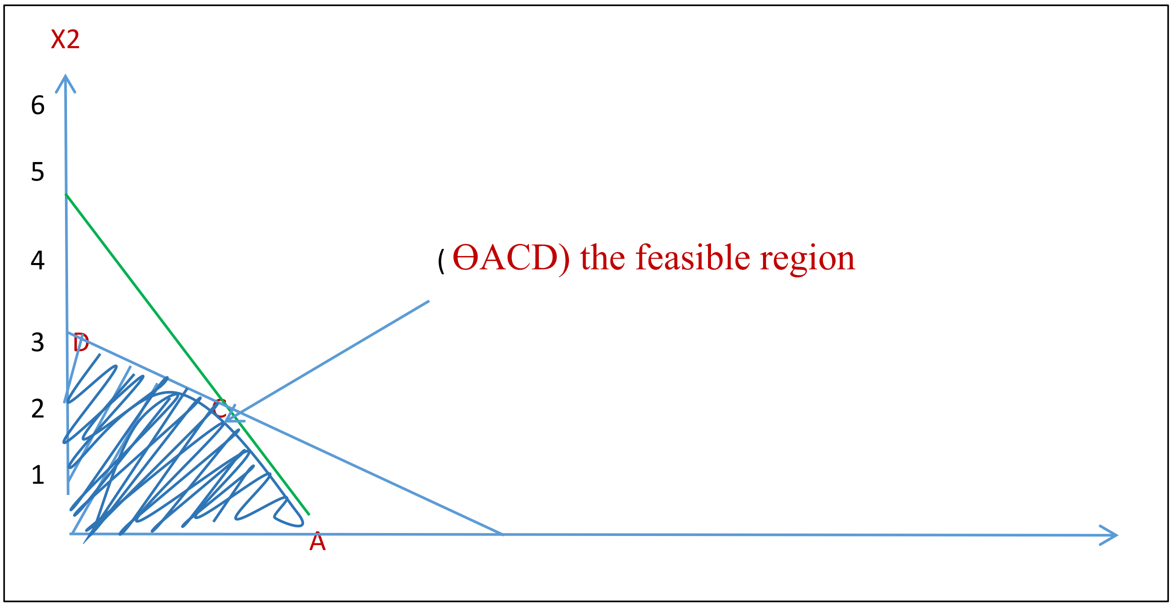
For each constraint, convert the inequalities into equations and plot the corresponding lines on the graph:

(Δ_1)......... $x_1 + 3x_2 = 9$

X1	0	9
X2	3	0

(Δ_2) ....$x_1 + x_2 = 5$

X1	0	5
X2	5	0



The region shaded limited by the polygon (ΘACD) is the area that satisfies all the constraints. This region is where all the inequalities overlap.

The vertices (corner points) (ΘACD) of the feasible region are potential candidates for the optimal solution.

Evaluate the Objective Function: Calculate the value of the objective function at each corner point to determine which point yields the maximum or minimum value:

The vertices	The coordinates		The objective function
	X1	X2	$Z= 6x_1 + 9x_2$
O	0	0	$Z_O=6(0)+9(0)=0$
A	5	0	$Z_A=6(5)+9(0)= 30$
C	**3**	**2**	$Z_C=6(3)+9(2)=36$
D	0	3	$Z_D=6(0)+9(3)=27$

The optimal solution is: $X_1=3$, $X_2=2$, Max $Z= 36$

Chapter four: Theory of the Simplex Method

4-1- Introduction:

Simplex method also called simplex technique or simplex algorithm was developed in 1947 by G.Dantzig an American mathematician. It has the advantage of being universal.

In principle, it consists of starting with a certain solution of which all that we know is that it is basic feasible i.e, it satisfies the constraints as well as non-negativity conditions ($x_j \geq 0$, j=1, 2, 3, ………). Then, we improve upon this solution at consecutive stages, we arrive at the optimal solution. The method also helps the decision maker to identify the redundant constraints, an unbounded solution, multiple solutions and an infeasible solution.

The simplex method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another in such a manner that the value of the objective function t at the succeeding vertex is less in a minimization problem (or more in a max problem) than at the proceeding vertex. this procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, this method leads to an optimal vertex in a finite number of steps.

4-2- Steps for solving LPM using the SIMPLEX method

The simplex method consists of:

(i) having a trial basic feasible solution to the constraint equations.

(ii) testing whether it is an optimal solution or not.

(iii) improving, if required, the first trial solution by a set of rules and repeating the process till an optimal solution is obtained.

The simplex technique will be explained by considering this example:

Example:

Max $Z = 3X_1 + 4X_2$

$$\text{ST:} \quad \begin{cases} X_1 + 2X_2 \leq 450 \text{ mn (time constraint of machine M1)} \\ 2X_1 + X_2 \leq 600 \text{ mn (time constraint of machine M2)} \end{cases}$$

$X_1, X_2 \geq 0$

SOLUTION

To solve this model by the simplex method, we have to follow these steps:

Step N°1: Express the problem in standard form:

The given problem is said to be expressed in standard form if the decision variables are non-negative, right-hand side of the constraints are non-negative and the constraints are expressed as equations.

Since the first two conditions are not with in the problem, non-negative slack variables Y_1, Y_2, are added to the left hand side of the two constraints respectively to convert them into equations.

Accordingly, the problem in standard form can be written as follows:

Max $Z= 3X_1 +4X_2 + 0Y_1+ 0Y_2$

ST: $\begin{cases} X_1+2X_2+ Y_1 = 450 \\ 2X_1+ X_2 + Y_2= 600 \end{cases}$ $\longrightarrow$ STANDARD FORM

$X_1, X_2, Y_1, Y_2 \geq 0$

Slack variables Y_1, Y_2 represent unutilized capacity or resources

In current problem Y_1 denotes the time (in mn) for which machine M1 remains unutilized, similarly Y_2 denotes the unutilized time for machine M2.

Step N°2: find initial basic feasible solution:

In the simplex method a start is made with a feasible which we shall get by assuming that the profit earned is zero.

The simplex tabular representation of the linear programming problem, organizing the coefficients of the variables, the right- hand side values, and the objective function coefficients.

Note that: Slack variables added to (≤) constraints to convert them into equalities, form columns with a single '1' and the rest 'Os' creating part of the identity matrix (unit matrix where n×n = number of constraints

Note that every simplex table will have identity matrix under the basic variables. The identity matrix is also called unit matrix or basis matrix. It is always a square matrix and its size is equal to the number of constraints.

When setting up the initial simplex tableau, the variables that construct the unit matrix are typically the:

- **Slack variables:**
  - These are added to "≤" constraints to convert them into equalities. In the initial tableau, they form columns with a single "1" and the rest "0s," creating part of the identity matrix.
- **Artificial variables:**
  - These are added to "≥" and "=" constraints when there's no readily available basic feasible solution. Similar to slack variables, they also form columns that contribute to the initial unit matrix.

Here's a breakdown of why and how:

- **Creating an Initial Basic Feasible Solution:**
  - The simplex method requires an initial basic feasible solution to start its iterative process. The unit matrix provides this. Each column of the unit matrix represents a

basic variable, and the corresponding right-hand side values give their initial values.

- **How They Form the Unit Matrix:**
  - When you add a slack variable to a "≤" constraint, its coefficient is "1" in that constraint's row and "0" in all other constraint rows. This creates a column with a "1" and zeros else where.
  - Similarly, artificial variables are introduced to create those necessary "1" and zero columns.
  - So, in the initial tableau, the columns corresponding to these slack and/or artificial variables will have the form of a unit matrix.

In essence, these variables provide the initial "handles" for the simplex method to manipulate as it searches for the optimal solution.

The initial basic tableau for the example is:

Basic variables	X_1	X_2	Y_1	Y_2	B
Y_1	1	2	1	0	450
Y_2	2	1	0	1	600
Z	3	4	0	0	0

In the simplex method a start is made with a feasible solution.

Step N°3: select the entering variable from the row of objective function

The entering variable is the non-basic variable. If there are more than one positive coefficient of *z*, then choose the most positive of them.

The entering variable column is known as the key column or pivot column which is shown marked with an arrow at the side.

Basic variables	X_1	X_2	Y_1	Y_2	B	$\frac{B}{X_2}$
Y_1	1	2	1	0	450	450/1=450
Y_2	2	1	0	1	600	600/1=600
Z	3	4	0	0	0	

Let the entering variable is X_2 where its coefficient = 4 (a positive number in the case of MAX Z)

Step N°4: select the leaving variable

Compute the ratios $\frac{B}{X_2}$ and choose the minimum of them. Let the minimum ratio be Y_1 (=450) Then, the vector Y_1 will leave the base YB.

The element lying at the intersection of key column and key row is called KEY or PIVOT element. The column to be entered is called *key column*

Pivot = 2

Basic variables	X_1	X_2	Y_1	Y_2	B	$\frac{B}{X_2}$
Y_1	1	2	1	0	450	450/1=450
Y_2	2	1	0	1	600	600/1=600
Z	3	4	0	0	0	

X_2 enters into the basis and Y_1 leaves the basis

To find the leaving variable we have to compute the ratio $\frac{B}{X_2}$ and we choose the minimum of them which is 450, SO the leaving variable is Y_1 it means that Y_1 leaves the base to be non-basic element.

The smallest positive ratio of the two equations is 450/2 (row Y1). Row Y1 now becomes the *pivot row* and 1 at the intersection of the pivot row and pivot column, becomes *the pivot.*

The element at the intersection of key row and key column is called *key element*

Step N°5: Drop the leaving variable and introduce the entering variable along with its associated value. h

Convert the pivot element 2 to unity and all other element in its column to zero by the following transformations.

New pivot equation = Old pivot equation / Pivot element

The remaining rows are formed by using the formula:

$$\text{New number} = \text{Old number} - \frac{\begin{pmatrix}corresponding\ nmber\\ in\ pivot\ row\end{pmatrix} \times \begin{pmatrix}corresponding\ number\\ in\ pivot\ column\end{pmatrix}}{the\ pivot}$$

X_2 enters into the basis and Y_1 leaves the basis. The iterative table is as follows:

Basic variables	X_1	X_2	Y_1	Y_2	B
X_2	1/2	1	1	0	225
Y_2	3/2	0	-1/2	1	375
Z	1	0	-2	0	900

Since all the coefficients of Z are not less than or equal to zero, the current basic feasible solution is not optimal.

STEP N°6: If it is not the optimal solution, repeat from step N°3

Basic variables	X_1	X_2	Y_1	Y_2	B
X_2	1/2	1	1/2	0	225
Y_2	3/2	0	-1/2	1	375
Z	1	0	-2	0	900

Key column

In this table only the coefficient of X_1 is positive, so this is the entering variable

To find the leaving variable we calculate $\frac{B}{X_1}$ AS FOLLOWS:

Basic variables	X_1	X_2	Y_1	Y_2	B	$\frac{B}{X_1}$
X_2	1/2	1	1/2	0	225	225×2/1=450
Y_2	$\boxed{3/2}$	0	-1/2	1	375	375×2/3= 250
Z	1	0	-2	0	900	

PIVOT ELEMENT= 3/2

X_1 enters into the basis and Y_2 leaves the basis. The iterative table is as follows:

Basic variables	X_1	X_2	Y_1	Y_2	B
X_2	0	1	2/3	-1/3	100
X_1	1	0	-3	2/3	250
Z	0	0	-5/3	-2/3	1150

Since all the coefficients of Z $cj \leq 0$, the current basic feasible solution is optimal.

The optimal solution is : $MAX\ Z = 1150$ $X_1 = 250$ $X_2 = 100$.

Chapter five: Duality Theory:

5-1- Introduction:

Duality in linear programming is a powerful concept where every linear programming problem (the primal) has a corresponding linear programming problem (the dual). Introducing the canonical form of the primal makes the relationship and the conversion to the dual more systematic and easier to understand.

5-2- Canonical Forms:

There are two main types of canonical forms for a linear programming problem:

1. Maximization Canonical Form:

Maximize: $c_i X_i$

Subject to: $A_i X_i \leq b_i$

$X_i \geq 0$

2. Minimization Canonical Form:

Minimize $c_i X_i$

Subject to: $A_i X_i \geq b_i$

$x \geq 0$

Where:

- x is the vector of decision variables.
- c is the vector of objective function coefficients.
- A is the matrix of constraint coefficients.
- b is the vector of the right-hand side values of the constraints.

5-3- Duality with Canonical Form:

When the primal problem is expressed in one of these canonical forms, the rules for constructing its dual are straightforward and symmetric:

Case 1: Primal is in Maximization Canonical Form:

Primal:	**Dual**
Maximize: $c_i X_i$	Minimize: $b_j Y_j$
Subject to: $A_i X_i \leq b$	Subject to: $ATy \geq c$
$X_I \geq 0$	$y_J \geq 0$

- For each primal constraint (≤), there is a corresponding non-negative dual variable (yi ≥0).
- For each primal variable (xj ≥0), there is a corresponding dual constraint (≥).
- The objective function coefficients of the primal (c) become the right-hand side of the dual constraints.

- The right-hand side values of the primal constraints (b) become the objective function coefficients of the dual.
- The constraint matrix A is transposed (AT).
- The direction of optimization is reversed (maximization becomes minimization).

<u>Case 2: Primal is in Minimization Canonical Form</u>:

Primal:	**Dual**
Minimize: cTx	Maximize: bTy
Subject to: Ax≥b	Subject to: ATy≤c
x≥0	y≥0

- For each primal constraint (≥), there is a corresponding non-negative dual variable (yi ≥0).
- For each primal variable (xj ≥0), there is a corresponding dual constraint (≤).
- The objective function coefficients of the primal (c) become the right-hand side of the dual constraints.
- The right-hand side values of the primal constraints (b) become the objective function coefficients of the dual.

- The constraint matrix A is transposed (AT).
- The direction of optimization is reversed (minimization becomes maximization).

Why Canonical Form is Helpful for Duality:

1. **Symmetry:** The relationship between the primal and dual is clear and symmetric when both are in canonical form. The dual of the dual is the primal.
2. **Consistent Rules**: Using the canonical form provides a consistent set of rules for converting the primal to the dual, making the process less prone to errors.

In summary, while you can derive the dual from a primal problem in any form, converting the primal to its canonical form (either maximization with ≤ constraints or minimization with ≥ constraints, and all variables non-negative) simplifies the process of formulating the dual by establishing a clear and systematic relationship between the two problems. This makes it easier to apply the duality transformation rules and understand the structure of the resulting dual problem.

Let's illustrate duality in linear programming with an example, starting with the primal problem in canonical form and then deriving its dual.

5-4- Example 01; Primal Problem (Maximization Canonical Form):

Maximize: $Z=6x_1+8x_2$

Subject to:

$$\begin{cases} 5x_1+2x_2\le 20 & \text{(Constraint 1)} \\ x_1+3x_2\le 15 & \text{(Constraint 2)} \end{cases}$$

$x_1\ge 0,\quad x_2\ge 0$

Deriving the Dual Problem:

Following the rules for converting a maximization canonical primal to a minimization canonical dual:

1. Dual Variables: Introduce a dual variable for each primal constraint. Let $y_1\ge 0$ correspond to the first constraint and $y_2\ge 0$ correspond to the second constraint. So, $y=[y_1\quad y_2]$.
2. Dual Objective Function: The dual objective function will be to minimize b^Ty:

Minimize: $W=20y_1+15y_2$

3. Dual Constraints: For each primal variable, create a dual constraint ($\ge$). The coefficients of the dual constraints are the columns of A^T, and the right-hand side values are the coefficients of the primal objective function (c).

$A^T=[52\quad 13]$

The dual constraints are:

$5y_1 + 1y_2 \geq 6$ (Corresponding to x_1)

$2y_1 + 3y_2 \geq 8$ (Corresponding to x_2)

4. Non-negativity of Dual Variables: Since the primal constraints were $\leq$, the dual variables are non-negative:

$y_1 \geq 0$

$y_2 \geq 0$

Dual Problem (Minimization Canonical Form):

$$\text{Minimize } W=20y_1+15y_2$$

Subject to:

$$\begin{cases} 5y_1+y_2\geq 6 \\ 2y_1+3y_2\geq 8 \end{cases}$$

$$y_1\geq 0 ,\ y_2\geq 0$$

Interpretation:

- The primal problem seeks to maximize profit (Z) given limited resources (represented by the constraints on x_1 and x_2).

- The dual problem seeks to minimize the total value of the resources used (W), where y1 and y2 can be interpreted as the shadow prices or the marginal value of one unit of the resource in the first and second primal constraints, respectively. The dual constraints ensure that the imputed value of the resources used to produce one unit of each primal variable is at least as much as the profit gained from that unit.

Key Duality Relationships Illustrated:

- Number of Variables and Constraints: The primal has 2 variables and 2 constraints, while the dual has 2 variables and 2 constraints.
- Objective Function: Maximization in the primal becomes minimization in the dual.
- Coefficients: The coefficients of the primal objective function (6, 8) become the right-hand side of the dual constraints. The right-hand side values of the primal constraints (20, 15) become the coefficients of the dual objective function.
- Constraint Matrix: The constraint matrix A in the primal is transposed (AT) in the dual.
- Inequality Direction: For a maximization primal with ≤ constraints and ≥0 variables, the dual is a minimization with ≥ constraints and ≥0 variables.

5-2- Example 2: Maximization Primal (non- canonical form)

Maximize: $Z=2x1+3x2+5x3$

subject to:

$$\begin{cases} x1+2x2+3x3\leq 10 & \text{(Constraint 1)} \\ 2x1-x2+x3\geq 8 & \text{(Constraint 2)} \\ x1+x2-x3\leq 5 & \text{(Constraint 3)} \\ -x1+3x2+2x3\geq 12 & \text{(Constraint 4)} \end{cases}$$

$$x1\geq 0,\ x2\geq 0,\ x3\geq 0$$

To convert this problem which is not in its canonical form we have to convert it in its canonical form then we convert it to dual form as follows;

Canonical form :

Maximize: $Z=2x1 \quad +3x2 \quad +5x3$

Subject to;

$$\begin{cases} x1+2x2+3x3 \leq 10 & \text{(Constraint 1)} \\ -2x1+x2-x3 \leq -8 & \text{(Constraint 2)} \\ x1+x2-x3 \leq 5 & \text{(Constraint 3)} \\ x1-3x2-2x3 \leq -12 & \text{(Constraint 4)} \end{cases}$$

$$x1\geq 0,\ x2\geq 0,\ x3\geq 0$$

Dual problem :

Minimize: $W=10y_1+8y_2+5y_3+12y_4$

Subject to:

$$\begin{cases} y_1-2y_2+2y_3+y_4 \geq 2 \\ 2y_1+y_2+y -3y_4 \geq 3 \\ 3y_1-1y_2-1y_3-2y_4 \geq 5 \end{cases}$$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0$

Remember the key relationships: the number of dual variables equals the number of primal constraints, and the number of dual constraints equals the number of primal variables. The objective function is reversed, the coefficients are swapped and transposed, and the inequality directions are flipped according to the primal's form.

Chapter Six: Transportation Problems

6-1- Introduction :

The origin of transportation models dates back to 1941 when F.L. Hitchcock presented a study entitled "The Distribution of a product from several sources to numerous localities". The presentation is regarded as the first important contribution to solving transportation problems. In 1947, T.C. Koopmans presented a study called "Optimum utilization of the transportation system". These two contributions are mainly responsible for the development of transportation models, which involve a number of shipping sources and a number of destinations

6-2- General Mathematical model of transportation problem:

Let there be m sources of supply, $S1$, $S2$,..., Sm having ai (i=1,2,...,m) units of supply (or capacity), respectively to be transported to n destinations, $D1$, $D2$,..., Dn with bj (j=1,2,...,n) units of demand (or requirement), respectively.

Let Cij be the cost of shipping one unit of the commodity from source i to destination j.

If xij represents the number of units shipped from source i to destination j, the problem is to determine the transportation schedules so as to minimize the total transportation cost while satisfying the supply and demand conditions.

The general mathematical linear programming model for a transportation problem can be formulated as follows:

Minimize the total transportation cost:

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} X_{ij}$

Subjec To: $\sum_{j=1}^{n} X_{ij} \leq a_i$ **Supply Constraints**:

$\sum_{i=1}^{m} X_{ij} \leq b_j$ **Demand Constraints**

$x_{ij} \geq 0 \quad \forall i,j$

6-3- The Transportation algorithm:

The algorithm for solving a transportation problem may be summarized in to the following steps:

Step 1: Formulate the problem and arrange the data in the matrix form. The formulation of the transportation problem is similar to the LP problem formulation. In the transportation problem, the objective function is the total transportation cost, and the constraints are the amount of supply and demand available at each source and destination, respectively.

Step2: Obtain an initial basic feasible solution.

In this chapter, the following two different methods are discussed to obtain an initial solution: North-West Corner Method, and Least Cost Method,

The initial solution obtained by any of the two methods must satisfy the following conditions:

(i) The solution must be feasible, i.e. it must satisfy all the supply and demand constraints (also called the rim conditions).

(ii) The number of positive allocations must be equal to m+n-1, where m is the number of rows and n is the number of columns.

Any solution that satisfies the above conditions is called a non-degenerate basic feasible solution; otherwise, degenerate solution.

Methods for finding an initial basic feasible solution:

1- **North-West Corner Method:**

Step 1:

Start with the cell at the upper left (north-west) corner of the transportation table (or matrix) and allocate a commodity equal to the minimum of the rim values for the first row and first column, i.e., min (a1, b1).

Step2:

(a) If the allocation made in Step 1 is equal to the supply available at first source (a1, in first row), then move vertically down to the cell (2,1),i.e., second row and first column. Apply Step 1 again for the next allocation.

(b) If the allocation made in Step 1 is equal to the demand of the first destination (b1 in the first column), then move horizontally to the cell (1,2),i.e., first row and second column. Apply Step 1 again for the next allocation.

(c)If $a1=b1$, allocate $x11=a1$ or $b1$and move diagonally to the cell (2,2).

Step3: Continue the procedure step by step till an allocation is made in the south- east corner cell of the transportation table.

Once the procedure is over, count the number of positive allocations. These allocations (occupied cells) should be equal (m+n)-1. If yes, then solution is non- degenerate feasible solution, Otherwise degenerate solution.

2- **Least Cost Method (LCM)**

The main objective is to minimize the total transportation cost, transport as much as possible through those routes (cells) where the unit transportation cost is lowest. This method takes into account the minimum unit cost of transportation for obtaining the initial solution and can be summarized as follows:

Step 1: Select the cell with the lowest unit cost in the entire transportation table and allocate as much as possible to this cell. Then eliminate (line out) that row or column in which either the supply or demand is fulfilled. If a row and a column are both satisfied simultaneously, then crossed off either a row or a column.

In case the smallest unit cost cell is not unique, then select the cell where the maximum allocation can be made.

Step2: After adjusting the supply and demand for all uncrossed rows and columns repeat the procedure to select a cell with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate as much as possible to this cell. Then crossed off that row and column in which either supply or demand is exhausted.

Step 3: Repeat the procedure until the available supply at various sources and demand at various destinations is satisfied. The solutions obtained need not be non-degenerate.

Step3: Test the initial solution for optimality.

In this chapter, the stepping stone method is discussed to test the optimality of the solution obtained in Step 2. The Stepping Stone

Method starts with an initial basic feasible solution (obtained using methods like the Northwest Corner Rule or Least Cost Method,) and systematically evaluates unoccupied cells to see if a better allocation can reduce the total transportation cost.

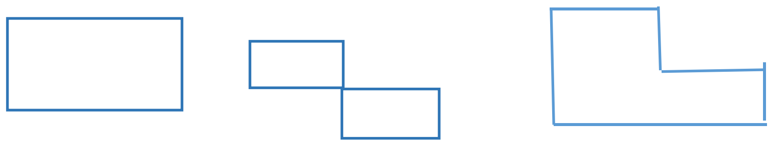
Here's a breakdown of the steps involved:

1. Evaluate Each Unoccupied Cell:

- For every unoccupied cell in the transportation table, you need to determine the change in the total transportation cost if one unit is allocated to that cell. This is done by following these substeps:
    - Trace a Closed Path (Loop): Starting from the unoccupied cell, trace a closed path using only *occupied* cells. The path should consist of alternating horizontal and vertical movements, and you must return to the starting unoccupied cell. You can "step" on occupied cells (the "stepping stones") to change direction.
    - Assign Signs: Assign a plus (+) sign to the unoccupied cell you are evaluating. Then, move along the closed path, assigning alternating minus (-) and plus (+) signs to the occupied cells at each corner of the path.

- Calculate the Net Change in Cost: Sum the transportation costs of the cells in the closed path, considering the signs you assigned. For example, if the costs in the loop are c_1, c_2, c_3, c_4 with signs $+,-,+,-$ respectively, the net change in cost (Δc) for the unoccupied cell is: $\Delta c = +c_1 - c_2 + c_3 - c_4$

There are three types of loops:



2. Identify the Cell with the Most Negative Net Change:

- After evaluating all unoccupied cells, identify the one with the most negative net change in cost. A negative value indicates that allocating one unit to this cell will decrease the total transportation cost.

3. Reallocate Units:

- If there is an unoccupied cell with a negative net change, you can improve the current solution by allocating as many units as possible to this cell.
- Look at the occupied cells in the closed path that have a minus (-) sign. Determine the smallest quantity allocated to these cells.

- Allocate this smallest quantity to the unoccupied cell you are evaluating (the one with the most negative net change).
- Adjust the allocations in the other cells of the closed path:
  - Subtract this smallest quantity from the occupied cells with a minus (-) sign.
  - Add this smallest quantity to the occupied cells with a plus (+) sign.
- This reallocation maintains the supply and demand constraints and results in a new basic feasible solution with a lower total transportation cost.

4. Repeat Steps 2-4:

- Continue evaluating all unoccupied cells in the new transportation table.
- If there are still unoccupied cells with negative net changes, repeat the reallocation process.

5. Check for Optimality:

- The process stops when all unoccupied cells have a net change in cost that is greater than or equal to zero ($\Delta c \geq 0$). At this point, no further reduction in transportation cost is possible, and the current allocation represents the optimal solution.

Step 4: Dating the solution.

If the current solution is optimal, then stop. Otherwise, determine a new improved solution. Repeat Step 3 until an optimal solution is reached.

6-4- Numerical example :

- **Balanced Transportation Problem Exercise: The Brick Delivery**

A construction company has three brick factories (Factory A, Factory B, and Factory C) with the following weekly production capacities:

- Factory A: 80 units
- Factory B: 100 units
- Factory C: 220 units

They need to supply bricks to three major construction sites (Site 1, Site 2, and Site 3) with the following weekly demand:

- Site 1: 150 units
- Site 2: 70 units
- Site 3: 180 units

The cost of transporting one unit from each factory to each site is given in the following table:

From/To	Site 1 (€/unit)	Site 2 (€/unit)	Site 3 (€/unit)
Factory A	4	2	3
Factory B	3	5	4
Factory C	6	1	2

Now, check if the problem is balanced:

- Total Supply: 80 + 100 + 220 = 400 units
- Total Demand: 150 + 70 + 180 = 400 units

The total supply = the total demand, so this is a balanced transportation problem.

Solution:

STEP 1: Write the linear programming model

Let X_{ij} : The number of units shipped from the factory i to the site j

Min CT= 4X11 + 2 X12 +3 X13 +3 X21 +5 X22 +4 X23 +6X31 +1X32+ 2 X33

$$\text{ST: } \begin{cases} X11 + X12 + X13 = 80 \\ X21 + X22 + X23 = 100 \\ X31 + X32 + X33 = 220 \end{cases}$$

(Supply constraints)

$$\begin{cases} X11 + X21 + X31 = 150 \\ X12 + X22 + X32 = 70 \\ X13 + X23 + X33 = 180 \end{cases}$$

(Demand constraints)

$$X11 , X12 , X13 , X21 , X22, X23 , X31 , X32 , X33 \geq 0$$

Formulate the transportation table.

We can represent the problem in a transportation table:

From/To	Site 1	Site 2	Site 3	Ɛ Supply
Factory A (Supply:)	4 X_{11}	2 X_{12}	3 X_{13}	80
Factory B (Supply:)	3 X_{21}	5 X_{22}	4 X_{23}	100
Factory C (Supply:)	6 X_{31}	1 X_{32}	2 X_{33}	220
Ɛ Demand	150	70	180	400=400

Step 2: Find an initial basic feasible solution using:

1- The North-West Corner Rule.

The North-West Corner Rule starts by allocating as much as possible to the cell in the top-left corner of the transportation table and then proceeds systematically.

The initial basic feasible solution using the North-West Corner Rule is:

From/To	Site 1 (Demand:)	Site 2 (Demand:)	Site 3 (Demand:)	Ɛ Supply
Factory A (Supply:)	4 80	2 ----	3 ----	80
Factory B (Supply:)	3 70	5 30	4 ----	100
Factory C (Supply:)	6 ----	1 40	2 180	220
Ɛ Demand	150	70	180	400=400

All supply and demand constraints are satisfied. The number of allocated cells is (3+3-1 = 5), which satisfies the condition for a basic feasible solution in a transportation problem with (m) sources and (n) destinations. Therefore, this initial solution is non-degenerate. Thus, an optimal solution can be obtained.

The total cost according to this method is:

CT= $(4\times 80) + (3\times 70)+(5 \times 30) +(1\times 40)+ (2\times 180)= 1080$ €

2- The Least Cost Method (LCM):

From/To	Site 1 (Demand:)	Site 2 (Demand:)	Site 3 (Demand:)	Ɛ Supply
Factory A (Supply:)	4 50	2 ----	3 30	80
Factory B (Supply:)	3 100	5 ----	4 ----	100
Factory C (Supply:)	6 ----	1 70	2 150	220
Ɛ Demand	150	70	180	400=400

ll supply and demand constraints are satisfied. The number of allocated ls is (3 + 3 - 1 = 5), which satisfies the condition for a basic feasible solution in a transportation problem with (m) sources and (n) destinations. Therefore, this initial solution is non-degenerate. Thus, an optimal solution can be obtained.

The total cost according to this method is:

CT= (4× 50) + (3× 30) + (3 ×100) +(1× 70) + (2× 150) = 960 €

<u>**NOTE**</u>: The North-West corner method ignores the cost information. However, the Least Cost method incorporates the cost factor directly into the allocation process.

Step 3: Test the initial solution for optimality using the stepping stone method.

Unoccupied cells	The Net Change in Cost	Closed Path
S_1D_2	$\Delta C= +2-3+2-1=0$	
S_2D_2	$\Delta C=+5-1+2-3+4-3=+4$	
S_2D_3	$\Delta C=+4-3+4-3= +2$	
S_3D_1	$\Delta C=+6-2+3-4=+3$	

all unoccupied cells have a net change in cost that is greater than or equal to zero ($\Delta c \geq 0$). At this point, no further reduction in transportation cost is possible, and the current allocation represents the optimal solution.

Conclusion: The optimal solution is:

X_{11}= 50 units shipped from factory A to site 1, X_{21}=100 units shipped from factory B to site1, X_{13}= 30 units shipped from factory A to site 3, X_{32}=70 units shipped from factory C to site 2, X_{33}= 150 units shipped from factory C to sit3.

Min total cost = 960 €

I. MATHEMATICAL FORMULATION OF LINEAR PROGRAMMING PROBLEM

Exercise N°1

A bakery produces two types of cookies: chocolate chip and oatmeal. Each chocolate chip cookie requires 2 cups of flour and 1 hour of labor, while each oatmeal cookie requires 1 cup of flour and 2 hours of labor. The bakery has a maximum of 100 cups of flour and 80 hours of labor available per week. The profit from each chocolate chip cookie is $3, and from each oatmeal cookie is $2.

❖ Write the linear programming model?

Exercise N°2

The company wants to maximize its profit from producing products A and B. Each unit of product A requires 50 minutes on machine M1 and 30 minutes on machine M2, while each unit of product B requires 24 minutes on machine M1 and 33 minutes on machine M2. The available processing time for machine M1 is 40 hours (2400 minutes) and for machine M2 is 35 hours (2100 minutes).

❖ Write the linear programming model?

Exercise N°3

A factory produces two products, A and B. The profit from product A is $3 per unit, and the profit from product B is $5 per unit. The factory has the following constraints:

1. **Resource Constraints**:
   - Each unit of product A requires 2 hours of labor.
   - Each unit of product B requires 3 hours of labor.
   - The total labor available is 12 hours.
2. **Demand Constraints**:
   - The factory can produce a maximum of 4 units of product A.
   - There is no limit on the production of product B.

❖ Write the linear programming problem?

Exercise N°4

A person wants to decide the constituents of a diet which will fulfill his daily requirements of proteins, carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yield per unit of these foods are given in the Table :

Food type	***Yield per unit***			***Cost per units ($)***
	Proteins	***Fats***	***Carbohydrates***	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

- ❖ Formulate the linear programming model for the problem.?

Exercise N°5:

A firm manufactures three products *A*, *B*, *C*. There is demand for at least 300, 250, 200 units of products *A*, *B* and *C* and the profit earned per unit is 50 €, 40€,70€, respectively. The relevant data is given in Table below:

Formulate the problem as a linear programming problem.

Raw material	***Requirements per unit of product (kg)***			***Total availability (KG)***
	A	**B**	**C**	
P	**6**	**5**	**9**	**5000**
Q	**4**	**7**	**8**	**6000**

Exercise N°6

A company manufactures two products *A* and *B*, which require, the following resources. The resources are the capacities machine *M*1, *M*2, and *M*3. The available capacities are 50, 25, and 15 hours respectively in the planning period. Product *A* requires 1 hour of machine *M*2 and 1 hour of machine *M*3. Product *B* requires 2 hours of machine *M*1, 2 hours of machine *M*2 and 1 hour of machine *M*3. The profit contribution of products *A* and *B* are 5€ and 4€ respectively

- ❖ Formulate a linear programming model?

Exercise N°7:

The objective of a diet problem is to ascertain the quantities of certain foods that should be eaten to meet certain nutritional requirement at a minimum cost. The number of milligrams of each of these vitamins contained with a unit of each food is given below:

vitamins	Gallon of MILK	Pound of BEEF	Dozens of EGGS	Minimum daily requirement
A	2	1	10	1 mg
B	100	10	12	50 mg
C	11	100	14	10 mg
Cost (€)	1€	1.10 €	0.5 €	----------------

- ❖ Write the linear programming formulation?

Exercise N°8:

A company has two grades of inspectors: type I and type II, who are to be assigned for a quality control inspection. It is required that at least 2000 pieces be inspected per 8 hours/day.

Grade I inspectors can check pieces at the rate of 50/ hour;

Grade II inspectors can check pieces at the rate of 40/hour;

The wage rate of grade I inspector is 4.5 €/hour and that of grade II is 2.5€/hour;

The company has available for the inspection job at most 10 grade I and 5 gradeII

- ❖ Formulate the problem?

Exercise N°9:

A manufacturer produces two types of models *M* and *N*. Each *M* model requires 4 hours of grinding and 2 hours of polishing, whereas each *N* model requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and

3 polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on model *M* is Rs. 3 and model *N* is Rs. 4. Whatever is produced in a week is sold in the market.

- How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week?

Sol N°9:

Solution: Let $x1$ be the number of model M to be produced and $x2$ be the number of model N to be produced.

Clearly $x1, x2 \geq 0$.

The manufacturer gets profit of Rs. 3 for model M and Rs. 4 for model N. So, the objective function is to maximize profit.

$P = 3x1 + 4x2$

It is given that each model M requires 4 hours and each model N requires 2 hours for grinding. The maximum available time for grinding is 40 hours and there are two grinders. So we have the constraint on the availability of the hours of grinding,

$4x1 + 2x2$ £ 80

Again model M requires 2 hours of polishing and model N requires 5 hours of polishing. The maximum available time for polishing is 60 hours and there are three polishers. Thus, the constraint for the hours available for polishing is,

$2x1 + 5x2$ £ 180

Thus, the manufacturer's allocation problem is to Maximize $P = 3x1 + 4x2$

subject to the constraints:

$4x1 + 2x2$ £ 80

$2x1 + 5x2$ £ 180

and $x1, x2 \geq 0$.

<u>SOL N°8/</u>

Solution: Let $x1$ and $x2$ be the number of Grade 1 and Grade 2 inspectors doing inspections in a

company, respectively. Clearly $x1, x2 \geq 0$.

One hour cost of inspection incurred by the company while employing an inspector = cost paid

to the inspector + cost of errors made during inspection. Thus, costs for

Inspector Grade 1 = 5 + 3 * 40 * (1 – 0.97) = Rs. 8.60

Inspector Grade 2 = 4 + 3 * 30 * (1 – 0.95) = Rs. 8.50

The inspectors of both the grade work for 8 hours a day. So the objective function (to minimize daily inspection cost) is:

Minimize $C = 8(8.60x1 + 8.50x2) = 68.80x1 + 68.00x2$

Now the constraint of the inspection capacity of the inspectors for 8-hours is

$8 * 40x1 + 8 * 30x2 \geq 2000$

Also, the company has only nine Grade 1 inspectors and eleven Grade 2 inspectors. So we have $x1$ £ 9 and $x2$ £ 11.

Thus, LPP is:

Minimize $C = 68.80x1 + 68.00x2$

subject to the constraints:

$320x1 + 240x2 \geq 2000$

$x1$ £ 9

$x2$ £ 11

and $x1, x2$ **$\geq$ 0.**

SOL N° 06

The contents of the statement of the problem can be summarized as follows:

Machines hours	*Products*		*Availability in*
	X	*Y*	
*M*1	0	2	50
*M*2	1	2	25
*M*3	1	1	15
Profit €. Per unit	5	4	

In the above problem, Products *X* and *Y* are competing candidates or variables.

Machine capacities are available resources. Profit contribution of products *X* and *Y* are given.

Now let us formulate the model.

Let the company manufactures x units of X and y units of Y. As the profit contributions of X and

Y are €.5/- and € 4/- respectively. The objective of the problem is to maximize the profit Z, hence objective function is:

Maximize $Z = 5x + 4y$ OBJECTIVE FUNCTION.

This should be done so that the utilization of machine hours by products x and y should not exceed the available capacity. This can be shown as follows:

For Machine $M1$ $0x + 2y \leq 50$

For Machine $M2$ $1x + 2y \leq 25$ and LINEAR STRUCTURAL CONSTRAINTS.

For machine $M3$ $1x + 1y \leq 15$

But the company can stop production of x and y or can manufacture any amount of x and y. It cannot manufacture negative quantities of x and y. Hence we have write, Both x and y are ≥ 0 . NON -NEGATIVITY CONSTRAINT.

As the problem has got objective function, structural constraints, and non-negativity constraints and there exist a linear relationship between the variables and the constraints in the form of inequalities, the problem satisfies the properties of the Linear Programming Problem.

The second semester exam of "Fundamentals of Operations Research"

Exercise N°1: (6pts)

A manufacturer produces two types of models *M* and *N*. Each *M* model requires 4 hours of grinding and 2 hours of polishing, whereas each *N* model requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on model *M* is 3€/Unit and model *N* is 4€/Unit. Whatever is produced in a week is sold in the market.

- ❖ How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week? Write the LPM.

Exercise N°2: (7pts)

Solve this linear programming model using the simplex method:

$$\text{Max } Z = 3X_1 + 4X_2$$

$$\text{ST:} \quad \begin{cases} X_1 + 2X_2 \leq 450 \text{ mn} & \text{(time constraint of machine M1)} \\ 2X_1 + X_2 \leq 600 \text{ mn} & \text{(time constraint of machine M2)} \end{cases}$$

$$X_1, X_2 \geq 0$$

Exercise N°3 : (7pts)

A company needs to transport goods from 3 warehouses to 3 retail stores. The supply at each warehouse and the demand at each store are given below. The goal is to minimize the transportation cost while meeting the demand of each store and ensuring the supply from each warehouse is fully utilized.

- Supply at Warehouses:
  - Warehouse 1: 100 uni
  - Warehouse 2: 60 uni
  - Warehouse 3: 40 uni

Demand at Retail Stores:

Store A: 90 units

Store B: 60 units

Store C: 50 units

- Transportation Costs (per unit):

	Store A	Store B	Store C
Warehouse 1	$5	$8	$6
Warehouse 2	$4	$6	$7
Warehouse 3	$3	$5	$4

Questions: 1- Arrange the data in the matrix form?

2- Formulate the balanced transportation problem as a linear programming model?

3-Using **the Northwest Corner Method** and **the Least Cost Method**, find an initial feasible solution for the transportation problem. Calculate the total costs? Explain the difference?

Model Answer and Marking Scheme for the Exam

EXERCICE N°1 (6pts)

--------------------- -----	Product M	Product N	Available hours per week
Grinding	4 hours	2 hours	$2 \times 40 = 80$ hours/ week
Polishing	2 hours	5 hours	$3 \times 60 = 180$ hours/week
Profit	3 €/ unit	4 €/ unit	----------------------------

Steps for building the Linear Programming Model:

Step N°1: Identify decision variables

Let: X_1 is the number of units of the product M produced per week; (0.5 p)

X_2 is the number of units of the product N produced per week. (0.5 p)

Step N°2: Formulate the objective function (the total profit) (0.5 p)

The total profit required per week = profit required by the product M (per week) + profit required by the product N (per week)

$$Z = 3\,X_1 + 4\,X_2$$

Step N°3 Formulate the constraints (grinding & polishing constraints)

Grinding constraint: the number of grinding hours required by the product M + the number of grinding hours required by the product N should be at most 80 hours / week(0.25 p)

$$4\,X_1 + 2\,X_2 \leq 80 \text{ hours}$$

Polishing constraint: the number of polishing hours required by the product M + the number of polishing hours required by the product N should be at most 180 hours / week (0.25 p)

$$2X_1 + 5\,X_2 \leq 180 \text{ hours}$$

Step N°3: non-negativity condition of the decision variables (0.5 p)

$$X_1 \geq 0$$

$$X_2 \geq 0$$

The linear programming model is:

$$MAX\ Z = Z = 3\ X_1 + 4\ X_2 \quad (1\ p)$$

$$St: \begin{cases} 4\ X_1 + 2\ X_2 \leq 80 \quad (1p) \\ 2X_1 + 5\ X_2 \leq 180 \quad (1\ p) \end{cases}$$

$$X_1 \geq 0 \quad (0.25\ p)$$

$$X_2 \geq 0 \quad (0.25\ p)$$

Exercise N°2: (7pts)

$$MAX\ Z = Z = 3\ X_1 + 4\ X_2$$

$$St: \begin{cases} 4\ X_1 + 2\ X_2 \leq 80 \\ 2X_1 + 5\ X_2 \leq 180 \end{cases}$$

$$X_1 \geq 0$$

$$X_2 \geq 0$$

To solve this LPM we convert it into its standard form:

$$MAX\ Z = Z = 3\ X_1 + 4\ X_2 + 0\ Y_1 + 0\ Y_2 \ (1\ p)$$

$$St: \begin{cases} 4\ X_1 + 2\ X_2\ + Y_1 = 450 \quad (0.5\ p) \\ 2X_1 + 5\ X_2\ + Y_2 = 600 \quad (0.5\ p) \end{cases}$$

$$X_1\ ,\ X_2\ ,\ Y_1\ ,\ Y_2 \geq 0 \quad (0.5\ p)$$

We start by the initial simplex tableau as follows:(3p)

Basic variables	X1	X2	Y1	Y2	B	B/X2
Y1	1	2	1	0	450	450/2 =225
Y2	2	1	0	1	600	600/1 = 600
Z	3	4	0	0	0	B/X1
X2	½	1	½	0	225	225/1/2=450
Y2	3/2	0	-2	1	375	375/3/2= 250
Z	1	0	-2	0	900	
X2	0	1	2/3	-1/3	100	
X1	1	0	-1/3	2/3	250	
Z	0	0	-5/6	-2/3	1150	

The optimal solution is:

X1= 250 units of the product M should be produced per week(0.5 p)

X2=100 units of the product N should be produced per week (0.5 p)

Max Profit Required (Per Week) = 1150 € (0.5 p)

EXERCISE N° 3: (7pts)

1° Formulate the transportation table: (1p)

	Store1	Store2	Store3	*Σ Supply*
Warehouse 1	$\boxed{5}$ X11	$\boxed{8}$ X12	$\boxed{6}$ X13	100
Warehouse 2	$\boxed{4}$ X21	$\boxed{6}$ X22	$\boxed{7}$ X23	60
Warehouse 3	$\boxed{3}$ X31	$\boxed{5}$ X32	$\boxed{4}$ X33	40
Σ Demand	90	60	50	200=200

2°/ Formulate the balanced transportation problem as a LPM:

Let X_{ij} the number of units shipped from the warehouse number i to the store number j

Where: i= 1,2,3 and j= 1,2,3 (0.5)

The linear programming model is :

$$\text{Min } Ct = 5X_{11} + 8X_{12} + 6X_{13} + 4X_{21} + 6X_{22} + 7X_{23} + 3X_{31} + 5X_{32} + 4X_{33} \quad (0.25p)$$

ST :

$$\begin{cases} X_{11} + X_{12} + X_{13} = 100 \ (0.25p) \\ X_{21} + X_{22} + X_{23} = 60 \ (0.25p) \\ X_{31} + X_{32} + X_{33} = 40 \ (0.25p) \end{cases} \quad \text{Supply constraints}$$

$$\begin{cases} X_{1} + X_{21} + X_{31} = 90 \ (0.25p) \\ X_{12} + X_{22} + X_{32} = 60 \ (0.25p) \\ X_{13} + X_{23} + X_{33} = 50 \ (0.25p) \end{cases} \quad \text{Demand constraints}$$

$X_{ij} \geq 0 \quad i=1,2,3 \qquad j= 1,2,3$ (0.25p)

2°/ find the initial solution:

- The initial feasible solution using **the North West Corner Method** is as follows:(1p)

	Store 1		Store 2		Store 3		Σ Supply
Warehouse 1	5	90	8	10	6	-----	100
Warehouse 2	4	-----	6	50	7	10	60
Warehouse 3	3	-----	5	-----	4	40	40
Σ Demand	90		60		50		200=200

We check the condition : the number of occupied cells = (m+n) – 1

m is the number of warehouses and n is the number of stores

so: according to the NWCMethod we have the number of occupied cells = 5

(M+N) = (3+3) – 1= 6-1= 5 ,We accept this initial solution (0.25p)

The total cost = 90(5) + 10(8) + 50(6) + 10(7) + 40(4) = 1060 € (0.25p)

- The initial feasible solution using the **Least Cost Method** is as follows:

	Store 1		Store 2		Store 3		Σ *Supply*
Warehouse 1	5	-----	8	50	6	50	100
Warehouse 2	4	50	6	10	7	-----	60
Warehouse 3	3	40	5	-----	4	-----	40
Σ *Demand*	90		60		50		200=200

We check the condition : the number of occupied cells = (m+n) – 1

m is the number of warehouses and n is the number of stores

so:

according to the Least Cost Method we have the number of occupied cells = 5

(M+N) = (3+3) – 1= 6-1= 5 ,We accept this initial solution (0.25p)

THE total cost = 50(8)+ 50(6) + 50(4) + 10(6) + 40(3) =1080 € (0.25p)

In transportation problems, **the Northwest Corner Method (NWC)** is a basic feasible solution method that does **not consider costs** during allocation—it simply follows a fixed rule: start from the top-left (northwest) cell and move right or down based on supply and demand.

The Least Cost Method (LCM), on the other hand, **tries to minimize cost** by allocating as much as possible to the cells with the lowest transportation cost first.

The Northwest Corner Method **can sometimes** lead to a lower total cost than the Least Cost Method **by chance**, due to the structure of the cost matrix and the specific supply/demand values. (0.5P)

References:

1. A. M. Natarajan, P. Balasubramanie, A. Tamilarasi, (2016), « Operations Research», Second Edition, Pearson.

2. Colin Palmer and Alexander E. Innes, (1980), «Operations Research by example». The MACMILLAN Press LTD. London and Basingstoke.

3. Er. Prem Kumar Gupta, D. S. Hira, (2014)., «Operations Research». S. CHAND. New Delhi.

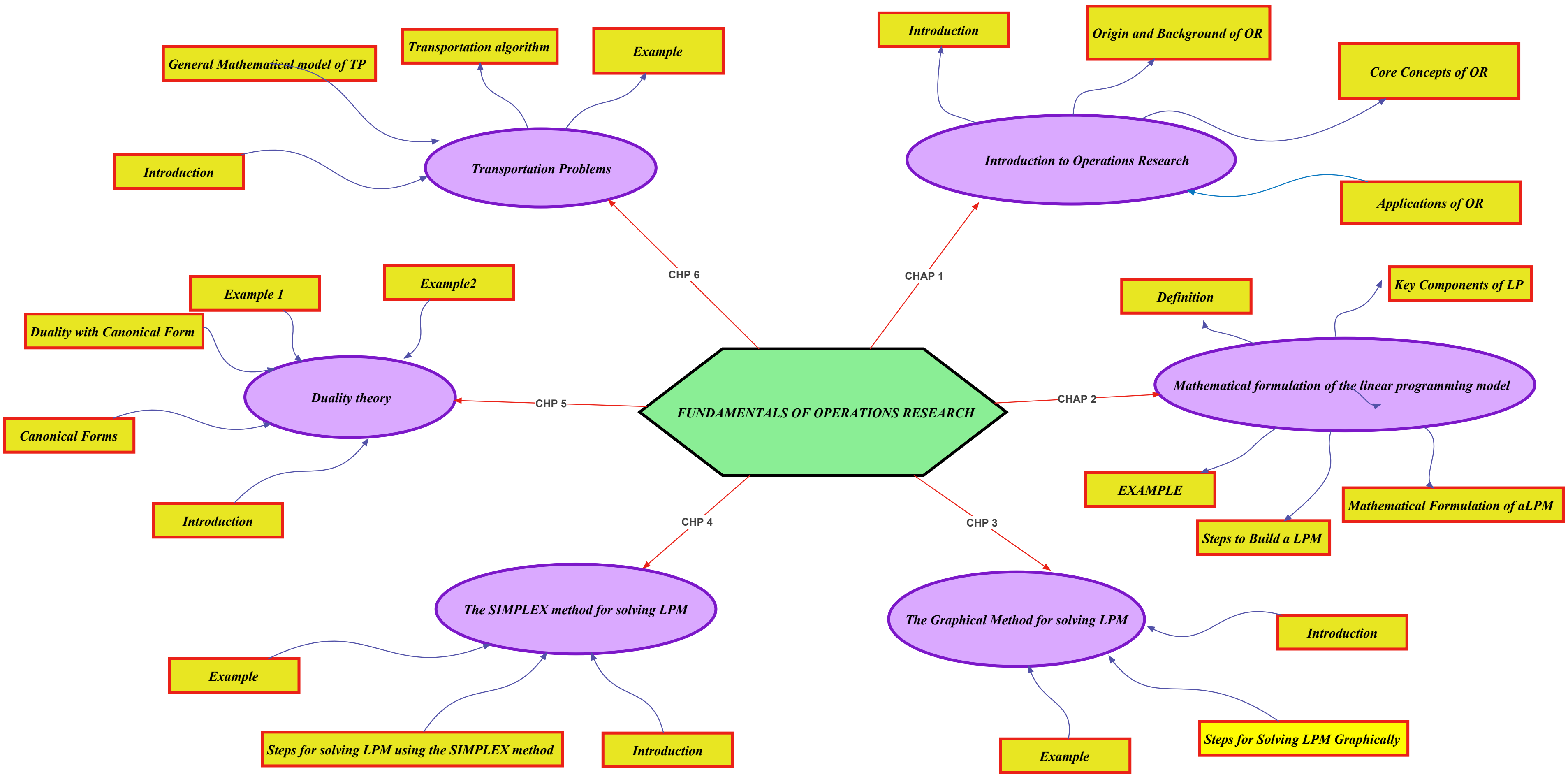
4. H. A. Eiselt. C & L. Sandblom, (2007). « 7Linear Programming and its Applications». Springer.

5. Nita H. Shah, Ravi M. Gor, Hardik Soni. (2010). «Operations Research». Eastern Economy Edition. New Delhi.

6. P. Rama Murthy. (2007). « Operations Research». Second Edition. New Age International Publishers. New Delhi.

7. Robert J. Vanderbei, (2020). « Linear Programming, Fondations and Extensions». International Series in Operations Research & Management Science. Fifth Edition. Springer.

8. Frederick S. Hillier, Gerald J. Lieberman(2014), «Introduction to Operations Research», Mc Graw-Hill.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