Chapter 2

Stresses and strains

What is Stress?

Stress is defined as force per unit area within materials that arises from externally applied forces, uneven heating, or permanent deformation and that permits an accurate description and prediction of elastic, plastic, and fluid behaviour.

Stress is given by the following formula:

$$\sigma = \frac{F}{A}$$

where, σ is the stress applied, ***F*** is the force applied and ***A*** is the area of the force application.

The unit of stress is N/m^2.

What is Strain?

Strain is the amount of deformation experienced by the body in the direction of force applied, divided by the initial dimensions of the body.

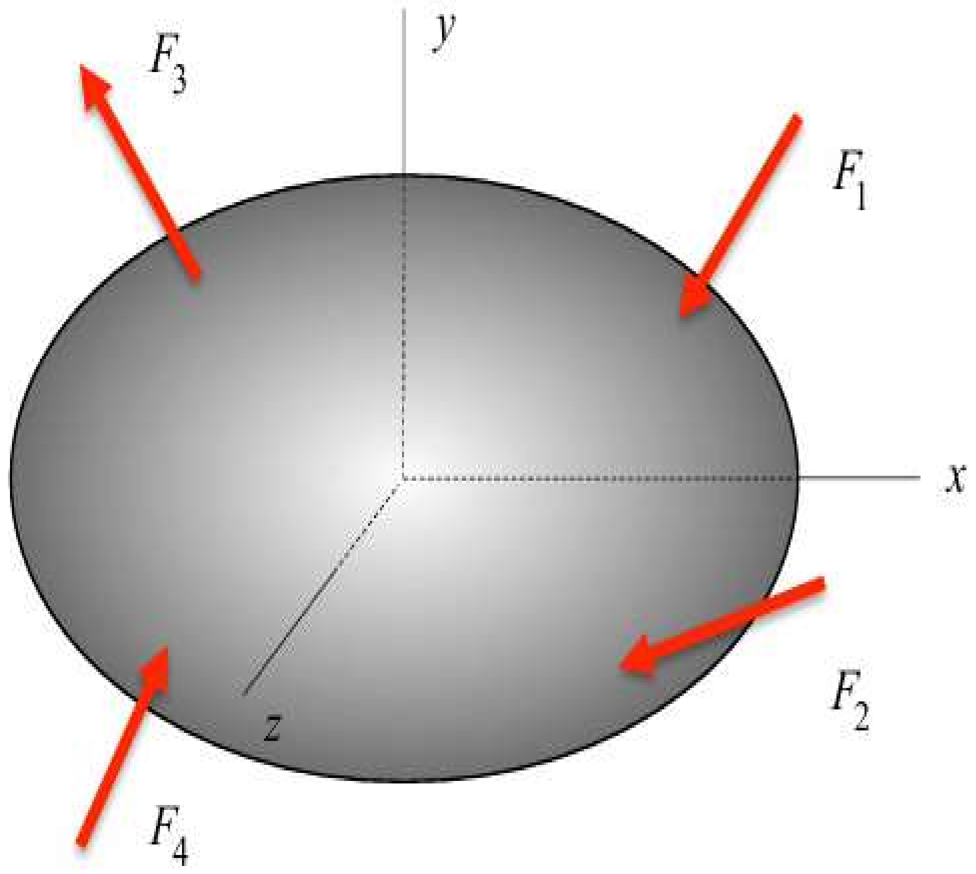
The following equation gives the relation for deformation in terms of the length of a solid:

$$\epsilon = \frac{\delta l}{L}$$

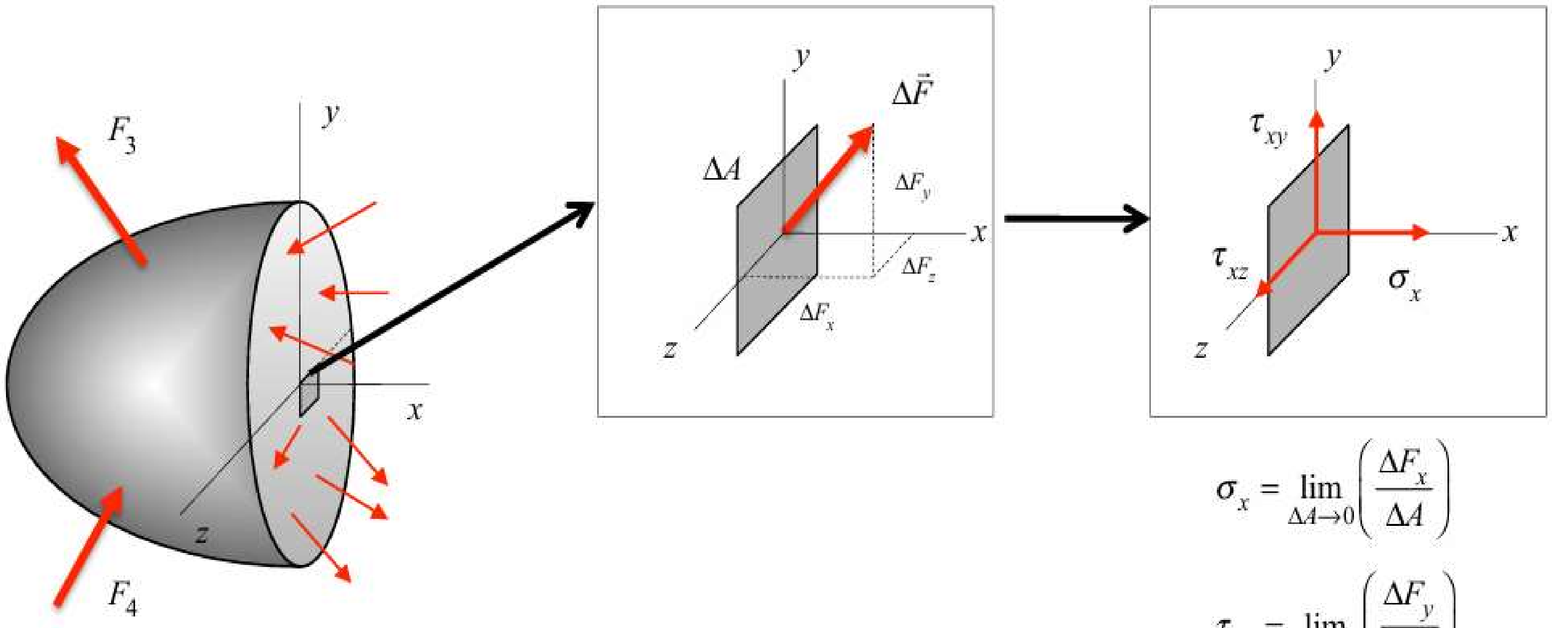
where ε is the strain due to the stress applied, δl is the change in length and **L** is the original length of the material.

The strain is a dimensionless quantity as it just defines the relative change in shape.

Consider a general 3-D loading on a component:



Making cut through body parallel to *yz*-plane:



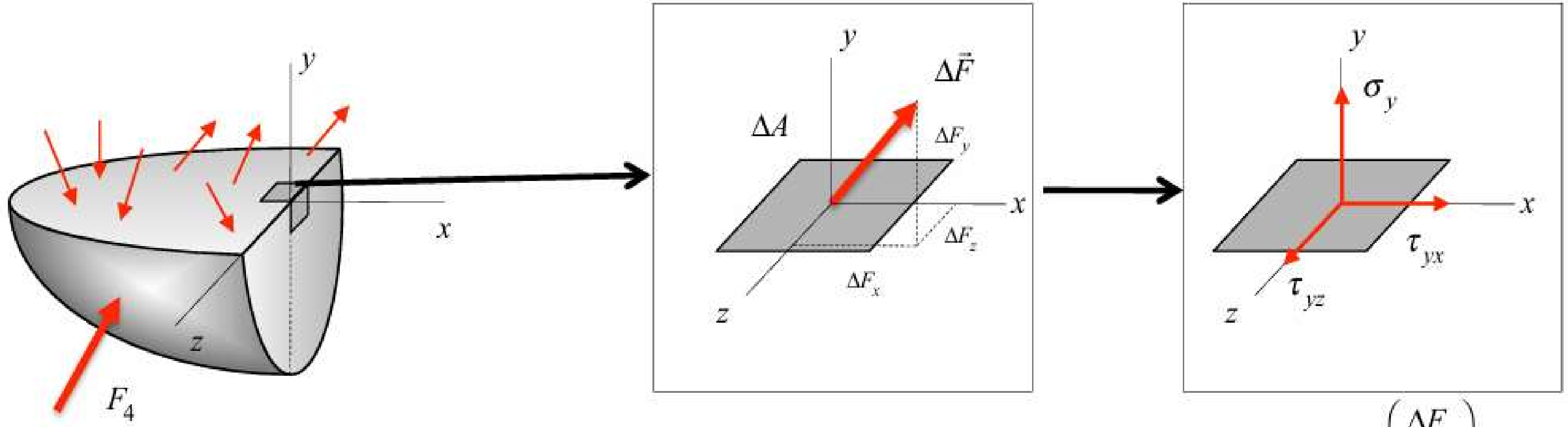
$$\sigma_x = \lim_{\Delta A \to 0}\left(\frac{\Delta F_x}{\Delta A}\right)$$

$$\tau_{xy} = \lim_{\Delta A \to 0}\left(\frac{\Delta F_y}{\Delta A}\right)$$

$$\tau_{xz} = \lim_{\Delta A \to 0}\left(\frac{\Delta F_z}{\Delta A}\right)$$

σ_x is the normal stress on the +*x*-face, and τ_{xy} and τ_{xz} are the components of shear stress on the *x*-face in the *y*- and *z*-directions, respectively.

Next, making a cut through body parallel to *xz*-plane:



$$\tau_{yx} = \lim_{\Delta A \to 0}\left(\frac{\Delta F_x}{\Delta A}\right)$$

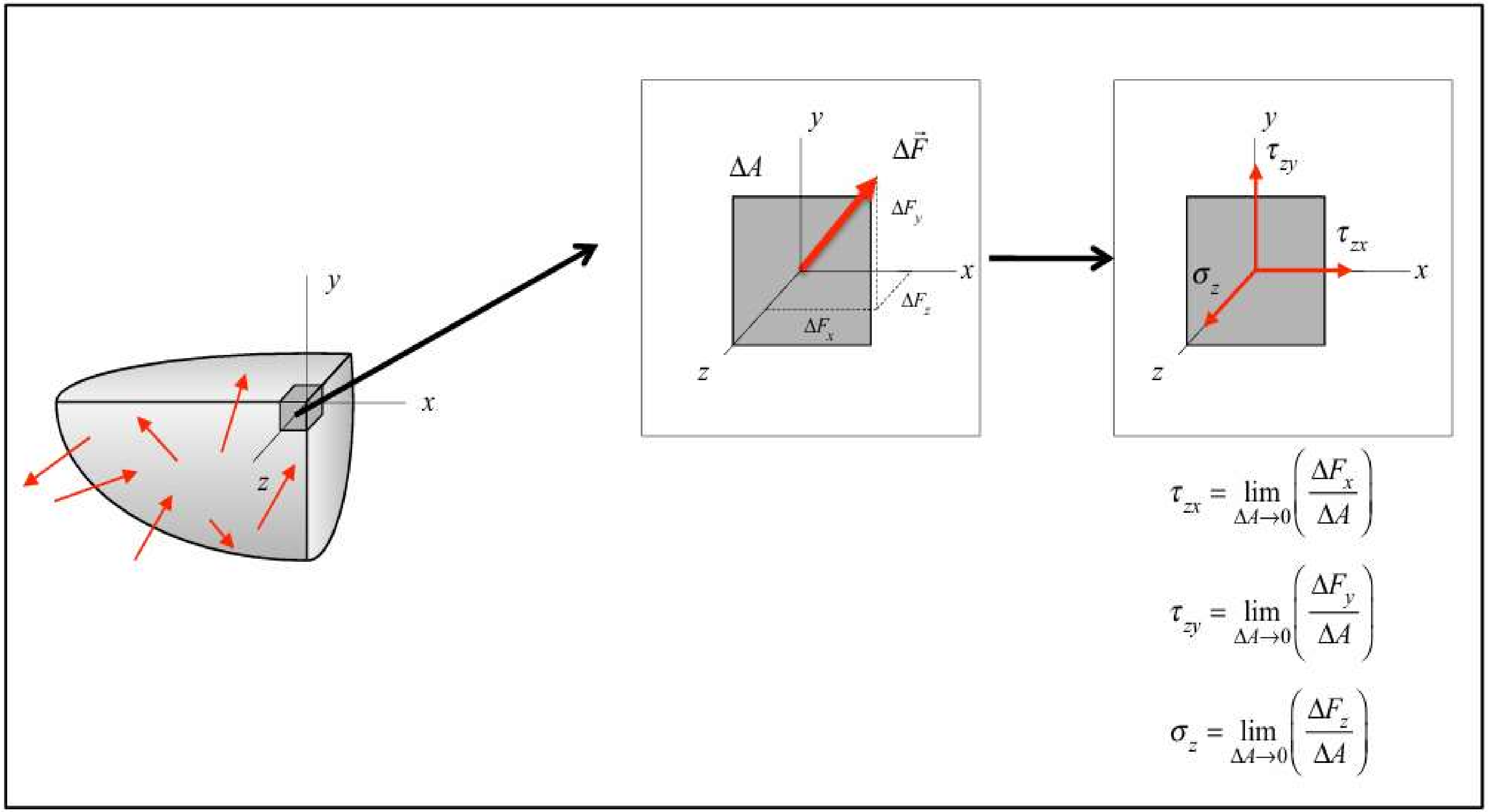
$$\sigma_y = \lim_{\Delta A \to 0}\left(\frac{\Delta F_y}{\Delta A}\right)$$

y

$$\tau_{yz} = \lim_{\Delta A \to 0}\left(\frac{\Delta F_z}{\Delta A}\right)$$

σ_y is the normal stress on the +*y*-face, and τ_{yx} and τ_{yz} are the components of shear stress on the +*y*-face in the *x*- and *z*-directions, respectively.

Next, making a cut through body parallel to *xy*-plane:



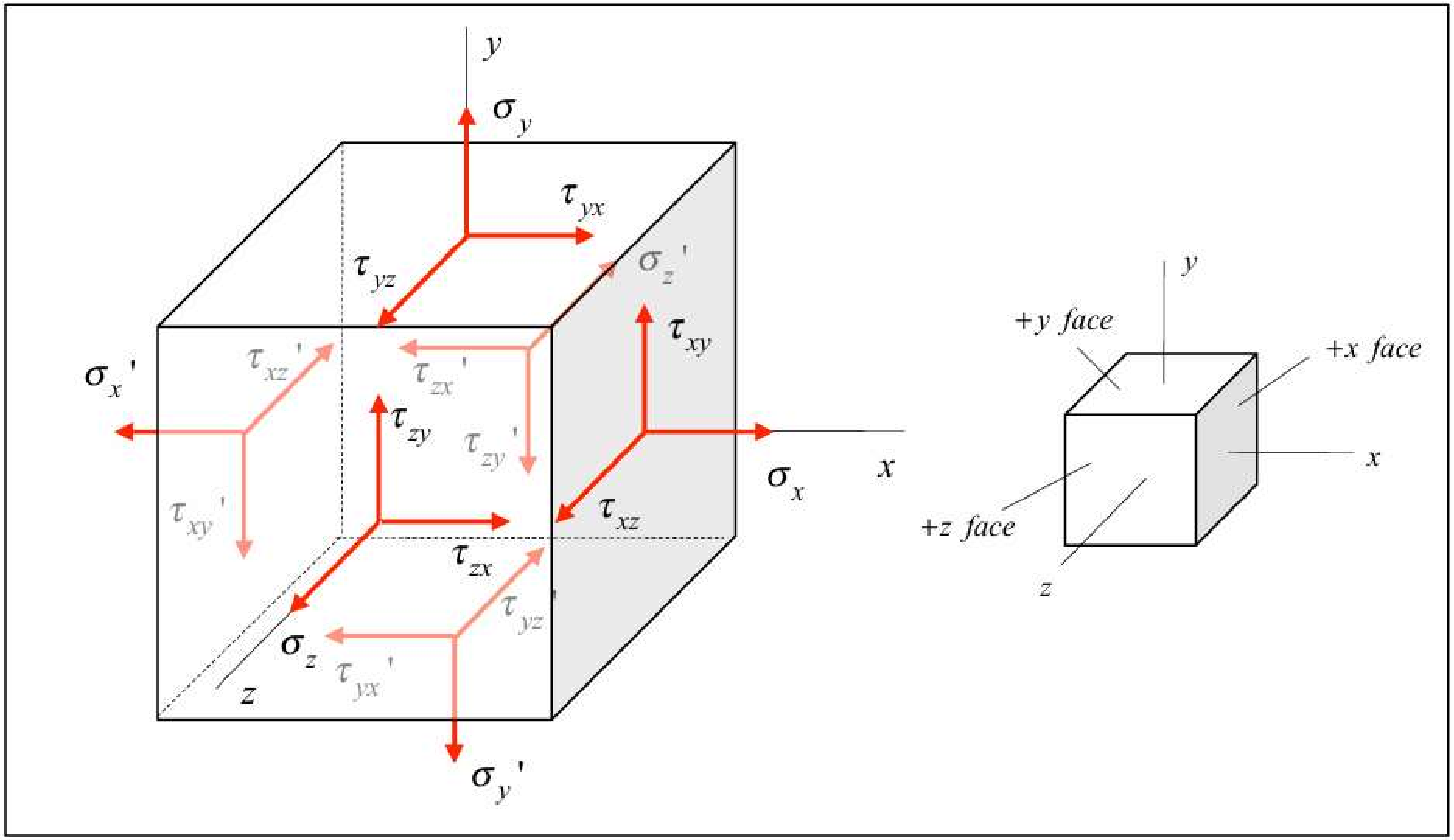
$$\tau_{zx} = \lim_{\Delta A \to 0}\left(\frac{\Delta F_x}{\Delta A}\right)$$

$$\tau_{zy} = \lim_{\Delta A \to 0}\left(\frac{\Delta F_y}{\Delta A}\right)$$

$$\sigma_z = \lim_{\Delta A \to 0}\left(\frac{\Delta F_z}{\Delta A}\right)$$

σ_z is the normal stress on the +*z*-face, and τ_{zx} and τ_{zy} are the components of shear stress on the +*z*-face in the *x*- and *y*-directions, respectively.

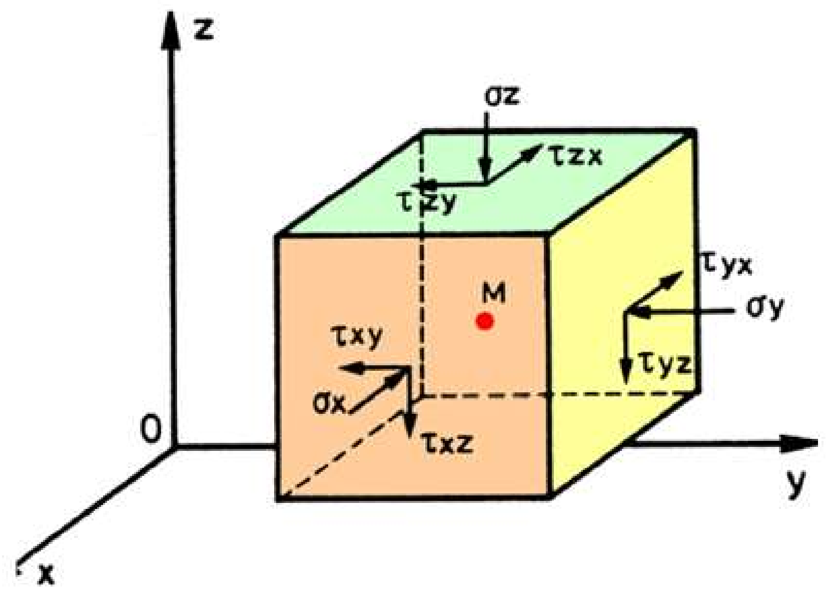
Suppose that we continue with an additional set of three cuts through the body at this point of interest, chosen here as the origin of the *xyz*-axes, with these cuts representing the $-x$, $-y$ and $-z$ planes. Following these cuts, we are left of a six-sided "stress element" whose sides are made up of the $\pm x$, $\pm y$ and $\pm z$ faces. As shown below, we have three components of stress (one normal and two shear) on each face.

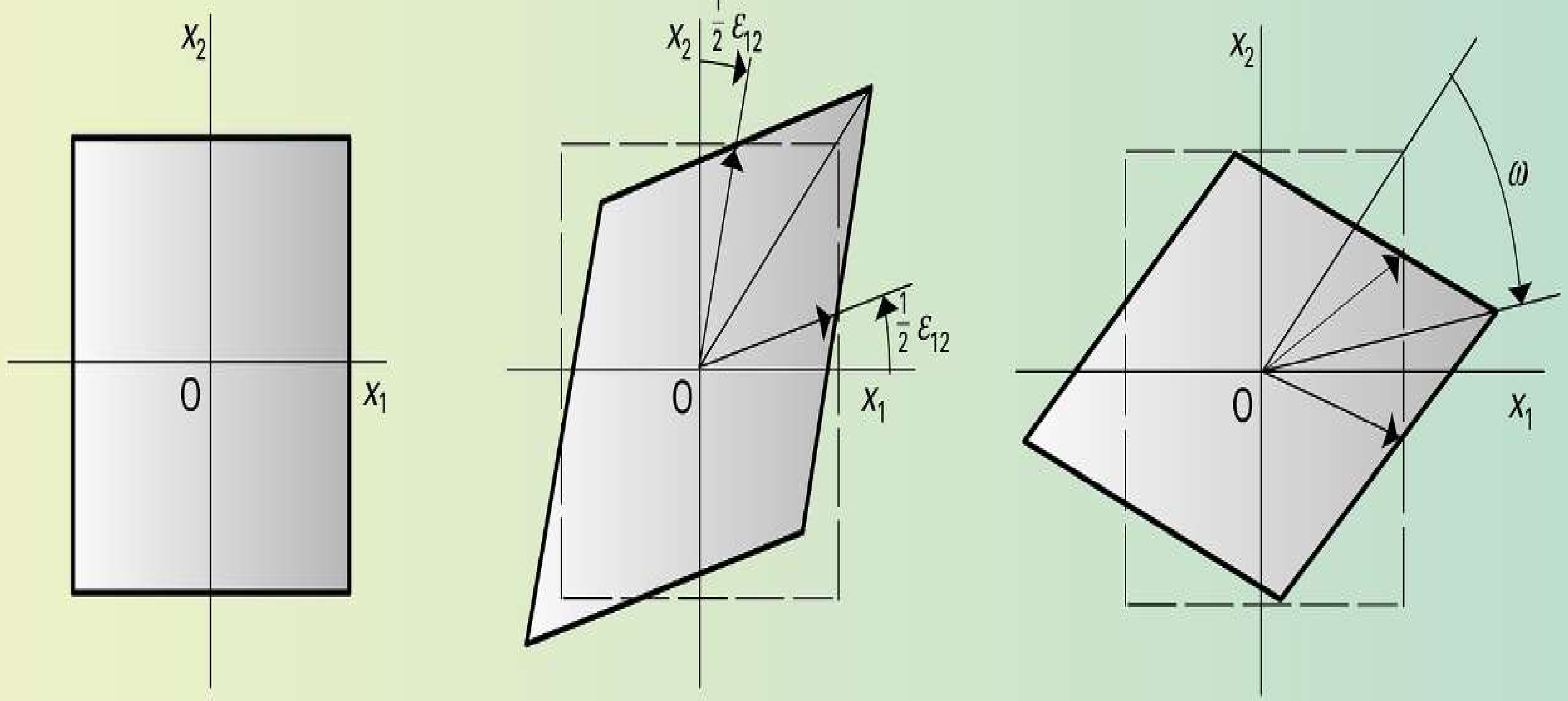


"stress element" at a point in the body

The stress state at point M is defined by a symmetric matrix called the stress tensor.

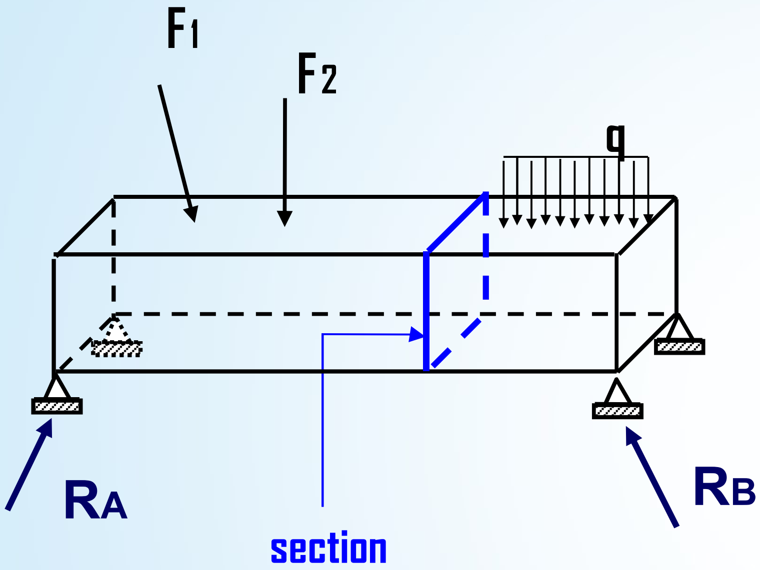
$$\sigma = (\sigma_{ij}) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

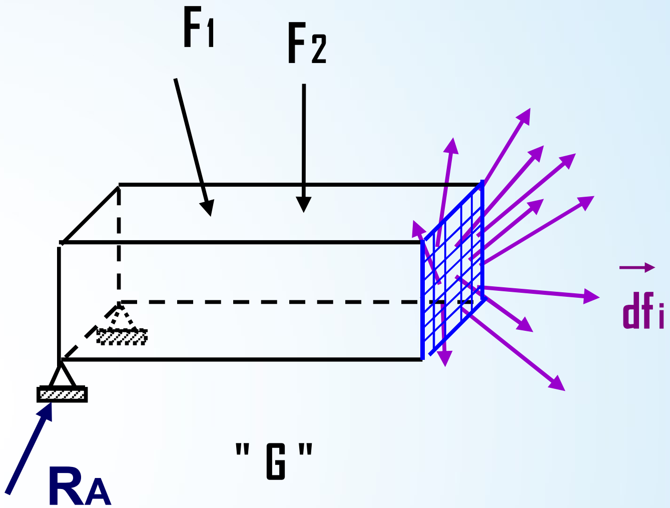


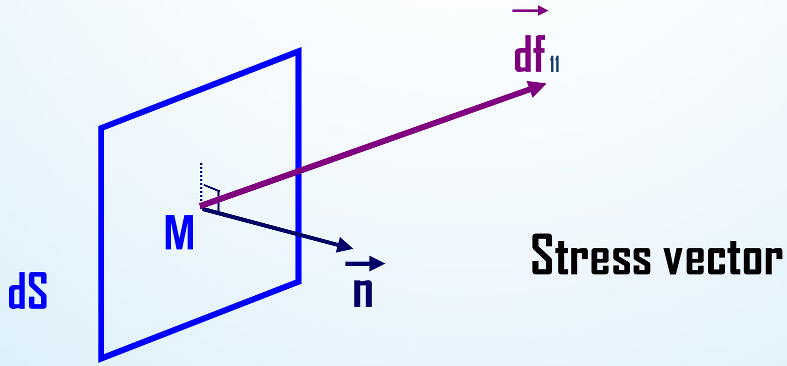


x_2
0
x_1
$\frac{1}{2}\varepsilon_{12}$
x_2
0
x_1
$\frac{1}{2}\varepsilon_{12}$
x_2
ω
0
x_1

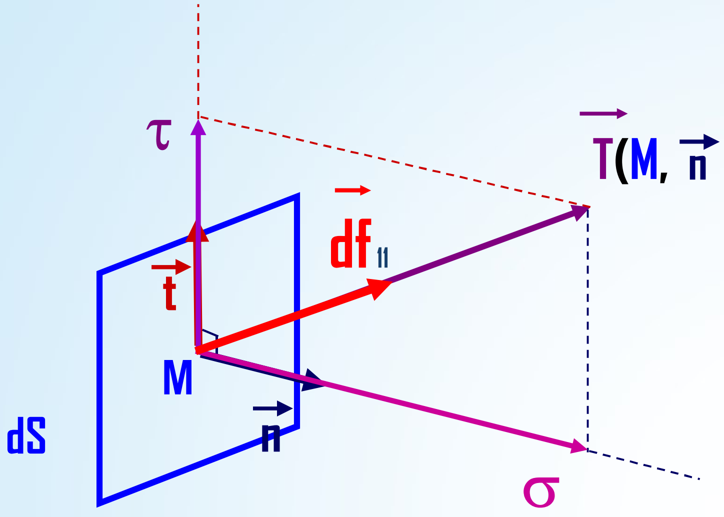
I – CONCEPTS OF STRESSES







2- Normal stress and tangential stress



$$\vec{T}(M, \vec{n}) = \frac{\vec{df}}{dS}$$

$$\vec{T}(M, \vec{n}) = \sigma.\vec{n} + \tau.\vec{t}$$

σ : Normal stress

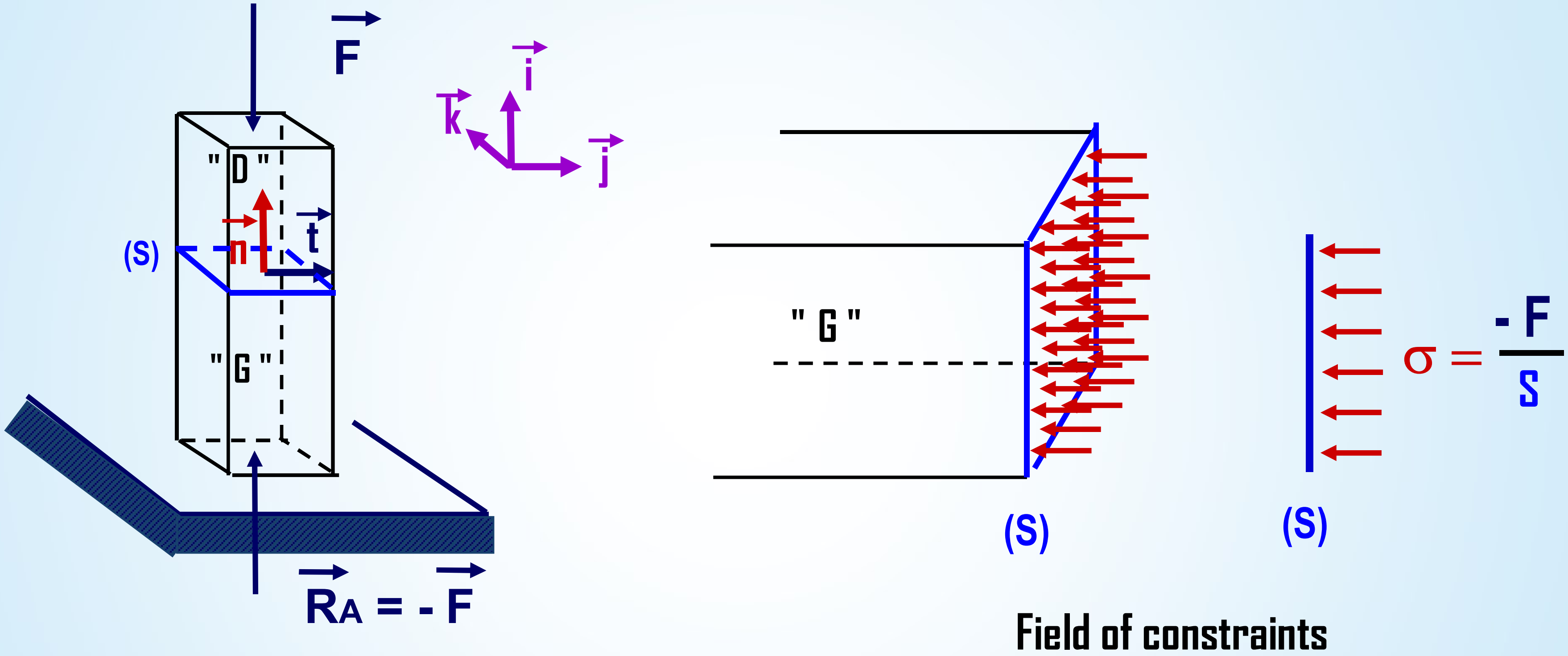
τ : tangential stress

$$\|\vec{T}\| = \sqrt{\sigma^2 + \tau^2}$$

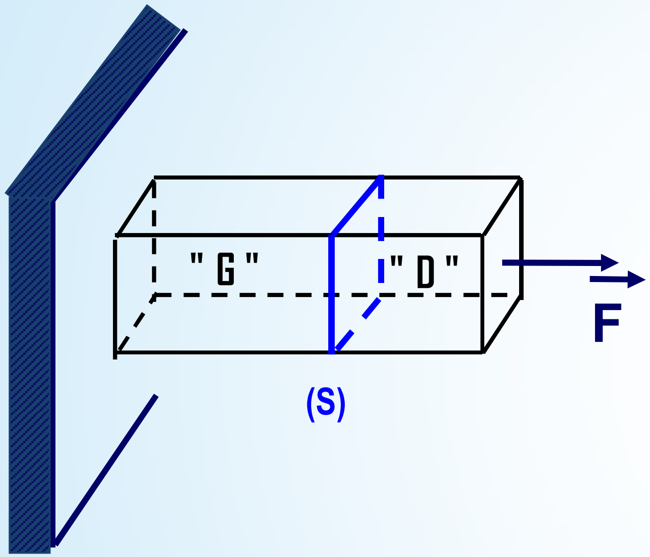
- The unit of stress is the ratio of a force to a unit area (N/mm2 = MPa).

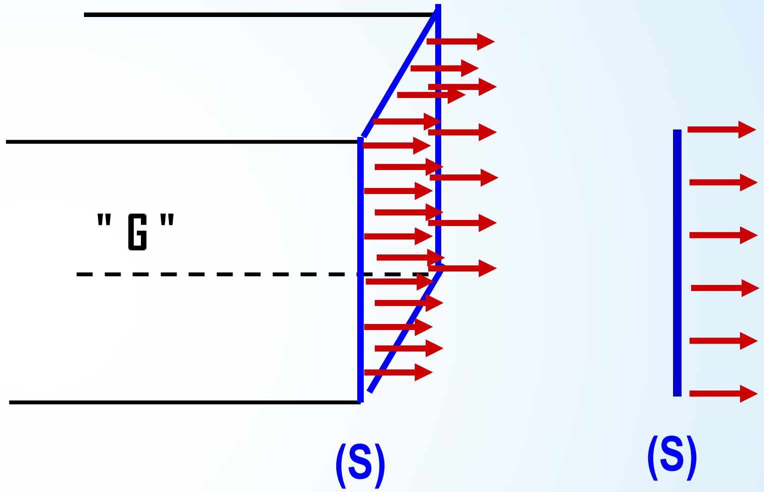
3- The different types of constraints (Stresses)

a) Compressive stress



b) Tensile stress

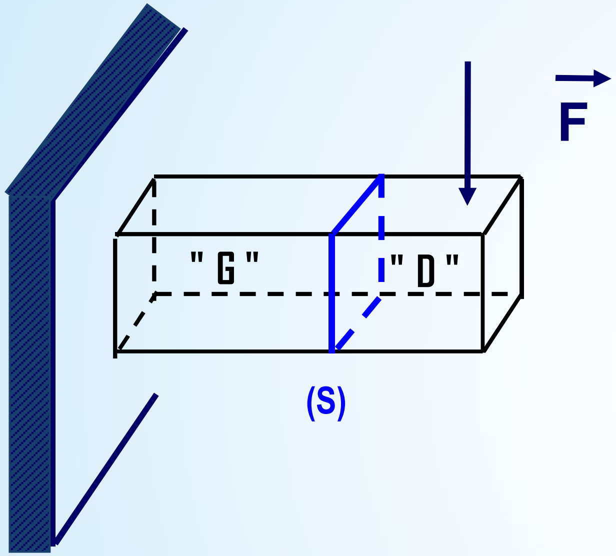


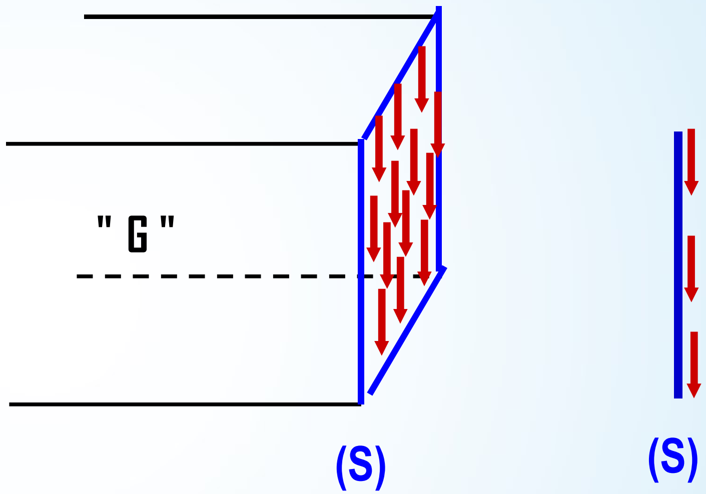


$$\sigma = \frac{F_{11}}{S}$$

Field of constraints

c) Shear stress :

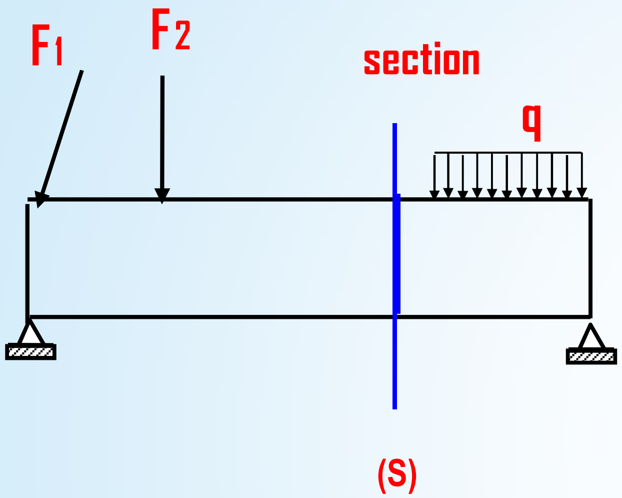


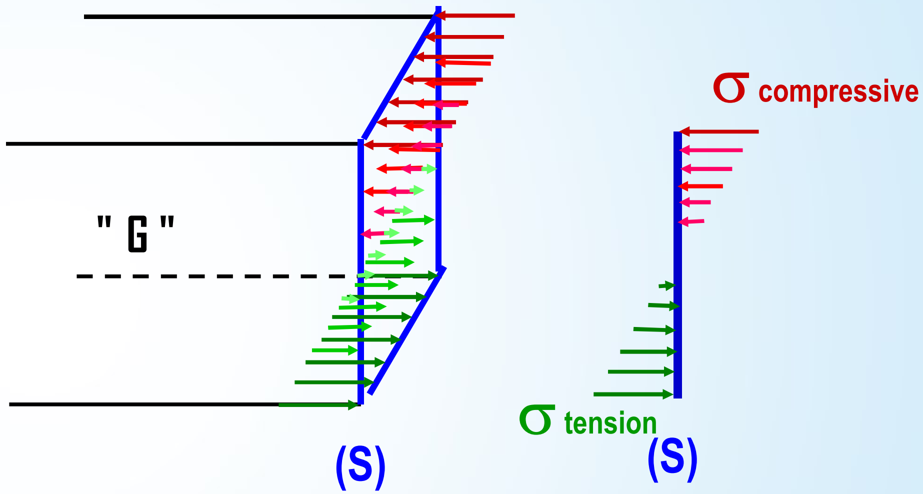


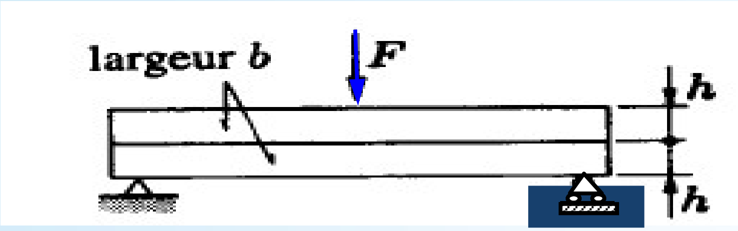
$$\tau = \frac{F}{S}$$

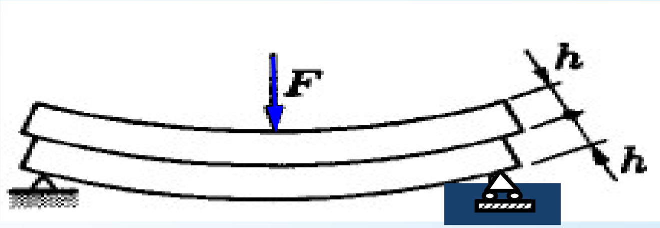
Field of constraints

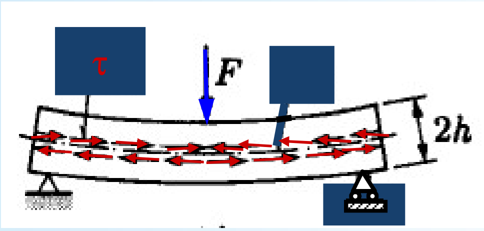
d) Bending stresses :











: Appearance of tangential stresses at the level of fibre contact

<u>Demonstration of shear stresses</u>

Normal stresses

of Compression

of tensile

Bending stresses :

Tangential stress

What is the purpose of stress calculations?

We must check that the stresses generated by external loads do not exceed the permissible stress limit for the material.

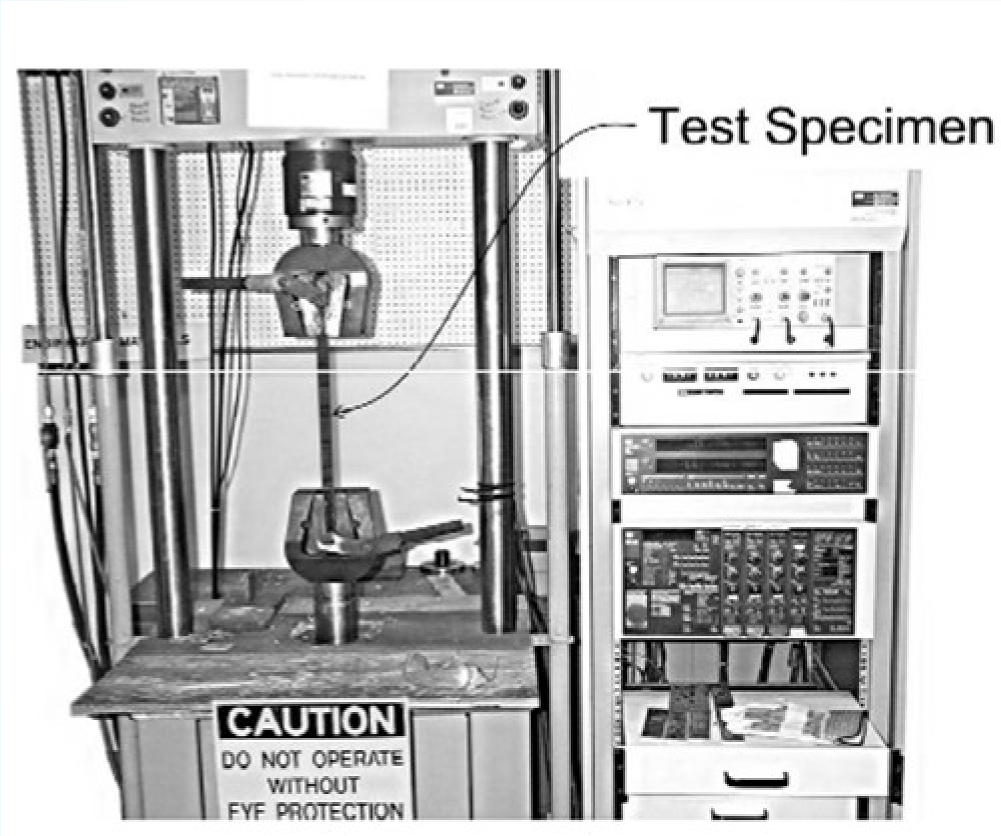
$$\sigma = \frac{F}{S} \leq \sigma_{ad}$$

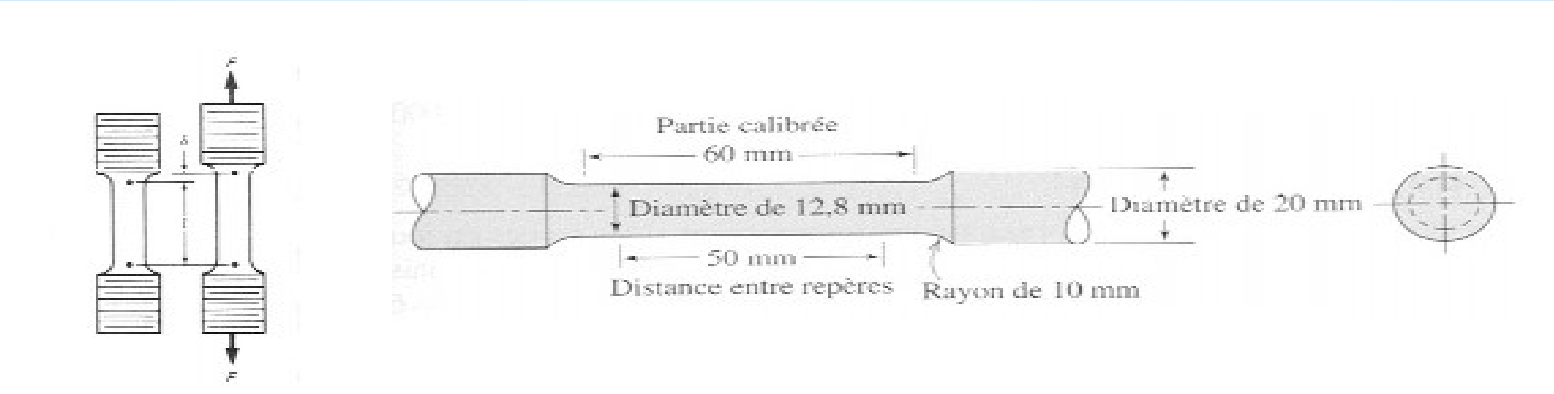
σ_{ad} : stress above which the piece is subject to deterioration of its mechanical and dimensional characteristics, or even failure.

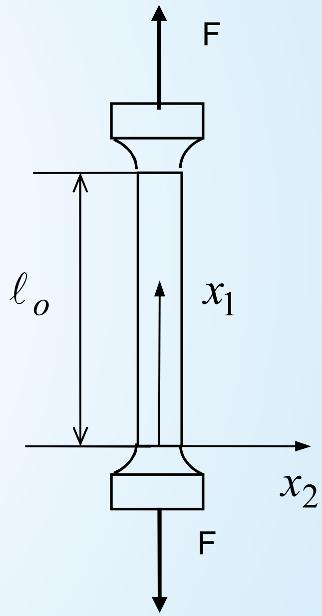
σ_{ad} is experimentally determined

MECHANICAL PROPERTIES OF MATERIALS

1- Tensile test





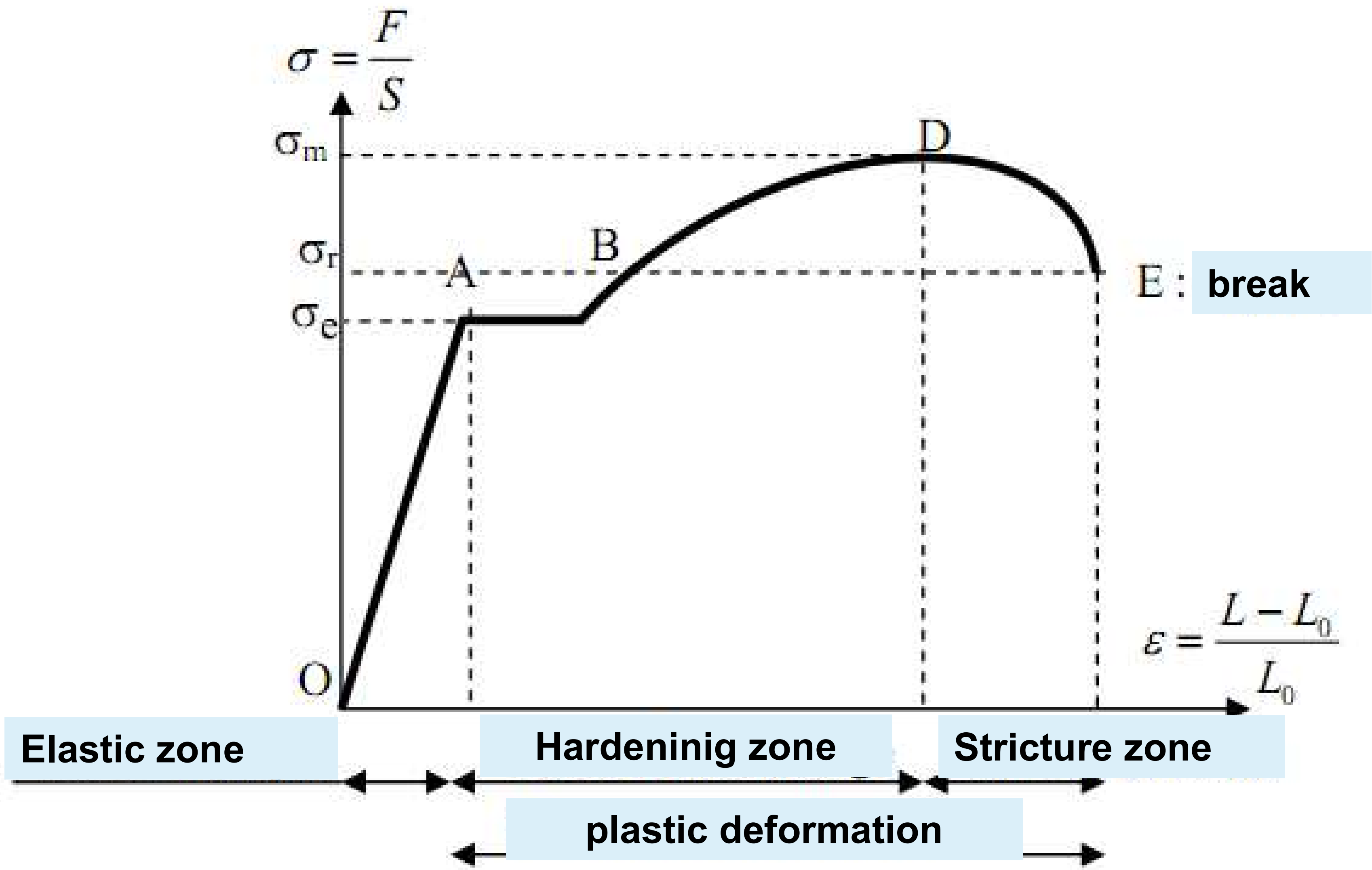


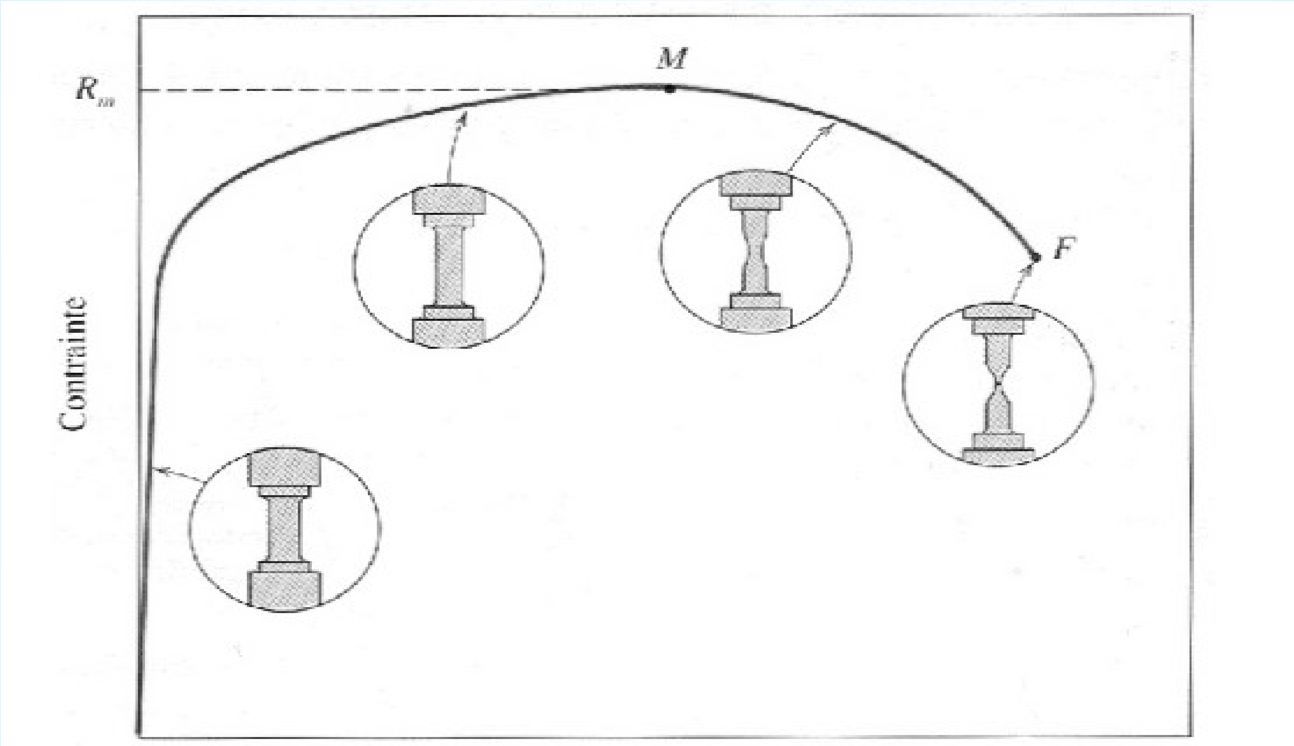
The specimen is subjected to an increasing force F, the relative elongation $\Delta\ell$ is measured and the test is continued until failure.

- The state of stresses and strains :

$$\sigma_1 = \frac{F}{S} \qquad \varepsilon_1 = \frac{\Delta\ell}{\ell_o}$$

Strain-stress diagram for steel





Stress-strain diagram for aluminium

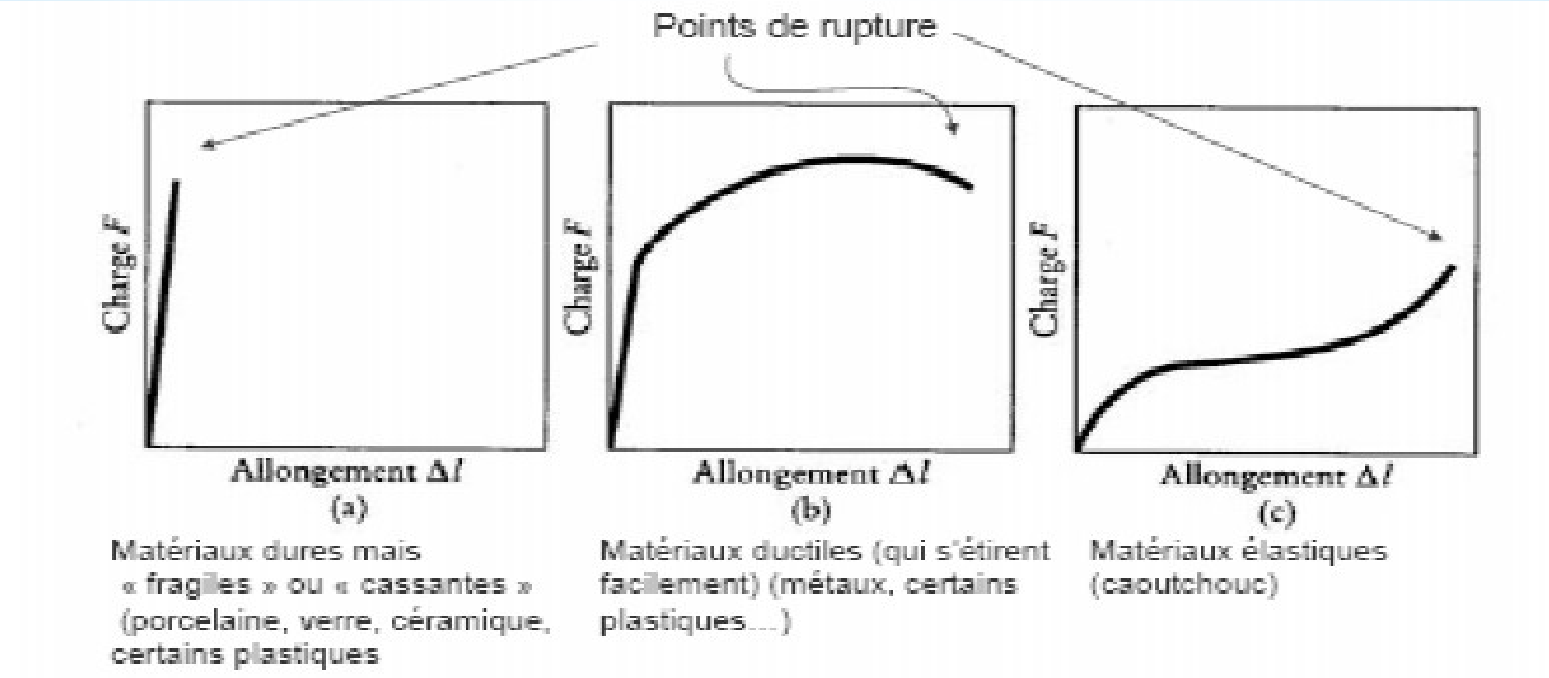
2- Ductile and fragile materials

-The behaviour of steel can be described in a stress-strain diagram composed of two phases: elastic behaviour + plastic behavior

- → **Ductile behavior**

- For glass, once the elastic limit is exceeded, breakage occurs:

- → **Fragile behaviour**



Stress-strain diagram for fragile, ductile and elastic materials

3- Admissible stress - Concept of safety factor

$$\sigma = \frac{F}{S} \leq \sigma_{ad} = \frac{\sigma_e}{f_s}$$

σ_{ad} : stress above which the component suffers deterioration of its mechanical and dimensional characteristics, or even breakage

- σ_e is determined experimentally.

For safety reasons, the stress must remain below an admissible limit stress.

σ_e : elastic limit

f_s : safety factor.

Exple: f_s =1,5 (concrete) ; f_s =1,15 (steel)

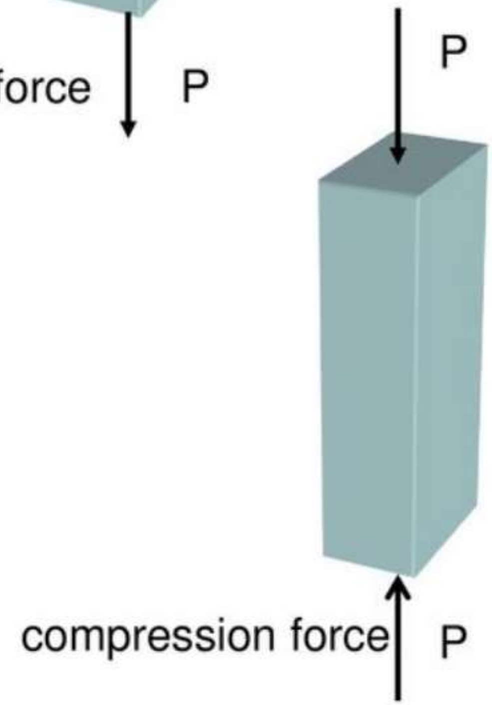
1. Stress

Tensile stress = $\dfrac{\text{force (pull)}}{\text{Cross-sectional area}}$

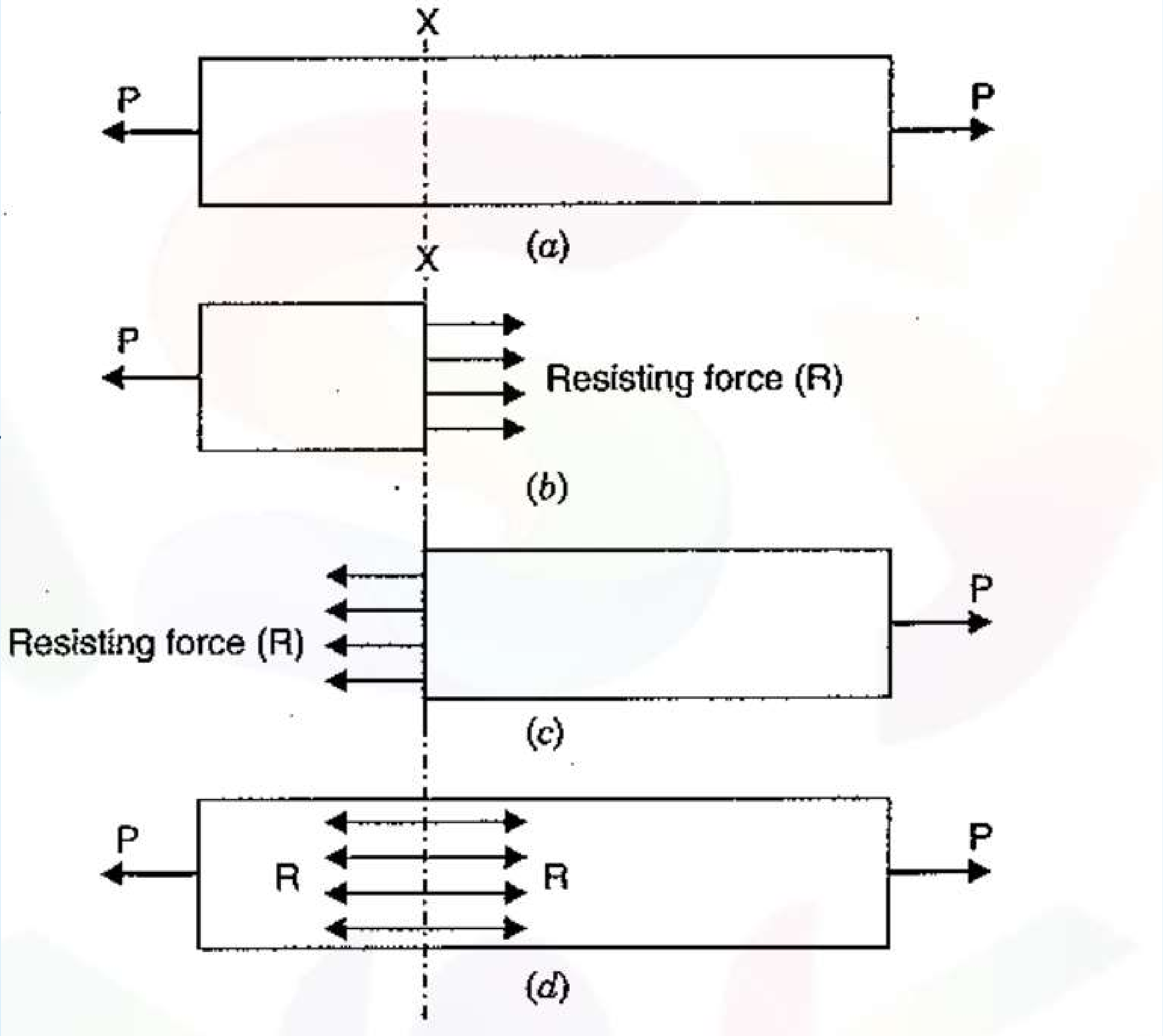
$$\sigma_t = \frac{P}{A}$$

Compression stress = $\dfrac{\text{force (push)}}{\text{Cross-sectional area}}$

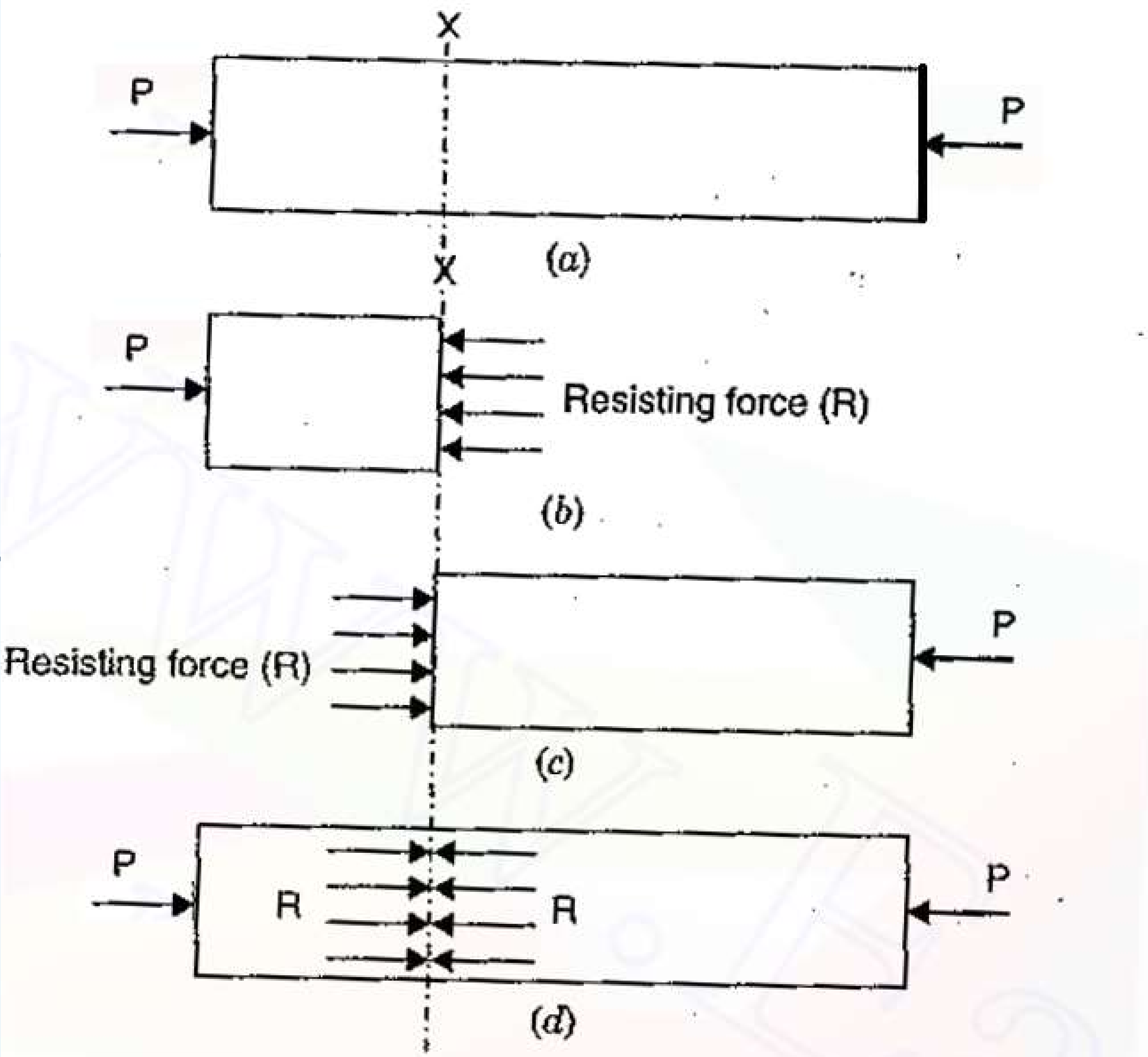
$$\sigma_c = \frac{P}{A}$$



Tensile stress



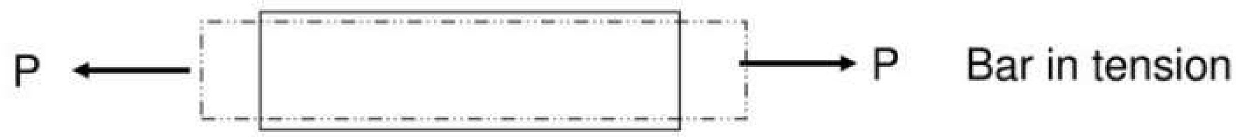
Compression stress



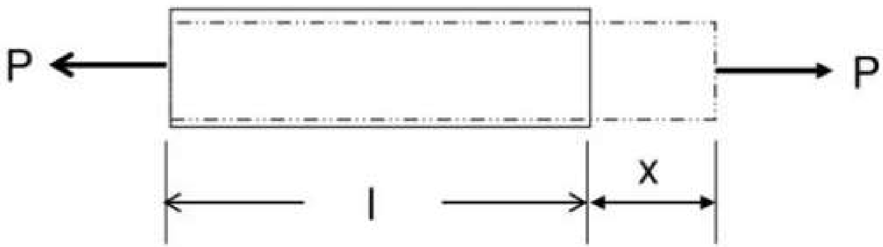
2. Strain

$$\text{strain} = \frac{\text{change in length } (x)}{\text{original length } (l)}$$

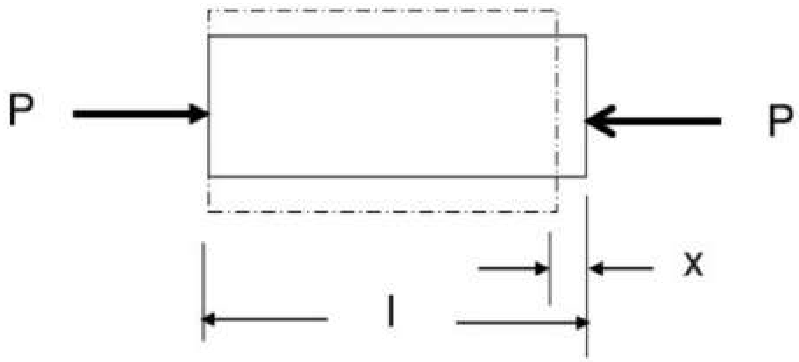
$$\varepsilon = \frac{x}{l}$$



$$\text{tensile strain } \varepsilon_t = \frac{x}{l}$$



$$\text{compressive strain } \varepsilon_c = \frac{x}{l}$$



3. Hook's law, Principal of superposition

Hook's law

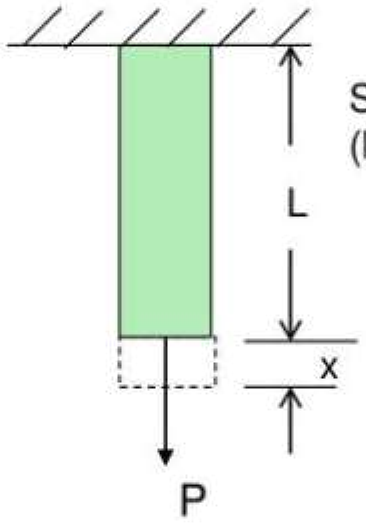
stress (load) α strain (extension)

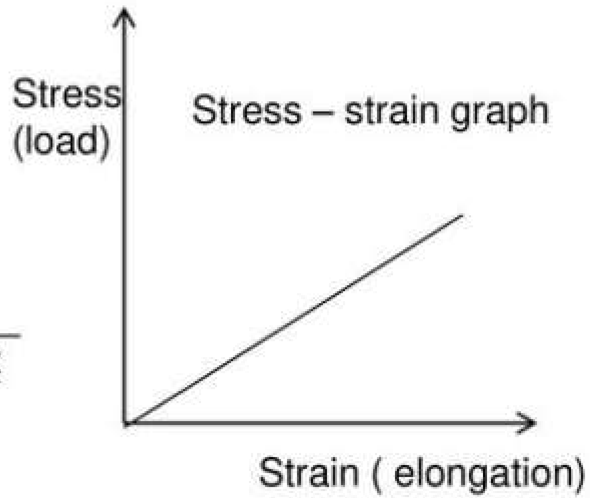
$$\text{mod}\,ulus\ of\ elasticity = \frac{stress}{strain}$$

young's module $E = \dfrac{\sigma}{\varepsilon}$

Substituting $\sigma = P / A$ and $\varepsilon = x / L$

$$x = \frac{P\,L}{A\,E}$$

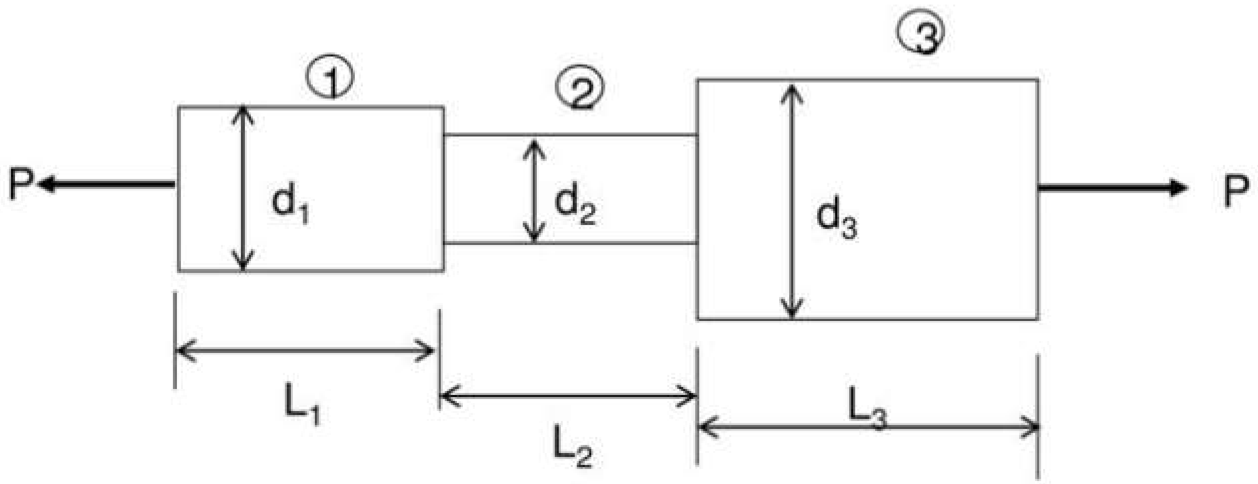




Principal of superposition

The effect of a system of forces acting on a body is equal to the sum of the effects these same forces applied.

4. Varying cross-section and loads



Loads $P = P_1 = P_2 = P_3$

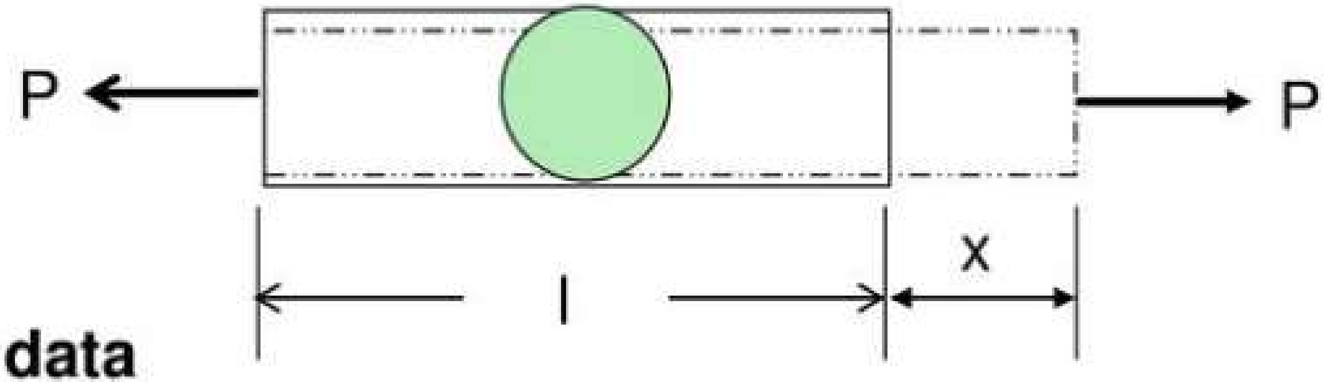
$P = \sigma_1 A_1 = \sigma_2 A_2 = \sigma_3 A_3$

The changes in length $x_1 = \varepsilon_1 L_1$, $x_2 = \varepsilon_2 L_2$, $x_3 = \varepsilon_3 L_3$

$$x_1 = \frac{P L_1}{A_1 E}, \quad x_2 = \frac{P L_2}{A_2 E}, \quad x_3 = \frac{P L_3}{A_3 E}$$

The total changes of length = $x_1 + x_2 + x_3$

Example 1



Given data

Tensile load P = 15,000 N
Steel rod diameter, d = 2 cm

To find

stress

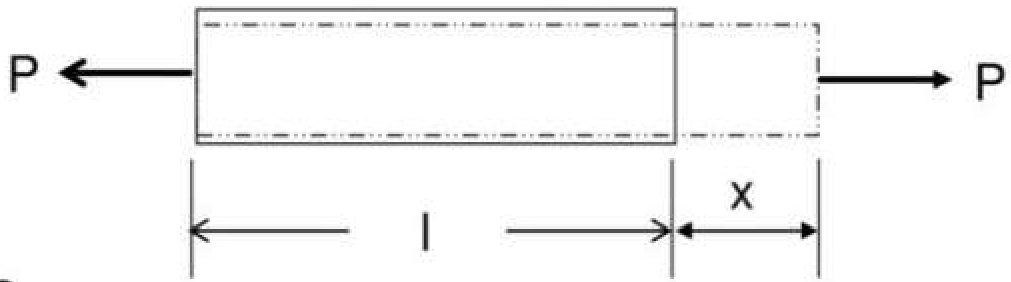
Calculation

$$\sigma_t = \frac{P}{A}$$

$$\sigma_t = \frac{15000}{(\pi/4) \times 2^2}$$

$$= 4777 \quad N/cm^2$$

Example 2



Given data

A wire , Tensile strain ε = 0.0002

Elongation x = 0.75 mm

To find

Length of wire l

Calculation

$$\varepsilon = + \frac{x}{l}$$

$$+0.0002 = + \frac{0.75 \text{ mm}}{l}$$

$$l = 3750 \text{ mm}$$

$$\text{length of wire} = 3.75 \text{ m}$$

- Exercise 1:

A steel bar with a 10 mm square cross-section is subjected to a tensile force of 10 kN. Knowing that the modulus of elasticity of steel is 200 GPa, calculate the deformation of the bar.

Solution exercise 1

- The cross-sectional area of the bar is $A = 10 \times 10 = 100\ mm^2 = 0.0001\ m^2$.
- The force applied is F = 10 kN = 10 000 N.
- The modulus of elasticity of the steel is $E = 200\ GPa = 200 \times 10^9\ Pa$.
- The deformation of the bar is given by the following formula:
- $\varepsilon = F / (A \times E)$
- $\varepsilon = 10{,}000 / (0.0001 \times 200 \times 10^9) = 0.0005$
- The deformation of the bar is therefore 0.0005.

- **Exercise 2:**

- A steel beam of 3 meters long and 20 mm square, is subjected to a compression force of 50 kN. Knowing that the modulus of elasticity of steel is 200 GPa, calculate the elongation of the bar.

Solution exercise 2

- The beam cross-section is A = 20 x 20 = 400 mm² = 0.0004 m².

  The applied force is F = 50 kN = 50,000 N.

  The modulus of elasticity of the steel is E = 200 GPa = 200 x 10^9 Pa.

  The initial length of the beam is L = 3 m.

  The deformation of the beam is given by the following formula:

  $\varepsilon = F / (A \times E)$

  $\varepsilon = 50\,000 / (0.0004 \times 200 \times 10^9) = 0.00625$

  The deformation of the beam is therefore 0.00625.

  The change in beam length is then given by the formula:

  $\Delta L = \varepsilon \times L$

  $\Delta L = 0.00625 \times 3 = 0.01875$ m

  The change in beam length is therefore 0.01875 m.

- **Exercise 3:**

- A steel bar of 2 meters long and 15 mm square is subjected to a compresive force of 80 kN. Knowing that the modulus of elasticity of steel is 200 GPa, calculate the elongation of the bar.

Solution exercise 3

The cross-sectional area of the bar is $A = 15 \times 15 = 225\ mm^2 = 0.000225\ m^2$.

The force applied is $F = 80\ kN = 80{,}000\ N$.

The modulus of elasticity of the steel is $E = 200\ GPa = 200 \times 10^9\ Pa$.

The initial length of the bar is $L = 2\ m$.

The elongation of the bar is given by the following formula:

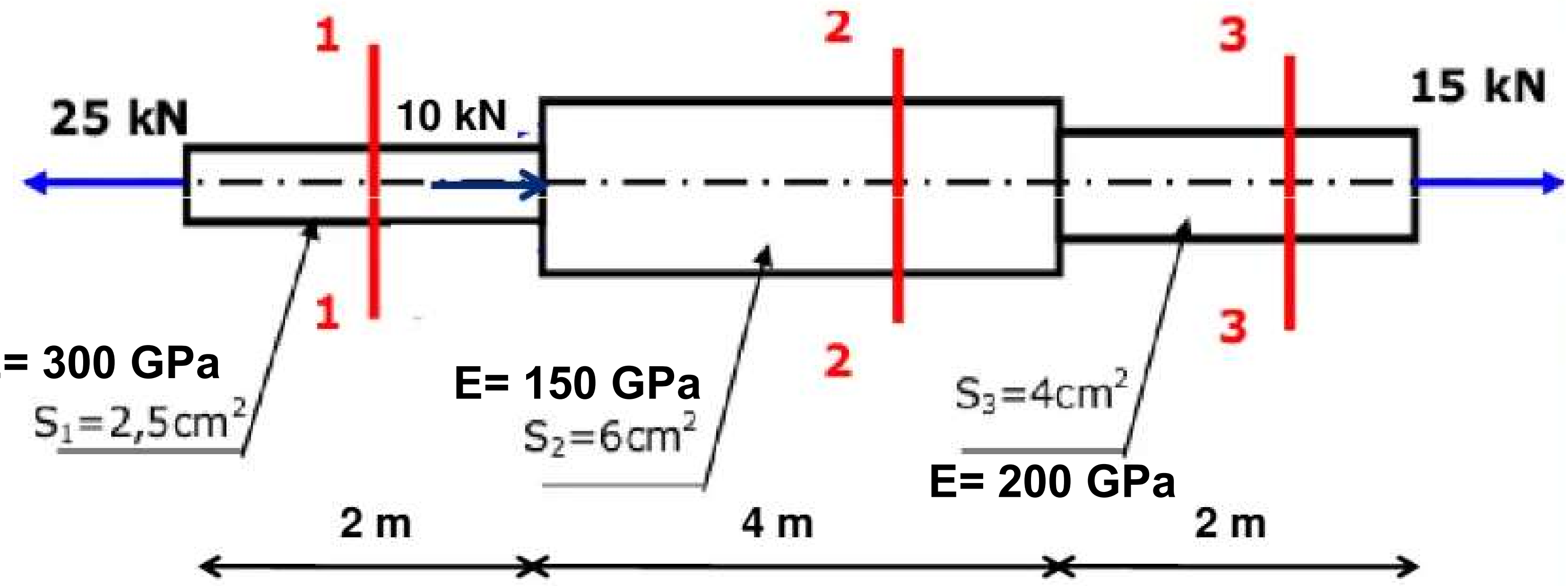
$\Delta L = (F \times L) / (A \times E)$

$\Delta L = (80{,}000 \times 2) / (0.000225 \times 200 \times 10^9) = 0.007111\ m$

The elongation of the bar is therefore 0.007111 m.

Exercice 4

- Either the bar shown in the diagram below
- Calculate the stresses at the sections 1-1, 2-2 and 3,3
- Calculate the elongations in the three segments of the bar as well as the total deflection



Exercise 5

- Calculate the forces and stresses in the bar subjected to a tensile force

- Draw the diagram of forces and stresses