Tutorial Sheet 3 : Real functions of a single variable

Exercice 1.

I. Give the domain of definition of the following functions :

1. $f(x) = \dfrac{5x+4}{x^2+3x+2}$
2. $f(x) = \sqrt{4-3x^2}$
3. $f(x) = \dfrac{x\sin\left(\dfrac{1}{x}\right)}{\sqrt{1-x^2}}$
4. $f(x) = \begin{cases} \ln x, & \text{if } x \geq 1 \\ \dfrac{1}{\sqrt{1-x^2}}, & \text{if } x < 1 \end{cases}$

Exercice 2.

Show that the function $f(x) = \dfrac{\cos x}{1+x^2}$ is bounded on $\mathbb{R}$.

Exercice 3.

I. Calculate the following limits :

$$\mathbf{1.}\lim_{x\to+\infty}\frac{\sqrt{x}+2-3x}{x}, \qquad \mathbf{2.}\lim_{x\to 2}\frac{x^2-4}{\sqrt{2}-\sqrt{x}}, \qquad \mathbf{3.}\lim_{x\to 0} e^{\frac{|x|}{x}}, \qquad \mathbf{4.}\lim_{x\to 1}|x-1|\cos\left(\frac{1}{x-1}\right)$$

$$\mathbf{5.}\lim_{x\to 0}\frac{e^{\alpha x}-e^{\beta x}}{x}, \quad \alpha,\beta\in\mathbb{R},$$

II. Using the definition of the limit of a function, show that :

$$\lim_{x\to 4}(2x-1) = 7, \qquad \lim_{x\to+\infty}\frac{3x-1}{2x+1} = \frac{3}{2}.$$

Exercice 4.

Study the continuity of the following functions :

$$\mathbf{1.}f_1(x) = \frac{1-\cos x}{x^2}, \qquad \mathbf{2.}f_2(x) = x^2\sin\left(\frac{1}{x}\right), \qquad \mathbf{3.}f_3(x) = \frac{\sin(\pi x)}{|x-1|}, \qquad \mathbf{4.}f_4(x) = \begin{cases} e^x, & x \geq 0 \\ \cos x, & x < 0 \end{cases}$$

$$\mathbf{5.}f(x) = \begin{cases} \frac{5x^2-2}{2} & \text{if } x \geq 1 \\ \cos(x-1) & \text{if } x < 1 \end{cases}, \qquad \mathbf{5.}(Supp)f_5(x) = \begin{cases} x^2\cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

$$\mathbf{6.}(Supp)f_6(x) = \begin{cases} x+1, & x \leq -1 \\ \cos^2\left(\frac{\pi x}{2}\right), & x > -1 \end{cases}, \qquad \mathbf{7.}(Supp)f_7(x) = \begin{cases} x^2\arctan\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Exercice 5. (Supp)

Let functions be defined by :

$$f(x) = \begin{cases} \frac{\sin x}{x-\pi}, & x \neq \pi \\ \alpha, & x = \pi \end{cases}, \qquad g(x) = \begin{cases} x+1 & \text{if } x \leq 1 \\ 3-\alpha x^2 & \text{if } x > 1 \end{cases}$$

Determine $\alpha \in \mathbb{R}$ such that f and g are continuous on its domain of definition.

Exercice 6.

Do the following functions admit a continuous extension at the points where they are not defined ?

$$f_1(x) = \frac{|x|}{x}$$

$$f_2(x) = \frac{(x-1)\sin x}{2x^2-2}$$

$$f_3(x) = \sin\left(\frac{1}{x}\right)$$

Exercice 7.

Study the differentiability of the following functions on their domain of definition.

$$f_1(x) = \begin{cases} x^2 \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}, \qquad f_2(x) = \begin{cases} x+1, & x \leq -1 \\ \cos^2\left(\frac{\pi x}{2}\right), & x > -1 \end{cases}, \qquad (\text{Supp}) f_3(x) = \begin{cases} \frac{x^2 \sin \frac{1}{x}}{\sin x}, & x \in]0, \pi[\\ 0, & x = 0 \end{cases}$$

$$f_4(x) = \begin{cases} x^2 + x, & x \leq 0 \\ \sin x, & 0 < x \leq \pi \\ 1 + \cos x, & x > \pi \end{cases}, \qquad (\text{Supp}) f_5(x) = \begin{cases} e^x - 1, & x \leq 0 \\ \sin \pi x, & 0 < x \leq 1 \\ -\pi \ln x, & x > 1 \end{cases}$$

Give the derivative of each function on its domain of differentiability.

Exercice 8.

Calculate the n^{th} derivative of the following functions :

1. $f(x) = e^{ax}$
2. (Supp) $f(x) = \sin x$
3. $f(x) = \cos x$
4. $f(x) = \frac{1}{1-x}$
5. (Supp) $f(x) = \frac{1}{1+x}$
6. (Supp) $f(x) = \frac{1}{1-x^2}$