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Course Compendium

Vibrations of discrete Systems with Solved Exercises

Designed for second-year Bachelor's students in Telecommunications

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Glossary

α : Damping coefficient

γ : Damping factor

D : Dissipation function

DOF : Degree of Freedom

E_t : Total energy of the system

f_0 : Natural frequency

$F(t)$: External force applied

i : current

$j_{/\triangle}$: Moment of inertia

k : Stiffness (spring constant)

L : Lagrangian

m : Mass

p : transition matrix.

q : Charge

q_i : Generalized coordinates

Q : Quality factor (inverse of damping)

SHM : Sinusoidal Harmonic Motion

T_0 : Natural period

T : Kinetic energy

U : Potential energy

$\overrightarrow{V}$, eigenvector

ω_0 : Natural angular frequency

ω : Damped angular frequency

Ω_r : Resonant angular frequency

δ : Logarithmic decrement

x : Displacement

$\dot{x}$: Velocity

$\ddot{x}$: Acceleration

Preface

This handout is designed for second-year students in the L2 Telecom program and aims to provide a clear and structured educational resource on the topic of mechanical vibrations. It serves as a supplement to the lectures and tutorials, offering both essential theoretical reminders and a selection of solved exercises, tailored to the level and needs of the program.

The first section presents the fundamentals of vibratory behavior in mechanical systems : single-degree-of-freedom oscillators, multi-degree-of-freedom systems, natural frequencies, resonance, damping, and more. These theoretical reminders are concisely formulated and supported by clear diagrams and precise mathematical formulations.

The second section is dedicated to solved exercises, aimed at reinforcing the concepts and illustrating standard resolution methods. The exercises have been chosen for their clarity, gradual progression, and relevance to the L2 Telecom curriculum.

This document aims to help students better understand the vibratory dynamics of systems, a critical skill for fields such as mechatronics, embedded systems, and telecommunication technologies sensitive to vibrations.

I hope this resource will be useful and serve as a reference for anyone interested in the study of vibrational phenomena. Readers opinions and comments are welcome.

General Introduction

In the field of telecommunications, vibration phenomena play a fundamental role in the modeling and operation of many physical devices. Elements such as electronic oscillators, mechanical resonators (quartz, MEMS), and passive filters use the principles of vibration to generate, stabilize, or modulate signals.

Understanding vibrations means learning to analyze the oscillatory motions that can appear in a system, predict their behavior, and develop solutions to control them. Indeed, in modern telecommunications networks, whether wired or wireless. It is essential to take the effects of vibrations into account, whether for the stability of a relay antenna, the reliability of a satellite, or the proper functioning of an optical system.

This handout aims to introduce the fundamental concepts of vibrations and oscillations, starting with the mathematical formulation of motions within the framework of Lagrange's equations and progressing to increasingly complex systems.

The first chapter focuses on introducing Lagrange's equations, a powerful mathematical tool used to describe the motion of dynamic systems. This approach is particularly useful for modeling complex systems, where Newton's laws would be difficult to apply directly. This chapter aims to familiarize students with the formulation of Lagrange's equations, which is essential for analyzing vibrations in both simple and complex systems.

The second chapter introduces free systems with one degree of freedom. We will analyze simple oscillators without damping or external excitation, where the motion is dictated solely by the initial conditions and properties of the system.

In the third chapter, we introduces damping in vibratory systems, focusing on how it dissipates mechanical energy and influences oscillatory behavior. The three main types of damping, underdamped, critically damped, and overdubbed are analyzed.
Methods for solving the equations of motion with damping are presented, emphasizing the role of damping in shaping the system's dynamic response and its importance in

practical engineering applications.

The fourth chapter explores the behavior of single degree of freedom systems under external periodic forces. It covers steady-state and transient responses, with emphasis on resonance, a condition where the excitation frequency matches the natural frequency, potentially causing large oscillations. Analytical methods are used to solve the forced vibration equations and interpret the system's frequency response.

The final chapter introduces systems with two degrees of freedom, including their modeling and analysis. We derive the coupled equations of motion and solve for natural frequencies and normal modes. The concept of modal analysis is introduced to understand how masses oscillate individually or together. This chapter forms the basis for understanding more complex, multi-degree-of-freedom systems.

Each chapter begins with a clear presentation of the learning objectives and the prerequisites required for a proper understanding of the content. To help students prepare for assessments, solved exercises are provided at the end of each chapter.

It should be emphasized that this document is by no means a substitute for in-person tutorial sessions TD. As with any material containing corrected exercises, the solutions are most beneficial to students who actively engage with the content, take the time to reflect, and attempt to solve the problems on their own before consulting the answers.

I hope that this collection, focused on the topic of vibrations, will serve as an effective tool for revision and learning for the majority of students.

Chapter I

Introduction to Lagrange's Equations

Chapter 1

Introduction to Lagrange's Equations

1.1 Introduction

The study of physical motion is traditionally grounded in Newton's laws. While Newtonian mechanics provides a clear and intuitive framework by applying the second law of motion to each particle in a system, this method quickly becomes impractical when dealing with complex systems. Situations involving multiple bodies, constraints, or internal forces that are difficult to model directly highlight the limitations of the Newtonian approach.

To address these challenges, analytical mechanics offers a more generalized and powerful formulation one of its most fundamental tools being Lagrange's equations. Developed in the $18th$ century by Joseph-Louis Lagrange's, this method approaches mechanics from an energy-based perspective, using two key quantities : The kinetic energy of the system, T, and the potential energy U,which are combined into a single function called the Lagrangian $L = T - U$.

One of the major advantages of Lagrangian mechanics lies in the use of generalized coordinates q_i, which are chosen to best describe the configuration of the system, taking into account any constraints. This allows us to reduce the number of variables and write the equations of motion in a unified and elegant form. Moreover, this formalism is not limited to mechanical systems : it extends naturally to electrical systems as well, such as LC circuits, which behave analogously to mass-spring oscillators.

This chapter aims to :

* Introduce the core concepts of analytical mechanics : generalized coordinates, degrees of freedom, and energy functions.
* Derive Lagrange's equations from the principle of least action or d'Alembert's principle.
* Apply the method to representative systems such as the horizontal mass-spring system, the simple pendulum, and systems with damping or external forcing.
* Provide the foundation for analyzing more complex vibrating systems, which play a crucial role in telecommunications, including filters, resonators, wave propagation, and more.

1.2 Basic Definitions

1.2.1 Oscillation

motion of a body that moves alternately from one side of an equilibrium position to the other.

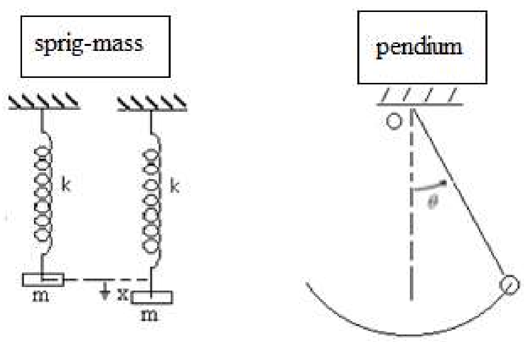


Figure I.1 : Examples of oscillatings systems.

1.2.2 Vibration

Example 1 *A wave is defined as the propagation of a disturbance. It is therefore a phenomenon that involves both space and time.*
** Electromagnetic wave : an oscillating disturbance of electric and magnetic fields that propagates through space, carrying energy in the form of electromagnetic radiation.*
** Sound wave = acoustic vibration : a pressure wave created either by the vibration of a membrane (e.g., a loudspeaker vibrating the air) or by the vibration of a string (e.g., a violin) that propagates through a medium.*

1.3 Periodic motion and its characteristic

1.3.1 Period

Vibrations or oscillations can occur in physical quantities of various kinds : pressure, temperature, velocity, acceleration, electric charge, or electric voltage.

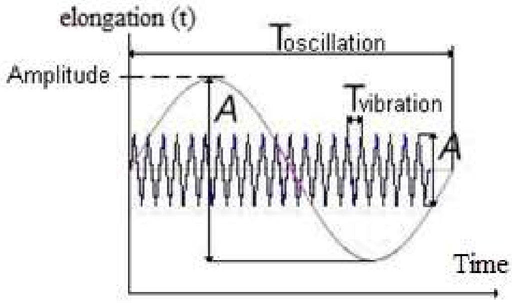


Figure I.2 : Difference between oscillation and vibration.

In all these cases, the physical phenomenon repeats itself identically at equal successive time intervals this repetition is called a cycle. By definition, these phenomena are called periodic (or cyclic), and the duration of one complete repetition (one cycle) is called the period of the motion. It is denoted by T and is expressed in the International System of Units (SI) in seconds (s).
For a rotational motion, one full turn constitutes a cycle.

For an oscillatory motion, the duration of one complete oscillation (forward and return) is the period.

* **Period (T)** : Duration of one complete cycle of a periodic phenomenon.

* **Frequency (f)** : Number of cycles per second. Expressed in $hertz(Hz)$: $f = \frac{1}{T}$.

* **Amplitude :** Maximum value of the oscillating quantity from its equilibrium position.

* **Angular frequency** (ω) : Measures how fast an oscillation occurs, in radians per second denoted by ω, related to frequency by $\omega = 2\pi f$.

1.4 Harmonic sinusoidal oscillation

A sinusoidal motion, also called sinusoidal oscillation or simply harmonic oscillation, is a type of periodic motion that follows a sinusoidal path over time. It is characterized by a regular and repetitive variation around an equilibrium position (Figure I.3). It is of form $x(t) = \sin(\omega_0 t + \varphi)$ .With :

φ : the phase shift relative to the origin of time (rad).

A : the maximum amplitude of the signal.

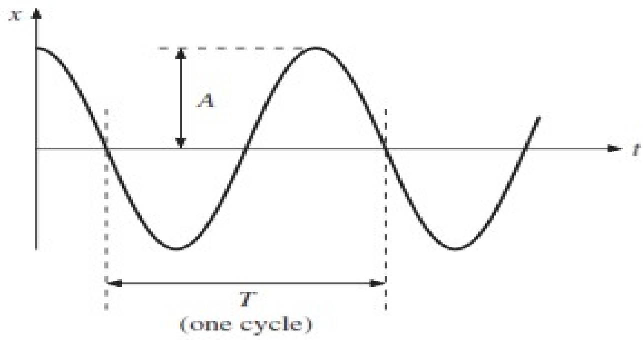


Figure I.3 : Form of sinusoidal motion.

1.5 Lagrange Method

The Lagrange method is a powerful analytical mechanics technique used to derive the equations of motion of a physical system based on its energies : kinetic energy and potential energy. It is especially useful for complex systems or systems with multiple

degrees of freedom, common in mechanics, robotics, and vibration analysis.

The Lagrange method appeared to be a more efficient means (calculations are faster) whatever the difficulty of the system considered for obtaining the equations of motion.

1.5.1 Case of conservative systems

The total energy of a mechanical system is the composition of $E_t = E_{kinetic} + E_{potential} = T + U$.

A system is conservative when there is no exchange of energy between it and the outside world. That is to say, it is not subject to any frictional force (loss) or excitation force (maintenance). The total energy of a conservative system is constant over time :

$$\frac{dEt}{dt} = 0 \qquad \forall t \tag{1.1}$$

To study the motion of a body, that is to say to find the mathematical equations (differentials) which govern the motion, it is first necessary to locate its position, using a system of axes in space.

In classical mechanics, a material point can be fully described in physical space at any given moment by three parameters : x, y, and z. These spatial coordinates represent the point's position and are referred to as its degrees of liberty.

However, analytical mechanics shows that it is often more advantageous to use a set of parameters q_i, called generalized coordinates, greater than three to describe the instantaneous configuration of a system. Here, $i = 1, 2, ..., n$. We then say that the system evolves in an n dimensional space.

Then the principle of the Lagrange method is one of the methods which allows us to characterize a motion by a Lagrangian formulation which produces a system of differential equations associated with the motion.

We define a mathematical function (whose variables are the generalized coordinates) called the Lagrange function L or "Lagrangian" given by :

$$L = E_c - E_p = T - U \tag{1.2}$$

When a system in motion does not exchange energy with the exterior it is conservative (conservation of its energy), we show in this (ideal) case that we obtain the n equations of motion thanks to :

$$\text{Lagrange equation : } \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad (i = 1, 2, \quad , n) \tag{1.3}$$

n = number of degrees of freedom = number of generalized coordinates – number of relationships between these coordinates.

Understanding Generalized Coordinates

In analytical mechanics, generalized coordinates are variables $q_1, q_2, ..., q_n$ chosen to uniquely define the configuration of a system, regardless of the type of motion or the constraints involved. Unlike the standard Cartesian coordinates x, y, z,which always represent positions in space, generalized coordinates can be **angles**, **distances**, or **any other parameter** that simplifies the description of the system.

1.5.2 Velocity-dependent damping forces

In reality, systems are naturally subjected to friction. When an oscillatory system is subjected to fluid friction forces of the form :

$$f = -\alpha v \tag{1.4}$$

α : viscous damping coefficient.
v : velocity of motion.
We show that we obtain the n equations of motion of this non-conservative system thanks to :

$$\text{Lagrange equation : } \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = 0, \qquad (i = 1, 2, \quad , n) \tag{1.5}$$

with :

L : the Lagrangian function $= T - U$.

D : dissipation function $= \dfrac{1}{2}\beta v^2$

n : number of degrees of freedom.

1.5.3 Case of a time-dependent external force (complete method)

In a vibrating system, an excitation force is an external force applied to induce or sustain oscillations. This force performs mechanical work on the system, meaning it supplies energy to it. This energy can compensate for losses due to friction (damping) and help maintain the oscillatory motion.

When an oscillatory system, in addition to being subjected to frictional forces derived from a dissipation function D, is excited (maintained) by a time-dependent external force F (t). We show that the n equations of motion of this non-conservative system are obtained using :

$$\text{Lagrange equation : } \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = \frac{\partial W}{\partial q_i}, \qquad (i = 1, 2, \quad , n) \qquad (1.6)$$

with :

L : the Lagrangian function $= T - U$.

D : dissipation function $= \int f dv = \int \beta v dv = \frac{1}{2}\beta v^2$

W : The work done by the excitation force $= \int F dx = Fx$.

1.6 Conclusion

Lagrange's equations provide an elegant and powerful approach for analyzing the motion of mechanical systems, especially when they involve constraints or non-Cartesian coordinates. By formulating problems in terms of the system's energies and generalized coordinates, this method allows for dynamic modeling without directly involving forces. This introduction lays the foundation for exploring more complex cases, such as damped or forced vibratory systems, which will be covered in the following chapters.

1.7 Corrected exercises

1.7.1 Exercise 1

A vibratory motion is characterized by the following displacement :

$$x(t) = 10\sin(50t + \pi/2)$$

Or x in centimeters (cm), t in seconds (s) and the phase in radians (rad).

1. Determine the maximum amplitude, the frequency and period of the motion.

2. Give the angular frequency,

3. Express the initial phase (phase shift at the origin).

4. Calculate the displacement x and the speed v at time $t = 0s$.

Solution

1. The equation $x\left(t\right) = 10\sin\left(50t + \dfrac{\pi}{2}\right)$ is of the form $x\left(t\right) = A\sin\left(\omega t + \varphi\right)$,so :

1. The maximum amplitude is $A = 10cm$,

2. The angular frequency is $\omega = 50rad/s$, $\quad \boldsymbol{\omega} = 2\pi f \Rightarrow f = \omega/2\pi = 50/2 \times 3.14 = 7.96$ Hz.

$T = 1/f = 1/7.96 = 0.125s$.

3. phase shift $\varphi = \pi/2$.

4. motion, velocity :

A $t = 0$, $x(0) = A\sin\varphi \Longrightarrow x = 10\sin\pi/2. \Longrightarrow x = 10cm.$

$v = \dfrac{dx}{dt} \Longrightarrow v = \omega A\cos\left(\omega(0) + \varphi\right) \Longrightarrow v = 0cm/s.$

1.7.2 Exercise 2

A mass-spring mechanical system oscillates harmonically with an amplitude $A = 2cm$ and a frequency $f = 1Hz$. At time $t = 0$, its displacement is at a maximum. Given that the mass is $m = 10kg$, calculate the kinetic energy, potential energy, and total energy of this oscillator at time $t = 2s$.

Solution

* **Kinetic energy**

$E_C = \frac{1}{2}mv^2$

The displacement at $t = 2s$ is given by :

$$x(2) = A\cos(\omega t) = 0.02\cos 2\pi 2 = 0.02\cos 4\pi = 0.02m$$

The Velocity at $t = 2s$:

$$v(2) = -A\omega\sin(\omega t) = -0.02 \times 2\pi\sin(4\pi) = 0$$

$$E_C = \frac{1}{2}mv^2 = 0.$$

* **Potential energy**

$E_P = \frac{1}{2}kx^2$

for a harmonic oscillator, we have :

$\omega = \sqrt{\frac{k}{m}} \longrightarrow k = \omega^2 m = (2\pi f)^2 m = 4\pi^2 \times 10 = 394.8N/m$

$$E_P = \frac{1}{2}kx^2 = \frac{1}{2} \times 394.8 \times (0.02)^2 = 0.079J$$

.

$$E_T = E_C + E_p = cste = 0.079J.$$

1.7.3 Exercise 3

We consider a material point constrained to move along a circle of radius R and center O, lying in the XOY plane.

1) Translate the constraint into one or more mathematical equations.

2)-What generalized coordinates can be used to locate this point ?

3)-What is the number of degrees of freedom of this point ?

Solution

1) Since the point must remain on the circle, its coordinates (x, y) must satisfy the equation of a circle :

$$x^2 + y^2 = R^2$$

This is the mathematical expression of the constraint.

2) The number of degrees of freedom of a mechanical system is given by the formula :

Degrees of freedom=Number of generalized coordinates-Number of constraint equations.

Possible generalized coordinates in the XOY plane : $x, y \longrightarrow 2$ coordinates.

Constraint equation :

$x^2 + y^2 = R^2 \longrightarrow 1$, Therefore :

$$\text{Number of degrees of freedom} = 2 - 1 = 1$$

1.7.4 Exercise 4

Determine for each system shown below, the number of degrees of freedom (DOF), the kinetic energy and the potential energy.

Solution

For the first system

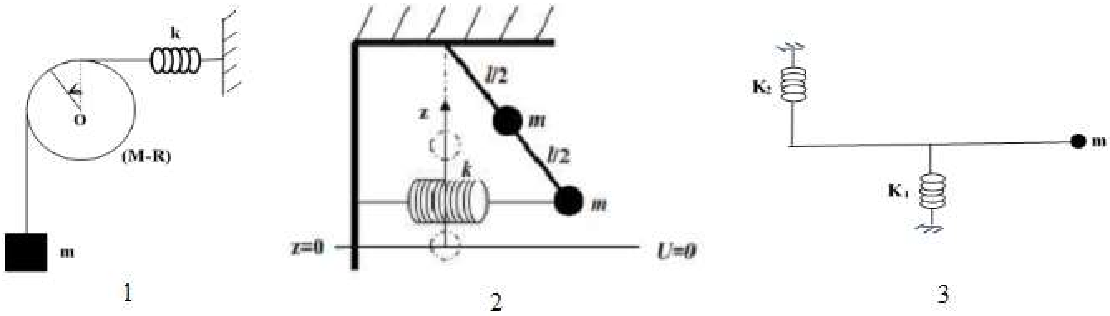
* **The number of DOF**

$d = N - r =$ number of generalized coordinates - number of relationships between generalized coordinates.

- Pulley of mass M have a rotational motion (θ)

- Body of mass m have a translational motion $(x) \rightarrow N = 2$.

- Moreover, we have $x = r\theta \longrightarrow r = 1 \Rightarrow d = 2 - 1 = 1$.



* **Kinetic energy and potential energy.**

$E_{cT} = E_{cm} + E_{cM}$.

$E_{cm} = \frac{1}{2}m\dot{x}^2$ with $x = r\theta \Rightarrow \dot{x} = r\dot{\theta}$

$E_{cM} = \frac{1}{2}j\dot{\theta}^2$ with $j = \dfrac{1}{2}MR^2$

$E_{cT} = \frac{1}{2}m\dot{x}^2 + \dfrac{1}{4}MR^2 \Longrightarrow E_{cT} = \dfrac{1}{2}R^2\left(m + \dfrac{1}{2}M\right)\dot{\theta}^2$.

$E_{pT} = E_{pm} + E_{pM} + E_{pr} = E_{pr} = \dfrac{1}{2}kx^2 = \dfrac{1}{2}r^2\theta^2$.

For the second system

Two identical masses attached respectively to the middle and to the end of a bar of negligible mass.

m have a rotation motion $(\theta) \longrightarrow r = 0 \Longrightarrow d = N - r = 1 - 0 = 1$

* **Kinetic energy and potential energy**.

$E_c = E_{cm} + E_{cm}$.

$E_c = \frac{1}{2}j_1\dot{\theta}^2 + \frac{1}{2}j_2\dot{\theta}^2$ with $j = ml^2$

$E_c = \dfrac{1}{2}m\left(\dfrac{l}{2}\right)^2\dot{\theta}^2 + \dfrac{1}{2}ml^2\dot{\theta}^2$.

$$E_c = \frac{5}{8}ml^2\dot{\theta}^2.$$

$E_{pT} = E_{pm} + E_{pm} + E_{pr} = mgh_1 + mgh_2 + \frac{1}{2}kx^2.$

with $\left\{ \begin{array}{c} h_1 = l - l\cos\theta \\ h_2 = l - \frac{l}{2}\cos\theta \\ x = l\sin\theta \end{array} \right\} \Longrightarrow$

$$E_p = mgl(1 - \cos\theta) + mgl(1 - \frac{1}{2}\cos\theta) + \frac{1}{2}k\left(l\sin\theta\right)^2.$$

For the third system

The number of degrees of freedom is equal to the number of generalized coordinates minus the number of constraint equations.

This system involves only rotational motion, and thus has one degree of freedom associated with the angular coordinate.$\longrightarrow d = 1$.

* **Kinetic energy and potential energy**

$E_c = E_{cm} = \frac{1}{2}j\dot{\theta}^2$

$$E_c = \frac{1}{2}m\left(l\right)^2\dot{\theta}^2.$$

$E_p = E_{p(spring1)} + E_{p(spring2)} = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2$ with $\longrightarrow \left\{ \begin{array}{c} x_1 = \frac{L}{2}\sin\theta \\ x_2 = L\sin\theta \end{array} \right\}$

$E_p = \frac{1}{2}k_1\left(\frac{L}{2}\sin\theta\right)^2 + \frac{1}{2}k_2\left(L\sin\theta\right)^2$

For small oscillations where $\theta << 10^o \longrightarrow \sin\theta \simeq \theta \Longrightarrow E_p = \frac{1}{2}k_1\left(\frac{L}{2}\theta\right)^2 + \frac{1}{2}k_2\left(L\theta\right)^2$

$$E_p = \frac{1}{2}L\left[\frac{1}{4}k_1 + k_2\right]\theta^2.$$

Chapter II

Free oscillations of single degree of freedom systems

Chapter 2

Free oscillations of single degree of freedom systems

2.1 Introduction

When the oscillating magnitude of a periodic phenomenon varies sinusoidally with time, the phenomenon is said to be sinusoidal. Then periodic phenomena (oscillation, vibration or wave) are studied using Fourier analysis (mathematics) which decomposes these periodic motions into a sum of sinusoidal functions.

A **free oscillator** is a system that oscillates in the **absence** of any exciting force. An oscillatory motion is said to be rectilinear with **single degree of freedom** when it occurs in a single direction in space, and knowledge of a single position variable is sufficient to determine its position. In the physical world, many systems exhibit vibratory or oscillatory motion, that is, periodic motion. Common examples include a swinging pendulum, a plucked guitar string, or a car bouncing on its springs. The simplest and most fundamental form of this type of motion is called Simple or Sinusoidal Harmonic Motion (**SHM**).

In this chapter we develop quantitative descriptions of SHM. We obtain equations for the ways in which the displacement, velocity and acceleration of a simple harmonic oscillator vary with time and the ways in which the kinetic and potential energies of the oscillator vary. To do this we discuss two particularly important examples of SHM : a mass oscillating at the end of a spring and a swinging pendulum. We then extend our discussion to electrical circuits and show that the equations that describe the movement of charge in an oscillating electrical circuit are identical in form to those that describe, for example, the motion of a mass on the end of a spring. Thus if we understand one type of harmonic oscillator then we can readily understand and analyse many other types. The universal importance of SHM is that to a good approximation many real oscillating systems behave like simple harmonic oscillators when they undergo oscillations of small amplitude. Consequently, the elegant mathematical description of the simple harmonic oscillator that we will develop can be applied to a wide range of physical systems.

2.2 The Simple harmonic Motion

A harmonic oscillator is an oscillator that is returned to its equilibrium position when moved a certain distance due to a restoring force opposing the motion.
A rectilinear motion is said to be **sinusoidal** when the abscissa x or the elongation of the mobile is a sinusoidal function of time :

$$x\left(t\right)=x_m\sin\left(\omega t+\varphi\right) \tag{2.1}$$

x_m : maximum amplitude or elongation of motion (m)
$\omega=\dfrac{2\pi}{T}=2\pi f$: The angular frequency and is expressed in radians per second.
$\omega t+\varphi$: phase of the motion at time t (rad) or argument in complex notation
φ : phase factor or phase shift (rad) or phase at the origin, which depends on initial conditions.

Remark 1 *We can also write that* $x\left(t\right)=x_m\cos\left(\omega t+\varphi'\right)=x_m\sin\left(\omega t+\varphi\right)$ *with* $\varphi'=\varphi+\dfrac{\pi}{2}$.

These two writings are equivalent to within a phase factor $=\frac{\pi}{2}$.
This equation describes a pure harmonic oscillation. The system oscillates around its equilibrium position with constant amplitude and frequency, provided no damping or external force is present. Understanding this simple motion is essential before studying more complex cases involving damping or external excitation.

2.2.1 velocity and acceleration in simple harmonic motion

To determine the velocity of the particle at any instant, we differentiate the position function $x\left(t\right)$,with respect to time t :

$$v(t) = \frac{dx}{dt} = \omega x_m \cos(\omega t + \varphi) \tag{2.2}$$

To find the acceleration, we take the second derivative of the displacement with respect to time :

$$\gamma(t) = \frac{d^2x}{dt^2} = -\omega^2 x(t) \tag{2.3}$$

which gives :

$$\ddot{x}\left(t\right) + \omega^2 x(t) = 0 \tag{2.4}$$

We obtain a second-order linear differential equation with constant coefficients without a second member, characteristic of this type of motion. We will now denote this ω by ω_0 = natural angular frequency of the **SHM** (Sinusoidal Harmonic Motion) which gives :

$$\ddot{x}\left(t\right) + \omega_0^2 x(t) = 0 \tag{2.5}$$

Equation (2.5) is the equation of SHM and all simple harmonic oscillators have an equation of this form. It is a linear second-order differential equation ; linear because each term is proportional to x or one of its derivatives and second order because the highest derivative occurring in it is second order.

2.2.2 A mass on a spring

A mass on a vertical spring

Our first example of a simple harmonic oscillator is a mass on a vertical spring as shown in Figure II.1. This mass moves without friction on the vertical plane.

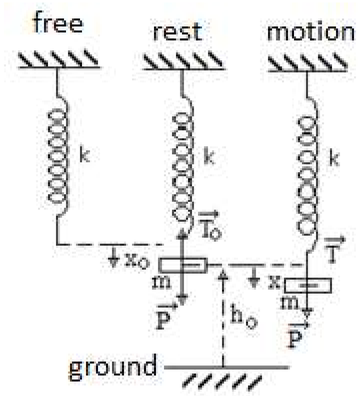


Figure II.1 : An oscillating mass on a vertical spring.

At $t = 0$, this point is moved away from its equilibrium position by a magnitude x and then released without initial velocity. We seek to determine its differential equation of motion. The equation of motion for a conservative system can be determined by :

1. Newton's law

2. Conservation of energy

3. Lagrange's method.

By applying Newton's method - Spring stiffness K (elasticity coefficient) = massless.

- Mass m at rest : spring stretched by x_0 ; x_0= static deformation : the distance traveled by the oscillator.

a) Static equilibrium

$$\sum \overrightarrow{F} = 0 \Longrightarrow \overrightarrow{p} + \overrightarrow{T_0} = 0 \Longrightarrow mg - kx_0 = 0 \tag{2.6}$$

$\overrightarrow{T_0}$: Spring return force proportional to deformation.

b)Motion

$\sum \overrightarrow{F} = m\overrightarrow{\gamma} = \overrightarrow{P} + \overrightarrow{T} \Longrightarrow mg - k(x + x_0) = m\ddot{x} \Longrightarrow -kx = m\ddot{x} \Longrightarrow$

$$\ddot{x} + \frac{k}{m}x = 0 \tag{2.7}$$

of the same type as

$$\ddot{x}(t) + \omega_0^2 x(t) = 0 \tag{2.8}$$

where

$$\omega_0^2 = \frac{k}{m} \tag{2.9}$$

is a constant. The reason for writing the constant as ω_0^2 will soon become apparent but we note that ω_0^2 is equal to the restoring force per unit displacement per unit mass.The solution is a **SHM.**

$$x(t) = x_m \sin(\omega_0 t + \varphi) \tag{2.10}$$

The amplitude of the oscillations of a free harmonic oscillator does not depend on time. Such oscillations are said to be **undamped**.

Remark 2 *A spring deforms during motion, there is no motion of mass, therefore a spring has no kinetic energy (moreover, the mass of a spring is not given in the exercises). The deformation of a spring is either an elongation (longer spring) or a compression (shorter spring) relative to its empty length. The **restoring force** of a spring on a mass is always negative (because it always acts in the opposite direction of the deform.*

Conservation of energy The energy of a harmonic oscillator is the sum of the kinetic (T) and potential (U) energy.

The kinetic energy of a system is the sum of the kinetic energies of the bodies that compose it. There are two forms of kinetic energy : translational kinetic energy and rotational kinetic energy.

* Any mass moving in **translation** with a speed v has kinetic energy $E_c = \frac{1}{2}mv^2$.

* Any mass moving in **rotation** with an angular velocity θ has kinetic energy $E_c = \frac{1}{2} j_{/\triangle} \dot{\theta}$. where $j_{/\triangle}$represents the moment of inertia of the system. The following table presents the standard formulas of the moment of inertia for different regular solids with respect to their principal axes of rotation.

Shape	Moment of inertia
Rod (length L, mass M)	$\frac{1}{2}ML^2$
Cylinder (radius R, mass M)	$\frac{1}{2}MR^2$
Sphere (radius R, mass M)	$\frac{2}{5}MR^2$
point mass	0

Tab II.1 : Moment of inertia of regular solids

* Any mass located at altitude h (height above the ground) has potential energy $E_p = mgh$

*For a spring with a stiffness constant (k) and a deformation (x). The potential energy is given by $E_p = \frac{1}{2}kx^2$

$E_t = E_{kinetic} + E_{potential} = T + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$.

The total energy of a conservative system is constant over time : $\frac{dEt}{dt} = 0 \qquad \forall t \Longrightarrow$

$m\ddot{x}\dot{x} + k\dot{x}x = 0$

$m\ddot{x} + kx = 0$ where

$$\ddot{x} + \frac{k}{m}x = 0 \tag{2.11}$$

$$\ddot{x}\,(t) + \omega_0^2 x(t) = 0 \tag{2.12}$$

Lagrange equation

We consider a vertical mass-spring system, consisting of a **mass** m, a **spring** with stiffness k,as shown in the figure II.1.

The total system energy $E_t = E_c + E_p$

$$E_c = E_c(m) = \frac{1}{2}m\dot{x}^2 \tag{2.13}$$

$$E_p = E_p(s) + E_p(m) = \frac{1}{2}kx^2 + mgh \tag{2.14}$$

where $h = h_0 - x$ (a relationship between h and x).

n =number of coordinates (x, h : therefore 2) - number of relations (1) $= 2 - 1 = 1$ degree of freedom.

The Lagrangian is given by :

$$L = E_c - E_p = \frac{1}{2}m\dot{x}^2 - mg\left(h_0 - x\right) - \frac{1}{2}k\left(x + x_0\right)^2 \tag{2.15}$$

$\dfrac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \Rightarrow \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} = m\ddot{x}$

$\dfrac{\partial L}{\partial x} = mg - k(x + x_0) = mg - kx_0 - kx = -kx \qquad (mg - kx_0) = 0$ equilibrium equation.

Hence the Lagrange equation :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 = m\ddot{x} + kx = 0 \tag{2.16}$$

$$\ddot{x}(t) + \frac{k}{m}x(t) = 0 \tag{2.17}$$

Of the same type as

$$\ddot{x}\left(t\right) + \omega_0^2 x(t) = 0 \tag{2.18}$$

with $\omega_0^2 = \dfrac{k}{m} \Longrightarrow$

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{k}{m}}. \tag{2.19}$$

The solution is a **SHM :**

$$x\left(t\right) = x_m \sin\left(\omega_0 t + \varphi\right) \tag{2.20}$$

x_m and φ are constants deduced from the initial conditions.

A mass on a horizontal spring

Our example of a simple harmonic oscillator is a mass on a horizontal spring as shown in Figure II.2. The mass is attached to one end of the spring while the other end is held fixed. The equilibrium position corresponds to the unstretched length of the spring and x is the displacement of the mass from the equilibrium position along the x-axis. We start with an idealised version of a real physical situation. It is idealised because the mass is assumed to move on a frictionless surface and the spring is assumed to be weightless. Furthermore because the motion is in the horizontal direction, no effects due to gravity are involved.

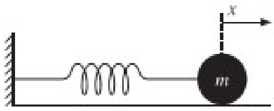


Figure II.2 : A simple harmonic oscillator

For small displacements the force produced by the spring is described by Hooke's law which says that the strength of the force is proportional to the extension (or compression) of the spring, i.e. $F \propto x$ where x is the displacement of the mass. The constant of proportionality is the spring constant k which is defined as the force per unit displacement. When the spring is extended, i.e. x is positive, the force acts in the opposite direction to x to pull the mass back to the equilibrium position.
Similarly when the spring is compressed, i.e. x is negative, the force again acts in the opposite direction to x to push the mass back to the equilibrium position.This situation is illustrated in Figure II.3 which shows the direction of the force at various points of the oscillation. We can therefore write :

$$f = -kx \tag{2.21}$$

where the minus sign indicates that the force always acts in the opposite direction to the displacement. All simple harmonic oscillators have forces that act in this way : (i) the magnitude of the force is directly proportional to the displacement; and (ii) the force is always directed towards the equilibrium position.

At $t = 0$, this point is moved away from its equilibrium position by a magnitude x and then released without initial velocity.

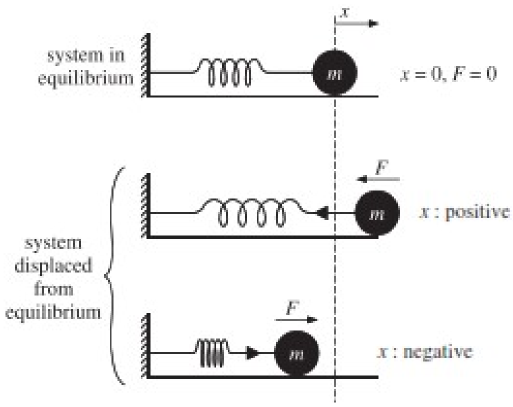


Figure II.3 : The direction of the force acting on the mass m
at various values of displacement.

Equation of Motion The system must also obey Newton's second law of motion which states that the force is equal to mass m times acceleration a,i.e.$F = m\gamma$. We thus obtain the equation of motion of the mass.

$$\sum F = m\ddot{x} \Rightarrow -kx = m\ddot{x} \Rightarrow \ddot{x} + \frac{k}{m}x = 0 \tag{2.22}$$

of the same type as

$$\ddot{x}\,(t) + \omega_0^2 x(t) = 0 \tag{2.23}$$

avec $\omega_0 = \sqrt{\dfrac{k}{m}}$ natural angular frequency of sinusoidal harmonic motion.

In the case of the presence of several springs, we proceed by calculating the equivalent stiffness. The stuffiness are linked either in series or in parallel (Opposition).

* **In series**

The equivalent stiffness for constants $k1$ and $k2$ such that :

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + ... \tag{2.24}$$

* **In parallel (opposition)**

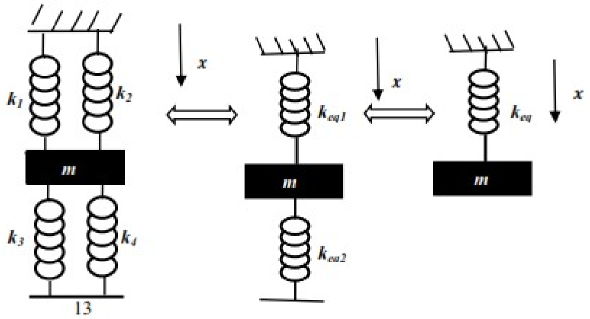


Figure II.4 : springs in parallel and in opposition.

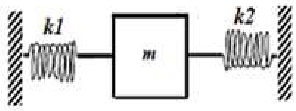


Figure II.5 : Springs in opposition.

Figure II.6 : Series springs

The equivalent stiffness is the sum of stuffiness $k1$ and $k2$ such that :

$$\begin{aligned} k_{eq1} &= k_1 + k_2, \qquad k_{eq2} = k_3 + k_4 \\ k_{eq} &= k_{eq1} + k_{eq2} \end{aligned} \tag{2.25}$$

Lagrange formalism

The Lagrangian of the system is : $L = E_c - E_p$ with
$E_c = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$
$E_p = \frac{1}{2}kx^2$
therefore $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$
Lagrange's equation is given as follows :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{2.26}$$

$$\partial L/(\partial \dot{x}) = m\dot{x} \rightarrow d/dt[\partial L/(\partial \dot{x})] = m\ddot{x} \qquad \partial L/\partial x = -kx$$

Lagrange's equation is given by :

$$m\ddot{x} + kx = 0 \rightarrow \ddot{x} + k/mx = 0 \tag{2.27}$$

of the form

$$\ddot{x}\,(t) + \omega_0^2 x(t) = 0 \tag{2.28}$$

The solution is a sinusoidal harmonic motion **(SHM)**

$$x\,(t) = x_m \sin\left(\omega_0 t + \varphi\right) \tag{2.29}$$

2.2.3 Stability of Equilibrium and Condition for Oscillation

When a mechanical system has one degree of freedom, its equilibrium position is defined by the condition that the resultant force is zero :

$$F\,(x) = .\frac{-dU}{dx} = 0 \tag{2.30}$$

where $U\,(x)$ is the potential energy of the system.

Equilibrium Position

A point x_0 represents an equilibrium position when :

$$\frac{dU}{dx}\,\lfloor_{x=x_0} = 0 \tag{2.31}$$

Nature of Equilibrium Based on Potential Energy

Stable Equilibrium

$$\frac{d^2U}{dx^2}\,\lfloor_{x=x_0} > 0 \tag{2.32}$$

The equilibrium corresponds to a local minimum of $U(x)$.A small displacement produces a restoring force that drives the system back toward x_0. The motion around this position becomes oscillatory.

Unstable Equilibrium

$$\frac{d^2U}{dx^2}\rfloor_{x=x_0}< 0 \tag{2.33}$$

The equilibrium corresponds to a local maximum of $U(x)$. A small disturbance moves the system away from its initial position.

Neutral Equilibrium

$$\frac{d^2U}{dx^2}\rfloor_{x=x_0}= 0 \tag{2.34}$$

The potential energy is constant near x_0 ; no restoring force acts on the system.

Condition for oscillation

For a system to oscillate freely about an equilibrium position, two conditions must be satisfied :

1. The potential energy U(x) must have a local minimum at this point. Mathematically :

$$\frac{dU}{dx} = 0 \qquad \text{and} \; \frac{d^2U}{dx^2} > 0 \tag{2.35}$$

This ensures that any small disturbance generates a restoring force bringing the system back to equilibrium.

2. Possibility of periodic movement around this position :

Around the minimum, the potential energy can be approximated by a parabola :

$$U(x) \approx U(x_0) + \frac{1}{2}k\,(x - x_0)^2 \tag{2.36}$$

where $k = \frac{d^2U}{dx^2}\rfloor_{x=x_0}$is the equivalent stiffness.

This quadratic form of $U(x)$ leads to a harmonic motion of pulsation :

$$\omega = \sqrt{\frac{k}{m}} \tag{2.37}$$

To understand when a mechanical system can oscillate freely around an equilibrium position, let's analyze the fundamental Mass–Spring System.
A mass m attached to a spring with stiffness k, moving without friction along a horizontal line.

Potential Energy :

$$U = \frac{1}{2}kx^2 \tag{2.38}$$

The minimum occurs at $x = 0$. Therfore :

$$\frac{dU}{dx} = 0 \qquad \text{and} \qquad \frac{d^2U}{dx^2} = k > 0 \tag{2.39}$$

This corresponds to a stable equilibrium.

Condition for Oscillation : near $x = 0$,the restoring force is :

$$F = -\frac{dU}{dx} = -kx \tag{2.40}$$

This force always acts to bring the mass back to equilibrium. The resulting motion is simple harmonic :

$$x\left(t\right) = x_m \sin\left(\omega_0 t + \varphi\right) \tag{2.41}$$

with

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{2.42}$$

The mass–spring system oscillates freely because its potential energy has a parabolic minimum near equilibrium.

2.2.4 The simple pendulum

Timing the oscillations of a pendulum has been used for centuries to measure time accurately. The simple pendulum is the idealised form that consists of a point mass m suspended from a massless rigid rod of length l, as illustrated in Figure II.7.

* **Generalized Coordinate**

We define the angle.$\theta(t)$ as the angular displacement from the vertical equilibrium position.

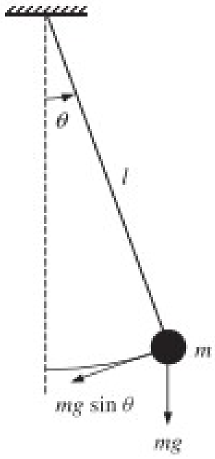


Figure II.7 : Simple pendulum

Equation of Motion

1) Using Newton's Second Law for Rotation

$$\sum \overrightarrow{M} = j_{/\triangle}\overrightarrow{\alpha} \Longrightarrow \overrightarrow{M}_p + \overrightarrow{M}_T = j_{/\triangle}\overrightarrow{\alpha} \tag{2.43}$$

with

$\overrightarrow{M}$: Moment of external forces, is the rotational equivalent of force. It measures the tendency of a force to rotate an object about an axis, pivot, or fulcrum. It is defined as : $\overrightarrow{M}_p = \overrightarrow{F}\overrightarrow{d}$

$\overrightarrow{d}$ is the position vector from the axis of rotation to the point of application of the force.

$\overrightarrow{F}$ is the applied force.

$j_{/\triangle}$: Moment of inertia $= ml^2$

$\overrightarrow{\alpha}$:Angular acceleration

We have : $\overrightarrow{M}_T = 0, \quad \overrightarrow{M}_p = \overrightarrow{F}\overrightarrow{d} = -mgl\sin\theta \Longrightarrow -mgl\sin\theta = ml^2\ddot{\theta} \Longrightarrow$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{2.44}$$

This equation does not have the same form as the equation of SHM, Equation (2.7), as we have $\sin\theta$ on the right-hand side instead of θ. However we can expand $\sin\theta$ in a power series in θ :

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ... \tag{2.45}$$

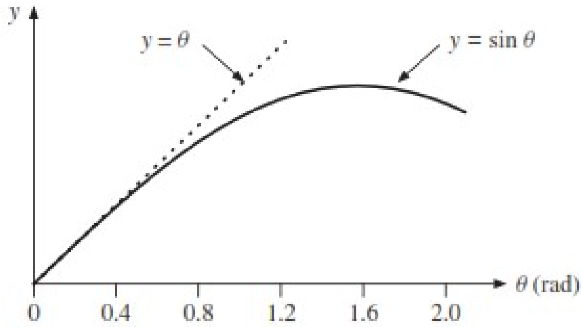


Figure II.8 : A comparison of the functions $y = \theta$ and $y = \sin\theta$ plotted against θ.

For small angular deflections the second and higher terms are much smaller than the first term. For example, if θ is equal to $0.1rad$, which is typical for a pendulum clock, then the second term is only 0.17% of the first term and the higher terms are much smaller still. We can see this directly by plotting the functions $y = \sin\theta$ and $y = \theta$ on the same set of axes, as shown in Figure II.8. The two curves are indistinguishable for values of θ below about $14rad$. Thus for small values of θ, we need retain only the first term in the expansion (2.45) and replace $\sin\theta$ with θ (in radians) to give

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{2.46}$$

of the same type as

$$\ddot{\theta}\,(t) + \omega_0^2\theta(t) = 0 \tag{2.47}$$

The solution is a sinusoidal harmonic motion **(SHM)**

$$\theta\,(t) = \theta_m \sin\left(\omega_0 t + \varphi\right) \tag{2.48}$$

with $\omega_0 = \sqrt{\dfrac{g}{l}}$ natural angular frequency of motion.

2) Using Lagrange formalism

* A point mass m undergoes an oscillatory rotational motion with an angular velocity $d\theta/dt$ about point O. located at a distance L. It has a moment of inertia $j_{/\triangle} = ml^2$, therefore a **rotational kinetic energy** $E_c = \frac{1}{2}j\dot{\theta}^2 = \frac{1}{2}ml^2\dot{\theta}^2$.

* On the other hand, the **potential energy** is given by : $E_p = mgh$ avec $h = h_0 + l - l\cos\theta = h_0 + l(1 - \cos\theta)$.

So $E_p = mgh_0 + mgl(1 - \cos\theta)$.

The Lagrangian of the system is given by

$$L = E_c - E_p = \frac{1}{2}ml^2\dot{\theta}^2 - mgh_0 + mgl(1 - \cos\theta) \tag{2.49}$$

hence the Lagrange equation is given by :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \tag{2.50}$$

$\dfrac{\partial L}{\partial \dot{\theta}} = j\dot{\theta} \Longrightarrow \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} = j\ddot{\theta} = ml^2\ddot{\theta}$

$\dfrac{\partial L}{\partial \theta} = -mgl\sin\theta$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = ml^2\ddot{\theta} + mgl\sin\theta = 0 \quad \text{non linear} \tag{2.51}$$

We recall that the Taylor series development of a function $f(x)$ in the neighborhood of x_0 is given by :

$$f(x)_{x\approx x_0} = f(x_0) + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{d^n f}{dx^n}\right)(x - x_0)^n \tag{2.52}$$

Application to the sin function :

$$\sin\theta \approx \theta - \frac{\theta^3}{6} + ... \tag{2.53}$$

So in the case of small oscillations $\theta \prec\prec 10^o$, we can make the approximation :
$\sin\theta \approx \theta$, ($\theta$ en $rd << 1$) $\longrightarrow ml^2\ddot{\theta} + mgl\theta = 0 \longrightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$.
Of the same type as $\ddot{\theta} + \omega_0^2\theta = 0$. The solution is a sinusoidal harmonic motion **(SHM) ;**

$$\theta(t) = \theta_m \sin(\omega_0 t + \varphi) \tag{2.54}$$

with $\omega_0 = \sqrt{\frac{g}{l}}$ proper pulsation of motion.

2.2.5 Oscillations in electrical circuits

In this section we consider oscillations in an electrical circuit. What we find is that these oscillations are described by a differential equation that is identical in form to Equation (2.7) and so has an identical solution : only the physical quantities associated with the differential equation are different. This illustrates that when we understand one physical situation we can understand many others. It also means that we can simulate one system by another and in this way build analogue computers, i.e. we can build an electrical circuit consisting of resistors, capacitors and inductors that will exactly simulate the operation of a mechanical system.

The LC circuit

The simplest example of an oscillating electrical circuit consists of an inductor L and capacitor C connected together in series with a switch as shown in Figure II.9.

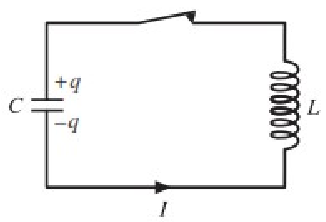


Figure II.9 : LC circuit.

As usual we start with an idealised situation where we assume that the resistance in the circuit is negligible. This is analogous to the assumption for mechanical systems that there are no frictional forces present.

Initially, the switch is open and the capacitor C subjected to the voltage V of the generator charges with an electric charge q :

$$V_c = V = q/c \tag{2.55}$$

When the switch is closed the charge begins to flow through the inductor and a current $i = dq/dt = \dot{q}$ flows in the circuit.

This is a time-varying current and produces a voltage across the inductor given by $V_L = \dfrac{di}{dt}$ We can analyse the LC circuit using Kirchhoff's law,which states that 'the sum of the voltages around the circuit is zero, So, we have :

$$V_c + V_L = 0 \tag{2.56}$$

$$L\frac{di}{dt} + \frac{q}{c} = 0 \tag{2.57}$$

giving

$$L\ddot{q} + \frac{q}{c} = 0 \iff \ddot{q} + \frac{1}{Lc}q = 0 \tag{2.58}$$

Of the same type as

$$\ddot{q} + \omega_0^2 q = 0 \tag{2.59}$$

The solution is a sinusoidal harmonic motion **(SHM),** $q(t) = q_m \sin(\omega_0 t + \varphi)$ with $\omega_0 = \sqrt{\dfrac{1}{LC}}$ natural angular frequency of motion, and $T_0 = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{LC}$ is the natural period of motion.

We can also consider the energy of this electrical oscillator. The energy stored in a capacitor charged to voltage V_c is equal to $\frac{1}{2}cv_c^2$. This is electrostatic energy.The energy stored in an inductor is equal $\frac{1}{2}Li^2$ and this is magnetic energy. Thus the total energy in the circuit is given by

$$\frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{c} \tag{2.60}$$

2.2.6 Electromechanical analogies

We observe through these examples that simple mechanical or electrical harmonic oscillations are described by the same type of 2nd order differential equation with constant linear coefficient in x or θ or q without second member. The solution to this type of equation is an **SHM**. We can then make analogies between mechanical and electrical quantities (see table II.2).

Mechanical system (mass + spring)	Electrical system (oscillating LC circuit)
$\ddot{x} + \frac{k}{m}x = 0$	$\ddot{q} + \frac{1}{LC}q = 0$
Elongation x	Charge q
Velocity $v = \frac{dx}{dt} = \dot{x}$	Current $i = \frac{dq}{dt} = \dot{q}$
Acceleration $\gamma(t) = \frac{d^2x}{dt^2} = \ddot{x}(t)$	$\frac{di}{dt} = \ddot{q}$
Mass m	Inductor L (choke or coil)
Stiffness k	$1/C$ (capacitor C)
Natural period $T_0 = 2\pi\sqrt{\frac{m}{k}}$	Natural period $T_0 = 2\pi\sqrt{L/C}$
Exchange of mechanical energy between mass and spring $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$	Exchange of electrical energy between coil and capacitor $\frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{c}$

Table II.2 : Analogy of electromechanical parameters.

2.3 Conclusion

In this chapter, we examined the behavior of mechanical systems with one degree of freedom undergoing free oscillations, meaning no external force or damping is present. Typical examples include the mass-spring system and the simple pendulum. These systems exhibit periodic motion governed by linear differential equations, with dynamics entirely determined by the system's physical parameters (mass, stiffness, length, gravity, etc.).

We emphasized key principles, including :

* The definition and significance of the natural (angular) frequency,

* The conservation of mechanical energy in undamped systems,

* The influence of initial conditions on the evolution of motion.

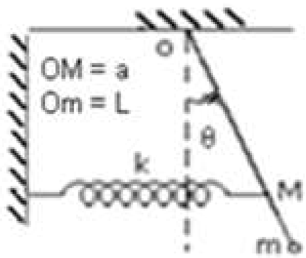
These concepts provide a solid foundation for approaching more complex oscillatory systems, including **damped**, **forced**, and **multi-degree-of-freedom** systems, which will be addressed in subsequent chapters.

2.4 Corrected exercises

2.4.1 Exercise 1

Consider a pendulum composed of a massless rigid rod of length L, a mass m, and a spring of stiffness k. At rest (equilibrium), the rod is vertical and the spring is undeformed. When the mass is displaced by an angle θ, the spring is stretched by a length x.

Determine the kinetic and potential energies of the oscillating mechanical system. Deduce the angular frequency (in the case of small oscillations).



Solution

1) Kinetic and Potential Energies of the System

$$E_T = E_c + E_p$$

$E_c{=}E_{cm} + E_{cS} = \frac{1}{2} j\dot{\theta}^2 = \frac{1}{2} ml^2\dot{\theta}^2$

$E_p = E_{pm} + E_{ps} = mgh + \dfrac{1}{2}kx^2 \longrightarrow \left\{ \begin{array}{c} h = l - l\cos\theta = l(1-\cos\theta) \\ x = a\sin\theta \end{array} \right\}$

So, $E_p = mgl(1-\cos\theta) + \dfrac{1}{2}k\,(a\sin\theta)^2$

$E_p = mgl - mgl\cos\theta + \dfrac{1}{2}k\,(a\sin\theta)^2$

2) Natural angular frequency

The total energy of a mechanical system is the composition of $E_T = E_c + E_p = T + U$. The total energy of a conservative system is constant over time :

$$\frac{dEt}{dt} = 0 \qquad \forall t$$

We have : $E_T = \frac{1}{2}ml^2\dot{\theta}^2 + mgl - mgl\cos\theta + \dfrac{1}{2}k\,(a\sin\theta)^2$

$\longrightarrow \dfrac{dEt}{dt} = ml^2\dot{\theta}\ddot{\theta} + mgl\sin\theta\dot{\theta} + ka^2\cos\theta\sin\theta\dot{\theta} = 0$

$= \dot{\theta}(ml^2\ddot{\theta} + mgl\sin\theta + ka^2\cos\theta\sin\theta) = 0$

$\dot{\theta} \neq 0 \Longrightarrow ml^2\ddot{\theta} + mgl\sin\theta + ka^2\cos\theta\sin\theta = 0$

In the small oscillation approximation, we have $\left\{ \begin{array}{c} \sin\theta \approx \theta \\ \cos\theta \simeq 1 \end{array} \right\} \Longrightarrow$

$$ml^2\ddot{\theta} + mgl\theta + ka^2\theta = 0$$

which gives the following equation :

$$\ddot{\theta} + \frac{mgl + ka^2}{ml^2}\theta = 0$$

of the same type as :

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

with

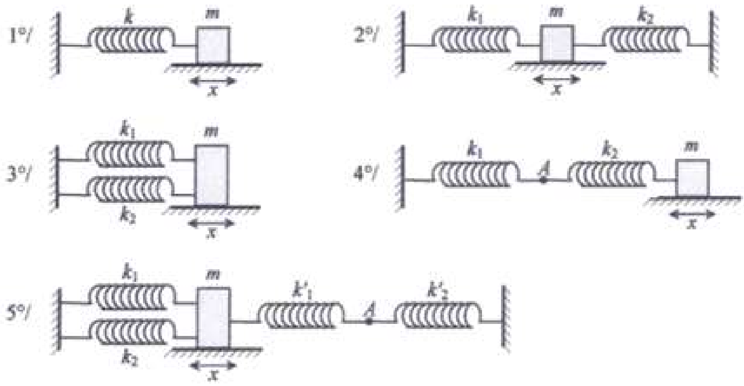
$$\omega_0^2 = \frac{mgl + ka^2}{ml^2} \Longrightarrow \omega_0 = \sqrt{\frac{mgl + ka^2}{ml^2}}.$$

2.4.2 Exercise 2

The following mechanical systems are considered :

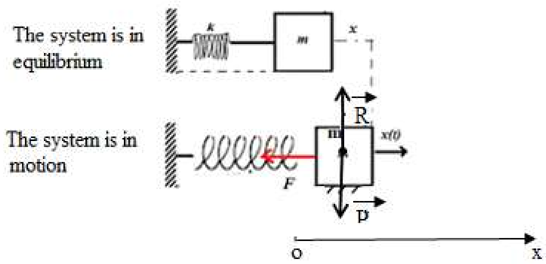
1) Determine the differential equation of motion for each system.

2) Give the solution to this differential equation.



Solution

1)The first system :



By applying the fundamental law of dynamics **(FLD)** :
$\sum \overrightarrow{F} = m\gamma \Longrightarrow \overrightarrow{F} + \underbrace{\overrightarrow{P} + \overrightarrow{R}}_{\|0} = m\gamma$

$Ox/ \Longrightarrow -kx = m\ddot{x} \Longrightarrow \ddot{x} + \dfrac{k}{m}x = 0$. We obtain a differential equation of motion of the form :

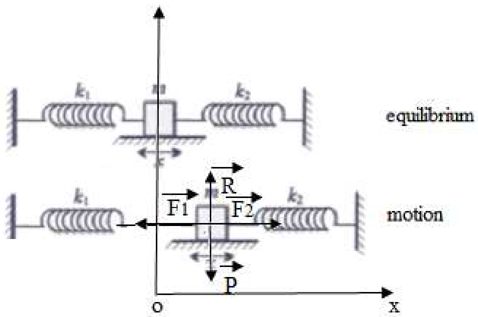
$$\ddot{x} + \omega_0^2 x = 0$$

avec $\omega_0 = \sqrt{\dfrac{k}{m}}$, natural angular frequenc yof the oscillator system.

2) The solution of this equation is of the form :

$$x(t) = A\cos(\omega_0 t + \varphi)$$

2) The second system :



FLD$\Longrightarrow \sum \overrightarrow{F} = m\gamma \Longrightarrow \overrightarrow{F}_1 + \overrightarrow{F}_2 + \underbrace{\overrightarrow{P} + \overrightarrow{R}}_{\|0} = m\gamma$

$Ox/ \Longrightarrow -F_1 + F_2 = m\ddot{x}$

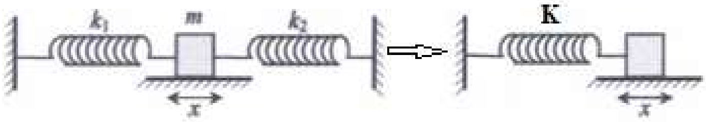
The spring of stiffness k_1 is elongated by $+x$.

The spring with stiffness k_2 is compressed by -$x \Longrightarrow \left\{ \begin{array}{c} F_1 = k_1(x) \\ F_2 = k_2(-x) \end{array} \right\}$

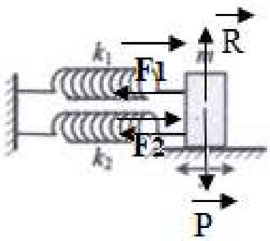
from where : $-k_1x - k_2x = m\ddot{x} \Longrightarrow \left\{ \begin{array}{c} m\ddot{x} + (k_1 + k_2)x = 0 \\ m\ddot{x} + Kx = 0 \end{array} \right\}$ $\longrightarrow$differential equation of motion of the form :

$$\ddot{x} + \frac{K}{m}x = 0, \text{ With} \qquad \mathbf{K} = k_1 + k_2.$$

A mass placed between two springs is equivalent to a system consisting of a mass m in series with a single spring whose equivalent stiffness is equal to the sum of the two individual stiffnesses.



3) The third system :



FLD$\Longrightarrow \sum \overrightarrow{F} = m\gamma \Longrightarrow \overrightarrow{F}_1 + \overrightarrow{F}_2 + \underbrace{\overrightarrow{P} + \overrightarrow{R}}_{\|0} = m\gamma$

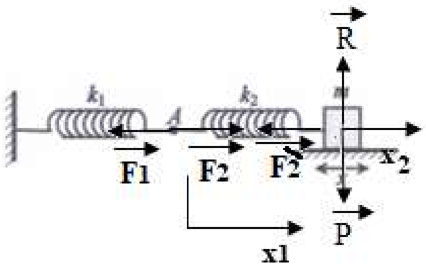
$Ox/ \Longrightarrow -F_1 - F_2 = m\ddot{x}$

$-k_1x - k_2x = m\ddot{x} \Longrightarrow \left\{ \begin{array}{c} m\ddot{x} + (k_1 + k_2)\,x = 0 \\ m\ddot{x} + Kx = 0 \end{array} \right\} \longrightarrow$ differential equation of motion of the form :

$$\ddot{x} + \frac{K}{m}x = 0, \text{ With} \qquad \mathbf{K} = k_1 + k_2.$$

We note that this system is the same as the previous one**.**

4) The fourth system :



FLD$\Longrightarrow \sum \overrightarrow{F} = m\gamma \Longrightarrow \overrightarrow{F}_1 + \overrightarrow{F}_2 + \underbrace{\overrightarrow{P} + \overrightarrow{R}}_{\|0} = m\gamma$

$$Ox/ \Longrightarrow \left\{ \begin{array}{l} -F_1 + F_2 = 0, \text{ (no mass at point A)} \\ -k_2(x_2 - x_1) = m\ddot{x}_2 \end{array} \right\}$$

$$-k_1x_1 + k_2(x_2 - x_1) = 0 \Longrightarrow -k_1x_1 + k_2x_2 - k_2x_1 = 0$$

Hence $\left\{ \begin{array}{l} k_2x_2 = x_1(k_1 + k_2) \\ k_2(x_2 - x_1) = m\ddot{x} \end{array} \right\} \Longrightarrow$

$$x_1 = \frac{k_2x_2}{k_1 + k_2}$$

$$-k_2(x_2 - x_1) = m\ddot{x}_2$$

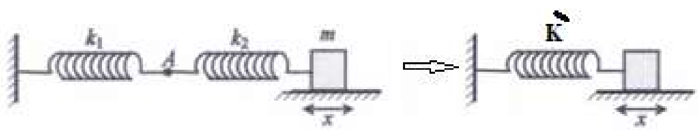
By substituting equation of x_1 into the second equation, we obtain :

$$-k_2(x_2 - \frac{k_2x_2}{k_1 + k_2}) = m\ddot{x}_2$$

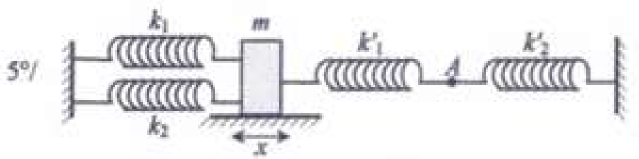
from where : $\dfrac{-k_2k_1}{k_2 + k_1}x_2 = m\ddot{x}_2 \Longrightarrow \ddot{x}_2 + \dfrac{k_2k_1}{m\,(k_2 + k_1)}x_2 = 0$

Let $x_2 = x$, we obtain a differential equation of motion of the form :

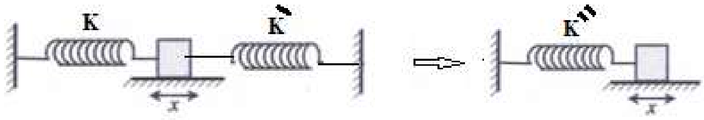
$$\ddot{x} + \frac{K'}{m}x = 0, \text{ with } K' = \frac{k_2k_1}{(k_2 + k_1)}, \text{That is } \frac{1}{K'} = \frac{1}{k_1} + \frac{1}{k_2}$$



5) **The fifth system**



This mechanical system is composed of system 3 and system 4. So, we have :

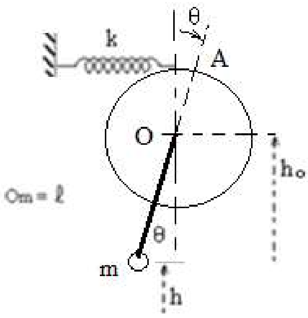


we obtain a differential equation of motion of the form :

$$\ddot{x} + \frac{K''}{m}x = 0, \text{ with } K'' = k + k' = (k_1 + k_2) + \frac{k_2 k_1}{(k_2 + k_1)}.$$

2.4.3 Exercise 3

Consider the following oscillatory system :



The cylinder of mass M and radius R can only rotate around its axis O. The spring of stiffness k is attached at A to the disc such that $OA = R$. The pendulum of length $Om = l$ (massless rod), attached to the cylinder at O axis of rotation (rotate together), carries a point mass m at its end.

At rest, the pendulum is vertical ($\theta = 0$) and the spring k is not deformed. To produce oscillations, the pendulum is moved θ away from the vertical. The cylinder then also rotates θ, and the spring is deformed by x (remember that $x = R\theta$), then the system is released (motion).

1) Determine the kinetic and potential energies of the system (as a function of θ).

2) Determine the differential equation of motion.

3) In the case of small oscillations, deduce the natural period of the oscillations.

Solution

1) Kinetic and Potential Energies of the System

$$E_c = E_M + E_m, \qquad \longrightarrow \left\{ \begin{array}{l} M : \text{Cylinder Mass} \\ m : \text{Point mass} \end{array} \right\}.$$

$$= \frac{1}{2} j_M \dot{\theta}^2 + \frac{1}{2} j_m \dot{\theta}^2$$

$$= \frac{1}{2}\frac{1}{2} M R^2 \dot{\theta}^2 + \frac{1}{2} m l^2 \dot{\theta}^2, \qquad \text{with } x = r\theta \Longrightarrow \dot{x} = r\dot{\theta}$$

$$\Longrightarrow E_c = \left(\frac{1}{4} M R^2 + \frac{1}{2} m l^2 \right) \dot{\theta}^2$$

$$E_p = E_{pM} + E_{pm} + E_{pk}$$

$$= mgh_1 + Mgh_2 + \frac{1}{2} k x^2, \quad \text{with} \left\{ \begin{array}{c} h = h_0 - l\cos\theta \\ x = R\theta \end{array} \right\}$$

$$\Longrightarrow E_p = mg\left(h_0 - l\cos\theta\right) + \frac{1}{2} k \left(R\theta\right)^2$$

2) Differential Equation of Motion

The differential equation of motion is derived using Lagrange's equation, based on the system's kinetic and potential energies :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

The Lagrangian is given by :

$$L = E_c - E_p$$

$$= \left(\frac{1}{4} M R^2 + \frac{1}{2} m l^2 \right) \dot{\theta}^2 - mg\left(h_0 - l\cos\theta\right) + \frac{1}{2} k \left(R\theta\right)^2.$$

$$\left\{ \begin{array}{l} \dfrac{\partial L}{\partial \dot{\theta}} = \left(\dfrac{1}{2} M R^2 + m l^2 \right) \dot{\theta} \Longrightarrow \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} = \left(\dfrac{1}{2} M R^2 + m l^2 \right) \ddot{\theta} \\ \dfrac{\partial L}{\partial \theta} = -mgl\sin\theta - kR^2\theta \end{array} \right\}$$

By substituting these expressions into equation (2.36), we obtain the motion equation :

$$\left(\frac{1}{2}MR^2 + ml^2\right)\ddot{\theta} + mgl\sin\theta + kR^2\theta = 0$$

3) Natural period of the oscillations.

When $\theta << 10^o \longrightarrow \left\{\begin{array}{c} \sin\theta \simeq \theta \\ \cos\theta = 1 \end{array}\right\}$

$(2.37) \Longleftrightarrow \left(\frac{1}{2}MR^2 + ml^2\right)\ddot{\theta} + -mgl\theta - kR^2\theta = 0$

$\Longleftrightarrow \left(\frac{1}{2}MR^2 + ml^2\right)\ddot{\theta} + -\left(mgl + kR^2\right)\theta = 0 \longrightarrow$

$$\ddot{\theta} + \frac{mgl + kR^2}{\frac{1}{2}MR^2 + ml^2}\theta = 0 \text{ of the same type as } \ddot{\theta} + \omega_0^2\theta = 0.$$

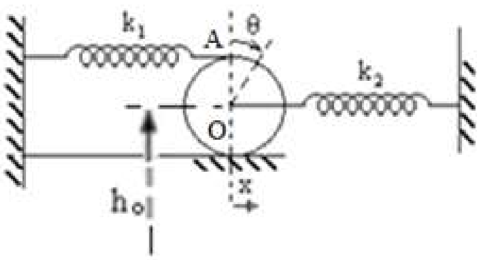
The solution corresponds to a sinusoidal harmonic oscillation **(SHM)**:

$$\theta(t) = \theta_m \sin(\omega_0 t + \varphi)$$

with $\omega_0^2 = \dfrac{mgl + kR^2}{\frac{1}{2}MR^2 + ml^2} \longrightarrow T_0 = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{mgl + kR^2}{\frac{1}{2}MR^2 + ml^2}}.$

2.4.4 Exercise 4

The mechanical oscillatory system shown in the figure is considered :



A cylinder of mass M and radius R rolls without slipping, meaning that when it rotates by an angle θ, its center of mass moves by a distance $x = R\theta$.

1) Determine the kinetic energy and potential energy of the system.

2) Deduce the Lagrangian, assuming $k_1 = k$, and $k_2 = 2k$.

Then, derive the equation of motion and determine the system's natural period.

Solution

The cylinder's moment of inertia : $J = \frac{1}{2}MR^2$.

The cylinder is characterized by a mass M and a radius R.

1) Kinetic and Potential Energies of the System

$E_c = E_{c(cylinder)} = \frac{1}{2}j\dot{\theta}^2 + \frac{1}{2}Mv^2$
$= \frac{1}{2}\frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}M^2\dot{\theta}^2$, with $x = R\theta \longrightarrow \dot{x} = R\dot{\theta}$

$$E_c = \frac{3}{4}MR^2\dot{\theta}^2 = \frac{3}{4}M\dot{x}^2$$

$E_p = E_{pcylinder} + E_{ps1} + E_{ps2}$

$= mgh_0 + \frac{1}{2}k_1x^2 + \frac{1}{2}k_2x^2$

$= mgh_0 + \frac{1}{2}k_1(x + R\theta)^2 + \frac{1}{2}k_2x^2$

$= mgh_0 + \frac{1}{2}k_1 4x^2 + \frac{1}{2}k_2x^2 \longrightarrow$

$$E_p = mgh_0 + \frac{1}{2}\left(4k_1 + k_2\right)x^2$$

2) Lagrangian L

$L = E_c - E_p = \frac{3}{4}M\dot{x}^2 - mgh_0 - \frac{1}{2}\left(4k + 2k\right)x^2$, for $k_1 = k$, and $k_2 = 2k$, we have :

$$L = \frac{3}{4}M\dot{x}^2 - mgh_0 - 3kx^2$$

Then, the equation of motion is given by :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$\left\{\begin{array}{l} \dfrac{\partial L}{\partial \dot{x}} = \dfrac{3}{2}M\dot{x} \Longrightarrow \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} = \dfrac{3}{2}M\ddot{x} \\ \dfrac{\partial L}{\partial x} = -6kx \end{array}\right\}$$

By substituting these expressions into equation (2.42), we obtain the motion equation :

$$\frac{3}{2}M\ddot{x} + 6kx = 0 \Longrightarrow \ddot{x} + \frac{4k}{M}x = 0$$

The solution corresponds to a sinusoidal harmonic oscillation **(SHM)**:

$$x\left(t\right) = x_m \sin\left(\omega_0 t + \varphi\right), \text{ with } \omega_0^2 = \frac{4k}{M}$$

$$\Longrightarrow T_0 = \frac{2\pi}{\omega_0} = \pi\sqrt{\frac{M}{k}}.$$

Chapter III

Damped free oscillations of one degrees of freedom systems

Chapter 3

Damped free oscillations of one degree of freedom systems

3.1 Introduction

In theory, free sinusoidal oscillations are perpetual motions with constant amplitude. However, in practice, the amplitude of such oscillations gradually decreases over time, and the system eventually comes to a complete stop. This behavior is due to the presence of damping, which results from various **frictional** forces encountered in nature. A common example is air resistance, which acts against the motion of the system.

These frictional forces dissipate energy and oppose motion, effectively functioning as **dampers** that reduce and eventually suppress oscillations.

There are two types of frictional forces : **solid** and **viscous**. In our program, we are only interested in viscous frictional forces, which, by definition, represent the action of a fluid on the motion of object :

$$\overrightarrow{f} = -\alpha \overrightarrow{v}. \tag{3.1}$$

with

α : damping coefficient.

$\overrightarrow{v}$: is the velocity of the mass,

where the minus sign indicates that the force always acts in the opposite direction to the motion. The constant α depends on the shape of the mass and the viscosity of the fluid and has the units of force per unit velocity.

In mechanics, the damper is represented by a symbol resembling a piston in a cylinder (figure III.1).

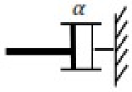


Figure III.1 : Schematic of a damper.

This chapter focuses on free oscillations in systems where damping is present, but no external force acts on the system. The energy initially stored in the system is slowly dissipated due to the damping effect, resulting in a motion that fades with time.

This behavior is typical in mechanical structures, electrical circuits, and various systems encountered in telecommunications, such as resonators and filters.

We will explore :

* The mathematical modeling of damping forces, particularly those proportional to velocity,

* The classification of damped oscillatory motion into underdamped, critically damped, and overdepend regimes,

* The solution of the second-order differential equation governing damped motion,

* The influence of the damping ratio on system behavior and response.

3.2 The equation of motion for a damped harmonic hoscillator

An example of a damped harmonic oscillator is shown in Figure III.2. It is similar to the simple harmonic oscillator described in Section (2.2.2) but now the mass is immersed in a viscous fluid. When an object moves through a viscous fluid it experiences a frictional force. This force dampens the motion : the higher the velocity the greater the frictional force. So as a car travels faster the frictional force increases thereby reducing the fuel economy, while the velocity of a falling raindrop reaches a limiting value because of the frictional force.

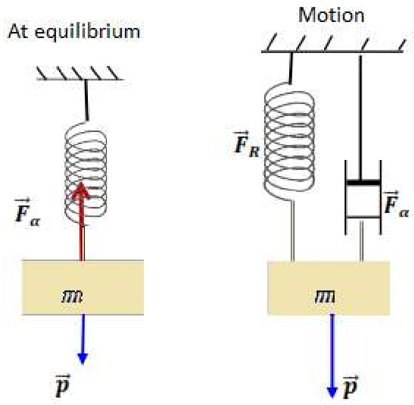


Figure III.2 : Damped Mass-Spring system.

3.2.1 Newton's Dynamic Principle

When there is no motion, the weight of mass m lengthens the spring by x_0 (static deformation of the spring),

$$\overrightarrow{P} + \overrightarrow{T}_0 = \overrightarrow{0} \tag{3.2}$$

we can then write the equilibrium equation :

$$mg - kx_0 = 0 \tag{3.3}$$

When we have a motion, x is the dynamic deformation of the spring. so, we have :

$$\sum \overrightarrow{F} = m\overrightarrow{\gamma} = \overrightarrow{P} + \overrightarrow{T} + \overrightarrow{f} \tag{3.4}$$

$$m\ddot{x} = mg - k(x_0 + x) - \alpha\dot{x} = mg - kx_0 - kx - \alpha\dot{x} \tag{3.5}$$

$$m\ddot{x} = -kx - \alpha\dot{x} \Longrightarrow \ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0. \tag{3.6}$$

We introduce the parameters $2\gamma = \frac{\alpha}{m}$, $\omega_0^2 = \frac{k}{m}$. In terms of these, Equation (3.6) becomes

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0. \tag{3.7}$$

Where γ is the damping factor, and ω_0 is the natural angular frequency. We find a second order linear differential equation with damping without a second member.

3.2.2 Lagrange method

The Lagrange equation for this system is given by :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0. \tag{3.8}$$

L is the Lagrangian of the system defined by $E_c - E_p$ with :

* kinetic energy : $E_c = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$.

* potential energy : $E_p = \frac{1}{2}kx^2$

while D is the dissipation function defined by : $D = \frac{1}{2}\alpha v^2 = \frac{1}{2}\alpha\dot{x}^2$.

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

we then have :

$$\left\{ \begin{array}{l} \dfrac{\partial L}{\partial \dot{x}} = m\dot{x} \Longrightarrow \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \\ \dfrac{\partial L}{\partial x} = -kx \\ \dfrac{\partial D}{\partial \dot{x}} = \alpha\dot{x} \end{array} \right\}$$

Substituting into equation (3.8), we have :

$$m\ddot{x} + kx + \alpha\dot{x} = 0 \Longrightarrow \ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0.$$

From where :

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0. \tag{3.9}$$

This is the equation of a damped harmonic oscillator. Equation (3.11) has different solutions depending on the degree of damping involved, corresponding to the cases of (*i*) light damping, (*ii*) heavy or over damping and (*iii*) critical damping. Light damping is the most important case for us because it involves oscillatory motion whereas the other two cases do not.

3.3 Solution of the equation of motion

Equation (3.11) is a second-order differential equation without a right-hand side. The set of solutions to this equation forms a 2 dimensional vector space. The general solution to this equation is written as a linear combination of two solutions forming a basis. This basis can be found by looking at the exponential solutions of the time.

$$\begin{aligned} x &= \exp^{rt} \\ \dot{x} &= r \exp^{rt} \\ \ddot{x} &= r^2 \exp^{rt} \end{aligned}$$

We replace the three terms in expression 3.11, we obtain :

$$r^2 \exp^{rt} + 2\gamma r \exp^{rt} + \omega_0^2 \exp^{rt} = 0, \qquad (3.10)$$

then :

$$r^2 + 2\gamma r + \omega_0^2 = 0. \qquad (3.11)$$

We obtain the characteristic equation (second degree equation in r), its solutions depend on the discriminant sign.
We calculate the discriminant Δ.
$\Delta = 4\gamma^2 - 4\omega_0^2 = 4\left(\gamma^2 - \omega_0^2\right) = 4\Delta'$.
Δ' includes the natural angular frequency and damping. Three cases are then studied according to the sign of the discriminant Δ'.

3.3.1 Case $\Delta' > 0$

$\Delta' = \gamma^2 - \omega_0^2 > 0 \longrightarrow \gamma^2 > \omega_0^2 \longrightarrow \alpha^2/4m^2 > k/m \longrightarrow \alpha^2 > 4km \longrightarrow$ **Heavy damping.**

The equation has two solutions :

$$r_{1,2} = -\frac{2\gamma \pm \sqrt{4\Delta'}}{2} = -\gamma\sqrt{\Delta'} = -\gamma \pm \beta \text{ avec } \beta = \sqrt{\Delta'}. \quad (3.12)$$

The general solution is a linear combination of the two solutions :

$$x_{1,2}(t) = \exp r^{1,2(t)} = A_1 \exp^{r_1 t} + A_2 \exp^{r_2 t} = A_1 \exp^{(-\gamma+\beta)t} + A_2 \exp^{(-\gamma-\beta)t}; \quad (3.13)$$

thus

$$x(t) = \exp^{-\gamma t}(A_1 \exp^{\beta t} + A_2 \exp^{-\beta t}) \quad (3.14)$$

A_1 and A_2 are constants that we determine using the Initial Conditions (IC).
The damping is so significant that there are no oscillations ; this system is said to be **overdamped**

3.3.2 Case $\Delta' = 0$

An interesting situation occurs when $\Delta' = \gamma^2 - \omega_0^2 = 0 \longrightarrow \gamma^2 = \omega_0^2 \longrightarrow \alpha^2/4m^2 = k/m \longrightarrow \alpha^2 = 4km$. This is the case of **critical damping.**
$\Delta' = 0 \longrightarrow \beta = \sqrt{\Delta'} = 0$
the solution is a double root $r_1 = r_2 = -\frac{2\gamma}{2} = -\gamma$. The solution is :

$$x(t) = \exp^{-\gamma t}(A_1 + A_2 t) \quad (3.15)$$

There are no oscillations and the system is said to be **critically damped.**
Here the mass returns to its equilibrium position in the shortest possible time without oscillating. Critical damping has many important practical applications. For example, a spring may be fitted to a door to return it to its closed position after it has been opened.

In practice, however, critical damping is applied to the spring mechanism so that the door returns quickly to its closed position without oscillating.

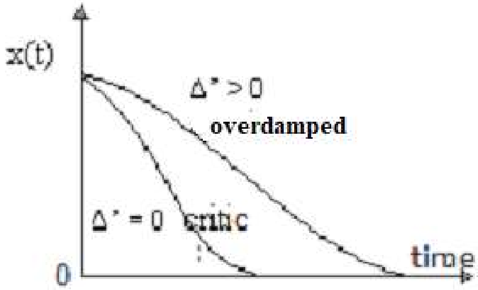


Figure III.3 : Critically damped system.

3.3.3 Case $\Delta' < 0$

$\Delta' = \gamma^2 - \omega_0^2 < 0 \longrightarrow \gamma^2 < \omega_0^2 \longrightarrow \alpha^2/4m^2 < k/m \longrightarrow \alpha^2 < 4km \longrightarrow$ **Light damping.**

we have no solutions in the set of real numbers, but as physically there are oscillations, we look for solutions in the set of complex numbers.

$$\Delta' = \gamma^2 - \omega_0^2 = -(\omega_0^2 - \gamma^2) = i^2(\omega_0^2 - \gamma^2) = i^2\omega^2 \tag{3.16}$$

with $\omega^2 = \omega_0^2 - \gamma^2$.

ω : Damped angular frequency $= \sqrt{\omega_0^2 - \gamma^2}$.

We have two complex roots :

$$r_{1,2} = -\gamma \pm \sqrt{\Delta'} = \gamma \pm i\omega \tag{3.17}$$

$$\begin{aligned} x_{1,2}(t) &= A_1 \exp^{r_1 t} + A_2 \exp^{r_2 t}. & (3.18)\\ &= A_1 \exp^{(-\gamma + i\sqrt{\omega_0^2 - \gamma^2})t} + A_2 \exp^{(-\gamma - i\sqrt{\omega_0^2 - \gamma^2})t} & (3.19)\\ &= \exp^{-\gamma t}\left[A_1 \exp^{i(\sqrt{\omega_0^2-\gamma^2})t} + A_2 \exp^{-i(\sqrt{\omega_0^2-\gamma^2})t}\right]. & (3.20)\end{aligned}$$

Whose real part can be written in the form :

$$x(t) = C \exp^{-\gamma t} \cos(\omega t - \varphi) \tag{3.21}$$

The parameters γ and ω are determined solely by the properties of the oscillator while the constants C and φ are determined by the initial conditions.

A graph of $x(t) = C \exp^{-\gamma t} \cos(\omega t - \varphi)$ is shown in Figure III.4 where the steady decrease in the amplitude of oscillation is apparent. The dotted lines represent the $\exp^{-\gamma t}$ term which forms an envelope for the oscillations.

This system is said to be **underdamped**. This decrease in oscillations is entirely due to dissipative forces via the damping coefficient γ.

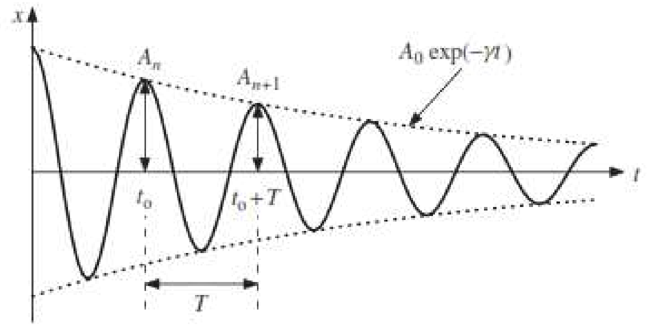


Figure III.4 : Damped oscillations.

The motion is damped periodic oscillatory with period of damped oscillatory motion $T = 2\pi/\omega \neq T_0 = 2\pi/\omega_0$, with $\omega = \sqrt{\omega_0^2 - \gamma^2} \rightarrow T = 2\pi/\omega = 2\pi/\sqrt{\omega_0^2 - \gamma^2} > T_0$ (Natural period of the motion).

Remark 3 *It should be noted that these 3 regimes (whatever Δ') end up stopping over time :* $\lim_{t\to\infty} x(t) = 0 \quad \forall \Delta' \rightarrow$ *These are transitional regimes.*

In summary we find three types of damped motion and these are illustrated in Figure III.5. They correspond to the conditions :

$\gamma^2 - \omega_0^2 < 0$, Light damping ; damped oscillations.

$\gamma^2 - \omega_0^2 > 0$, Heavy damping ; exponential decay of displacement.

$\gamma^2 - \omega_0^2 = 0$,Critical damping ; quickest return to equilibrium position without oscillation.

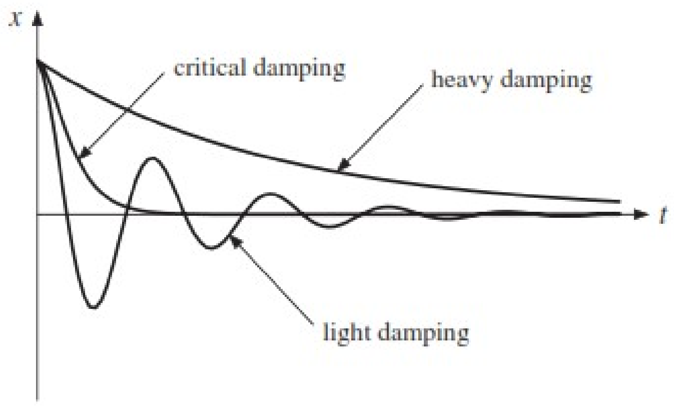


Figure III.5 : The motion of a damped oscillator for the cases of light damping, heavy damping and critical damping.

To appreciate the physical origin of these different types of motion, we recall that γ^2 is the damping term while ω_0^2 is proportional to the spring constant. When the damping term is small compared with k/m,the motion is governed by the restoring force of the spring and we have damped oscillatory motion. Conversely, when the damping term is large compared with k/m the damping force dominates and there is no oscillation. At the point of critical damping the two forces balance. We finally note that the relative size of γ^2 compared with ω_0^2 also determines the response of the oscillator to an applied periodic driving force, as we shall see in Chapter 4.

3.3.4 The quality factor Q of a damped harmonic oscillator

It is useful to have a figure of merit to describe how good an oscillator is, where we imply that the smaller the degree of damping the higher the quality of the oscillator. Moreover we would like a figure of merit that is dimensionless and can readily be applied to any oscillator whether it is mechanical, electrical or otherwise. This is called the quality factor Q of the oscillator and is defined as :

$$Q = \frac{\omega_0}{2\gamma} \tag{3.22}$$

The quality factor quantifies the quality of an oscillatory system. The lower the damping, the greater the quality of the system. Q is correspondingly greater.

3.3.5 Logarithmic decrement

The logarithmic decrement δ is the natural logarithm of the ratio of the amplitudes $x(t_0)$ and $x(t_0 + T_a)$. It characterizes the decrease in amplitude during a period

$$\delta = \ln \frac{x(t_0)}{x(t_0 + nT_a)} = \ln \frac{A \exp^{-\gamma t_0}}{A \exp^{-\gamma(t_0 + nT_a)}} \tag{3.23}$$

$$\delta = \ln(\exp^{\gamma n T_a}). \tag{3.24}$$

We then have :

$$\delta = n\gamma T_a \tag{3.25}$$

Remark 4 *The pseudo-period and the logarithmic decrement are only meaningful if the regime is pseudo-periodic.*

3.3.6 Damped electrical oscillations

In our mechanical example of a mass moving through a fluid we saw that the fluid offered a resistance that damped the motion. In the case of an electrical oscillator it is the resistance in the circuit that impedes the flow of current. An electrical oscillator is shown in Figure III.6. It consists of an inductor L and capacitor C as before (see Figure II.8) but now there is also the resistor R. We charge the capacitor to voltage $V_c = \frac{q}{c}$, and then close the switch. Kirchoff's law gives

$$L\frac{di}{dt} + Ri + \frac{q}{c} = 0 \tag{3.26}$$

$$L\ddot{q} + R\dot{q} + \frac{1}{c}q = 0 \tag{3.27}$$

This has the identical form to Equation (3.9), and we recognise the analogous quantities we met before : q is equivalent to x, L to m and k to $1/C$. However, we see that R is analogous to the mechanical damping constant α and so R/L is the equivalent of ■ (= α/m).

Since the above differential equations have identical forms, their solutions also have identical forms. The importance of this is that we can use our results for the mechanical oscillator to immediately write down the corresponding results for the electrical case.

Again we emphasise the exact correspondence between the equations and solutions that describe the mechanical and electrical systems, so that mechanical systems can be simulated by electrical circuits. Such analogue computers can greatly facilitate the design and development of mechanical systems.

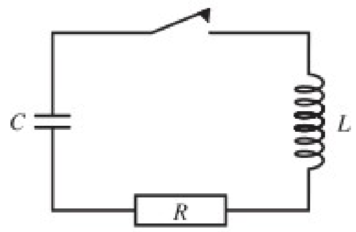


Figure III.6 : The circuit of a damped electrical oscillator.

3.4 Application Example

when a mass m is moved away from its equilibrium position by a displacement x_0 and then released without initial velocity. It takes on a **damped oscillatory motion** with a pseudo period of $1s$. We note that after 5 pseudo periods, the amplitude A is equal to 20% of the initial amplitude. Deduce the numerical value of γ.

Solution

$$\begin{aligned} T_a &= 1s, n = 5 \Longrightarrow \delta = \ln \frac{x(t_0)}{x(t_0 + nT_a)} = 5\gamma T_a = 5\gamma \\ &\Longrightarrow \ln \frac{A}{0.2A} = 5\gamma \Longrightarrow \ln \frac{1}{0.2} = 5\gamma \Longrightarrow \gamma = 0.32. \end{aligned}$$

3.5 Conclusion

This chapter provided an in-depth analysis of the dynamic behavior of single degree of freedom systems subjected to damped free vibrations. By introducing the damping factor, we classified the system responses into underdamped, critically damped, and overdepend regimes. Through the solution of the second-order differential equation, we highlighted the influence of damping on the natural frequency, pseudo-periodic response, and the exponential decay of amplitude over time. This study forms a fundamental basis for understanding vibratory behavior in mechanical systems, particularly in the contexts of design, vibration diagnostics, and passive structural control.

3.6 Corrected exercises

3.6.1 Exercise 1

We consider the free damped oscillator governed by the equation : $m\ddot{x} + \alpha\dot{x} + kx = 0$.
m : mass
k : stiffness constant
x : the dynamic displacement of m.
The motion $x(t)$ of m is such that the system is at equilibrium. We launch m with an initial speed $v_i = 25cm/s$.
So at $t = 0$, we have $x(t = 0) = 0$, $\dot{x}(t = 0) = v_i$.
1) Calculate the natural period of the system.
$NA : m = 150g$, $k = 3.8N/m$.
2) Show that if $\alpha = 0.6\ kg/s$, the mass m has a damped oscillatory motion. In this case,

solve the equation with the initial conditions. Calculate the pseudo period of the motion.

Solution

The motion equation is given by :

$$m\ddot{x} + \alpha\dot{x} + kx = 0 \Longrightarrow \ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$
$$\Longrightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

$$\omega_0^2 = \sqrt{\frac{k}{m}} = \sqrt{\frac{3.8}{0.15}} \Longrightarrow \omega_0 = 5rad/s.$$

2) We have $\alpha = 0.6\ kg/s \Longrightarrow \gamma = \dfrac{\alpha}{2m} = \dfrac{0.6}{2 \times 0.15} = 2$

We make a change of variable $x = \exp^{rt}$.

$$r^2 + 2\gamma r + \omega_0^2 = 0$$

$\Delta' = \gamma^2 - \omega_0^2$

$= 4 - 25 = -21 < 0 \rightarrow$low damping

The solution is :

$$x(t) = C\exp^{-\gamma t}\cos(\omega t - \varphi)$$

ω is called damped angular frequency$\rightarrow$ $\omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{25 - 4} \Longrightarrow \omega = 4.58rad/s.$

Then, the period of damped oscillatory motion is given by $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{4.58} \Longrightarrow T = 1.37s.$

From the initial conditions :

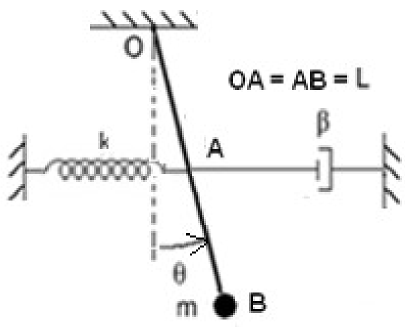
$$x(t = 0) = 0 \Longrightarrow x(0) = C\cos\varphi = 0 \Longrightarrow \varphi = \pm\frac{\pi}{2} since C \neq 0$$
$$\dot{x}(t = 0) = v_i \Longrightarrow \omega C = 25 \Rightarrow C = \frac{25}{4.58} = 5.4cm.$$

So

$$x(t) = 5.4\exp^{-2t}\cos(4.58t + \frac{\pi}{2}).$$

3.6.2 Exercise 2

Consider the pendulum shown in the figure :



The mass m is a point mass. The massless rod OB, of length $2L$, rotates about point O by an angle θ relative to its vertical equilibrium position. At rest $(\theta = 0)0$ the spring is undeformed. A damping device applies a viscous friction force at point A.

1) Find the equation of motion for this system.

In the case of small oscillations, give the corresponding differential equation.

2) Given : $m = 0.5kg; k = 4N/m; \alpha = 12kg/s; L = 0.5m; g = 10m/s^2$

Calculate the damping coefficient γ, the natural angular frequency ω_0, and give the corresponding transient solution of the system.

Solution

1) Motion equation :

The system undergoes a rotational motion, so the Lagrange equation is written as follows :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = 0 \tag{3.28}$$

Hence, the Lagrangian $L = E_c - E_p$

$E_c = \frac{1}{2} j_m \dot{\theta}^2, \quad j_m = ml^2 = m(2l)^2.$

$E_c = \frac{1}{2} m4l^2$

$$E_c = 2ml^2\dot{\theta}^2 \tag{3.29}$$

$$E_p = E_{pm} + E_{ps} = mgh + \frac{1}{2}kx^2 \qquad \text{with} \left\{ \begin{array}{c} x = OA\sin\theta = l\sin\theta \\ h = h_0 - 2l\cos\theta \end{array} \right\}$$

$$E_p = mg(h_0 - 2l\cos\theta) + \frac{1}{2}k\left(l\sin\theta\right)^2$$

$$E_p = mgh_0 - mg2l\cos\theta + \frac{1}{2}kl^2\sin\theta^2 \tag{3.30}$$

$$L = E_c - E_p = 2ml^2\dot{\theta} - mgh_0 + mg2l\cos\theta - \frac{1}{2}kl^2\sin\theta^2$$

$$\frac{\partial L}{\partial\dot{\theta}} = 4ml^2\dot{\theta} \longrightarrow \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 4ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta} = -mg2l\sin\theta - kl^2\cos\theta\sin\theta$$

$$D = \frac{1}{2}\alpha v^2 = \frac{1}{2}\alpha\dot{x}^2 \qquad \text{with } x = l\sin\theta, \text{ using } (fg(t))' = f'(g)\,g'(t) \longrightarrow \dot{x} = l\cos\theta\dot{\theta}$$

$$D = \frac{1}{2}\alpha l^2\cos^2\theta\dot{\theta}^2 \Longrightarrow \frac{\partial D}{\partial\dot{\theta}} = \alpha l^2\cos^2\theta\dot{\theta}$$

$$(3.21) \Longleftrightarrow 4ml^2\ddot{\theta} + mg2l\sin\theta + kl^2\cos\theta\sin\theta + \alpha l^2\cos^2\theta\dot{\theta} = 0$$

In the case of small oscillations $\theta << 10^0 \Longrightarrow \left\{ \begin{array}{c} \sin\theta \approx \theta \\ \cos\theta \simeq 1 \end{array} \right\} \longrightarrow$

$$4ml\ddot{\theta} + mg2\theta + kl\theta + \alpha l\dot{\theta} = 0$$

$$\ddot{\theta} + \frac{\alpha}{4m}\dot{\theta} + \left(\frac{2mg + kl}{4ml}\right)\theta = 0 \tag{3.31}$$

of the same type as :

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2\theta = 0, \qquad \text{with} \qquad 2\gamma = \frac{\alpha}{4m}, \qquad \omega_0^2 = \left(\frac{2mg + kl}{4ml}\right)$$

2) To solve the differential equation, we perform a change of variable, hence, the characteristic equation.

$$r^2 + 2\gamma r + \omega_0^2 = 0$$

Numerical Application : $2\gamma = \dfrac{\beta}{4m} = \dfrac{12}{4 \times 0.5} = 6 \Longrightarrow \gamma = 3$

$\omega_0^2 = \left(\dfrac{2mg + kl}{4ml}\right) = 12rd/s.$

$$\begin{aligned} \Delta' &= \gamma^2 - \omega_0^2 \\ &= 9 - 12 = -3 < 0 \rightarrow \text{low damping.} \end{aligned}$$

The solution is :

$$x(t) = C \exp^{-\gamma t} \cos(\omega t - \varphi)$$

ω is called damped angular frequency$\rightarrow$ $\omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{12 - 9} \Longrightarrow \omega = 1.73 rad/s \longrightarrow$

$$x(t) = C \exp^{-3t} \cos(\sqrt{3}t - \varphi).$$

3.6.3 Exercise 3

The system shown in the adjacent figure consists of a disk with mass $M = 2kg$ and radius R, rotating around its fixed axis. An inextensible, massless string drives the disk without slipping along its edge. A mass $m = 1kg$ is attached to the end of the string, along with a spring of stiffness $k = 25N/m$ and a damper with damping coefficient $\alpha = 2kg/s$. Another spring with the same stiffness k is attached to the opposite end.

1) Find the kinetic energy E_c, the potential energy E_p and the dissipation function D as a function of the variable θ.

2) Give the differential equation of motion, and find the response in the lightly damped regime.

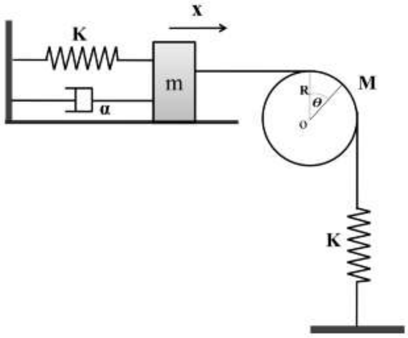
3) After 30 pseudo-periodic oscillations, the total energy of the vibratory system decreases by 70% of its initial value.

a) Calculate the logarithmic decrement δ.

b) After how many pseudo-periodic oscillations does the amplitude of the motion become equal to 60% of its initial value?

4) What is the value of α for which the motion becomes critical ?

5) Is the total energy of the system conservative ? Explain.



Solution

1) System Definitions :

* A disk of mass $M = 2kg$ and radius R rotates around a fixed axis. Its moment of inertia is $j = \frac{1}{2}MR^2$.

* A string wrapped around the disk (no slipping) pulls a mass $m = 1kg$ in translation, so the linear displacement is $x = R\theta \Rightarrow \dot{x} = R\dot{\theta}$.

a) Kinetic Energy $E_c(\theta)$:

The total kinetic energy includes the rotational energy of the disk, and the translational energy of the mass :

$$E_c = E_{c,disk} + E_{c,mass} = \frac{1}{2}j\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 \tag{3.32}$$

$$= \frac{1}{4}MR^2\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\theta}^2$$

$$E_c(\theta) = \frac{1}{2}(\frac{1}{2}M + m)R^2\dot{\theta}^2 \tag{3.33}$$

b) Potentiel Energy $E_p(\theta)$:

Each spring is stretched (or compressed) by a displacement $x = R\theta$, so :

$$E_p = \frac{1}{2}kR^2\theta^2 + \frac{1}{2}kR^2\theta^2 = kR^2\theta^2 \tag{3.34}$$

c) Dissipation Function

$$D = \frac{1}{2}\alpha v^2 = \frac{1}{2}\alpha R^2\dot{\theta}^2 \tag{3.35}$$

2) Lagrange's equation with dissipation :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = 0 \tag{3.36}$$

Lagrangian :

$$L = E_c - E_p = \frac{1}{2}(\frac{1}{2}M + m)R^2\dot{\theta}^2 \tag{3.37}$$

Compute each term of equation 3.28 :

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= (\frac{1}{2}M + m)R^2\dot{\theta} \Longrightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = (\frac{1}{2}M + m)R^2\ddot{\theta} & (3.38)\\
\frac{\partial L}{\partial \theta} &= -2kR^2\theta \\
\frac{\partial D}{\partial \dot{\theta}} &= \alpha R^2\dot{\theta} & (3.39)
\end{aligned}$$

Plug into the equation :

$$(\frac{1}{2}M + m)R^2\ddot{\theta} + 2kR^2\theta + \alpha R^2\dot{\theta} = 0 \tag{3.40}$$

Divide through by $(\frac{1}{2}M + m)R^2$

$$\ddot{\theta} + \frac{\alpha}{(\frac{1}{2}M + m)}\dot{\theta} + \frac{2k}{(\frac{1}{2}M + m)}\theta = 0 \tag{3.41}$$

of the same type as :

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2\theta = 0 \tag{3.42}$$

$$\text{with } 2\gamma = \frac{\alpha}{(\frac{1}{2}M + m)} = \frac{2}{\left(\frac{1}{2} \times 2 + 1\right)} \Longrightarrow \gamma = 0.5$$

$$\omega_0^2 = \left(\frac{2k}{(\frac{1}{2}M + m)}\right) = \frac{2 \times 25}{(\frac{1}{2} \times 2 + 1)} = 25 rad/s.$$

Lightly Damped Regime ; If $\gamma < \omega_0$ (light damping), the general solution is :

$$x(t) = C \exp^{-\gamma t} \cos(\omega t - \varphi) \tag{3.43}$$

ω is called pseudo pulsation$\rightarrow$ $\omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{25 - (0.5)^2} \Longrightarrow \omega = 4.97 rad/s \longrightarrow$

$$x(t) = C \exp^{-0.5t} \cos(\sqrt{4.97}t - \varphi). \tag{3.44}$$

3) a) In an underdamped system :

The energy of the system (kinetic + potential) decays exponentially over time.

After n oscillations, the energy drops as :

$$\frac{E_n}{E_0} = \exp^{-2n\delta} \tag{3.45}$$

Given $\frac{E_n}{E_0} = 1 - 0.70 = 0.30$ (i.e., 30% of the initial energy remains),
$n = 30$

$$\exp^{-2n\delta} = 0.30 \Longrightarrow -2n\delta = \ln(0.30) \Longrightarrow \delta = -\frac{1}{2n} \ln(0.30)$$

$$\delta = -\frac{1}{2 \times 30} \ln(0.30) \simeq 0.02$$

b) To find the number of pseudo-periodic oscillations n after which the amplitude becomes 60% of its initial value, we use the exponential decay law :

$$\frac{A_n}{A_0} = \exp^{-n\delta} \tag{3.46}$$

$\frac{A_n}{A_0} = 0.60$
Previously computed $\delta = 0.02$

Solving for n :

$$0.60 = \exp^{-n0.02} \Longrightarrow \ln(0.60) = -0.02n$$

$$n = -\frac{\ln(0.60)}{0.02} \approx 25.4 \tag{3.47}$$

So, the amplitude drops to 60% of its initial value after approximately 25 oscillations.

4) Critical Damping Condition :

Critical damping occurs when : $\Delta' = \gamma^2 - \omega_0^2 = 0 \Longrightarrow \gamma^2 = \omega_0^2 = 0 \Longrightarrow$

$$\frac{\alpha^2_{critic}}{\left[2(\frac{1}{2}M + m)\right]^2} = \frac{2k}{(\frac{1}{2}M + m)}$$

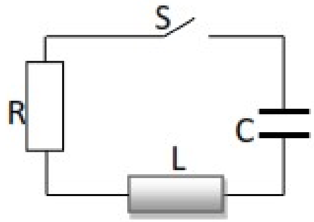
$$\alpha^2_{critic} = \frac{2k\left[2(\frac{1}{2}M + m)\right]^2}{(\frac{1}{2}M + m)} \tag{3.48}$$

5) No, the total energy is not conserved due to damping (friction losses from the damper). Energy is dissipated over time, as shown by the exponential decay in amplitude and energy.

3.6.4 Exercise 4

The figure shows a series circuit. Initially the capacitor is charged, then the switch S is closed. We are given $L = 1H$ and $C = 0.01\mu F$. Denote by $q(t)$ the charge on the capacitor at time t.
1) Write the differential equation that describes the circuit in terms of the variable q. Specify the values of γ and ω_0.
2) For which values of R does the system oscillate ? What is the critical resistance R_{cr}?

3) Plot approximately $q(t)$ for $R = 100\Omega$ and $R = 500\Omega$.



Solution

1) Differential equation and parameters γ and ω_0

For a series RLC circuit, the charge $q(t)$ satisfies

$$L\ddot{q} + R\dot{q} + \frac{1}{c}q = 0$$

where $\dot{q} = i(t)$ is the current.

Divide by L to put it in standard form :

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{Lc}q = 0$$

or more commonly

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = 0$$

with

$$\gamma = \frac{R}{2L} = \frac{R}{2} \quad \omega_0 = \frac{1}{\sqrt{Lc}} = \frac{1}{\sqrt{1 \times 0.01 \times 10^{-6}}} = 10000 rad/s$$

2) Oscillation condition and critical resistance

The behaviour depends on the damping compared to the natural frequency :

Underdamped (oscillatory) when $\gamma - \omega_0 < 0$

Critically damped when when $\gamma - \omega_0 = 0$

Overdamped when $\gamma - \omega_0 > 0$

Critical resistance R_{cr} found from $\gamma = \omega_0 \Rightarrow \dfrac{R_{cr}}{2} = 10000 \Rightarrow$

$$R_{cr} = 10000 \times 2 = 20000\Omega$$

So the circuit oscillates (i.e. underdamped) whenever

$$R_{cr} < 20000\Omega$$

For the values asked $R = 100\ \Omega$ and $R = 500\Omega$, both are much less than 20000Ω, so both cases are underdamped oscillatory.

3) Expression $q(t)$ for $R = 100\Omega$ and $R = 500\Omega$

General underdamped solution. For $\gamma < \omega_0$ the damped angular frequency is $\omega = \sqrt{\omega_0^2 - \gamma^2}$. The general solution is

$$q(t) = C \exp^{-\gamma t} \cos(\omega t - \varphi)$$

1) Compute the damping factor and the damped angular frequency

$$\gamma = \frac{R}{2L}, \qquad \omega = \sqrt{\omega_0^2 - \gamma^2}$$

Given $L = 1H$:

For $R = 100\Omega$

$$\gamma = \frac{100}{2} = 50, \qquad \omega = \sqrt{(10000)^2 - (50)^2} = 10000 rad/s$$

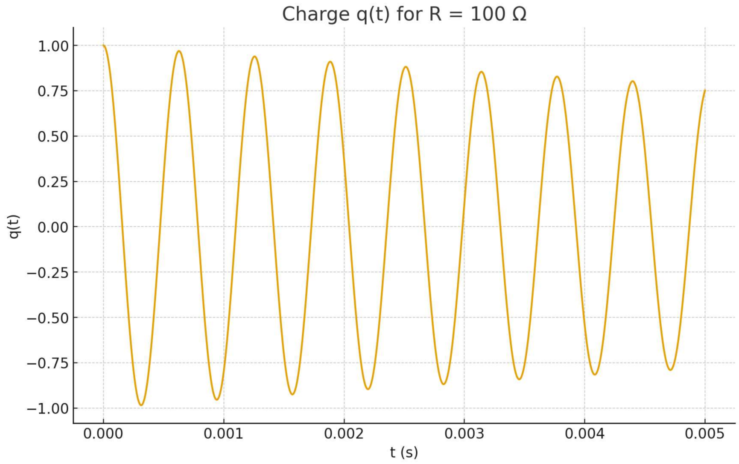
For $R = 500\Omega$

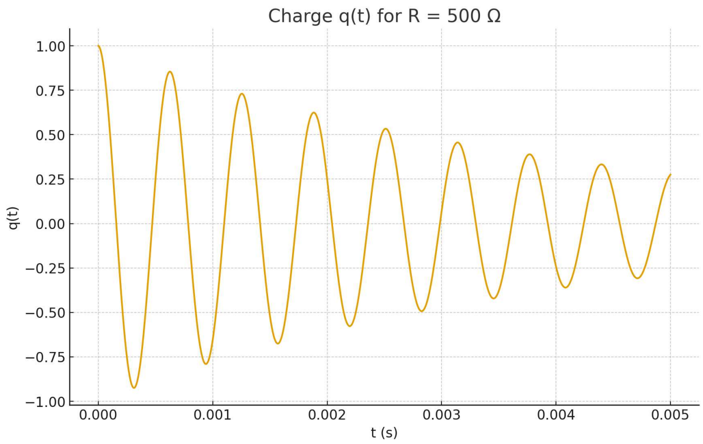
$$\gamma = \frac{500}{2} = 250, \qquad \omega = \sqrt{(10000)^2 - (250)^2} = 9997 rad/s$$

For $R = 100\Omega$, $q(t) = C \exp^{-50t} \cos(\omega t - \varphi)$. The damping is weak → oscillations persist, only slowly decaying.

For $R = 500\Omega$, $q(t) = C\exp^{-250t}\cos(\omega t - \varphi)$. The damping is stronger $\rightarrow$ oscillations die out much faster.

So oscillations remain almost unchanged in frequency, but the amplitude decays faster when R is bigger.





Chapter IV

Damped Forced oscillations of single degree of freedom systems

Chapter 4

Forced oscillations of single-degree-of-freedom systems

4.1 Introduction

In mechanical and structural systems, oscillations or vibrations are common phenomena that occur due to various forces acting on the system. When a system is subjected to an **external periodic force**, the motion that results is known as **forced oscillation**.

A single-degree-of-freedom (SDOF) system refers to a system that can be described by a single coordinate, often representing displacement, to characterize its motion. This kind of system is particularly useful in understanding basic mechanical oscillations, such as the motion of a mass attached to a spring or a simple pendulum under the influence of external forces.

Forced oscillations occur when the system is driven by an external force that compels it to oscillate at a frequency that is generally different from its natural frequency. The behavior of the system in such scenarios can be analyzed to predict its response, which depends on factors such as the damping in the system and the frequency of the applied force.

In the following sections, we will explore the theoretical foundations of forced oscillations in SDOF systems, including the concepts of resonance, damping, and the effect of varying external forces on system behavior.

4.2 Differential equation of motion of a forced system

If we want an oscillatory motion to persist, we must maintain it by constantly providing it with energy from the outside thanks to an exciting force (always > 0). The differential equation of forced oscillations of one-degree-of-freedom systems is given by :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = F_{ext}(t) \tag{4.1}$$

* For a translational motion the equation is written :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} + \frac{\partial D}{\partial \dot{x}_i} = F_{ext}(t) \tag{4.2}$$

* For a rotational motion the equation is written :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} + \frac{\partial D}{\partial \dot{\theta}_i} = M(F_{ext}) \tag{4.3}$$

F_{ext} : External exciting force.
$M(F_{ext})$: Moment of applied force, characterizes the ability of a force to turn an object around a point.

4.2.1 Example of a forced damped system (mass-spring-damper system)

In the adjacent figure, the mass m is attached to a spring K and a damper β.
Applying the fundamental principle of dynamics, we have :

$$\sum \overrightarrow{F} = m\overrightarrow{\gamma} = \overrightarrow{P} + \overrightarrow{R} + \overrightarrow{T} + \overrightarrow{f} + \overrightarrow{F}$$
$$m\ddot{x} = -kx - \beta\dot{x} + F(t) \longrightarrow \ddot{x} + \frac{\beta}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos\Omega t$$

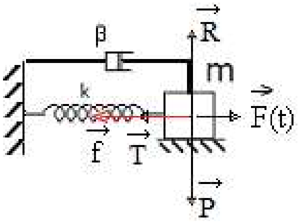


Figure IV.1 : Forced damped system.

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\Omega t \tag{4.4}$$

With $2\gamma = \frac{\beta}{m}$, $\omega_0^2 = \frac{k}{m}$.

We find a second-order linear differential equation (already seen) but with a second member. We say **complete equation**. The general solution X_g of this equation is therefore the sum of the solution of the equation without second member noted X_{SSM} and of a particular solution X_p of the same type as the second member.

$$X_g = X_{SSM} + X_p \tag{4.5}$$

X_{SSM} : The solution of the damped oscillation, see the 3 possible regimes according to the sign of the discriminant. This solution is called transient state because : $\underset{t\to\infty}{x(t)} = 0$ $\forall\Delta'$.
But the system is maintained with the exciting force $F(t) = Focos\Omega.t$ (second member), so oscillates with the frequency $\Omega \neq \omega \neq \omega_0$.
This means that if we cut $F(t)$, the system will oscillate with the pseudo pulsation ω (in the case of low damping) for a certain time then this transient state will disappear.

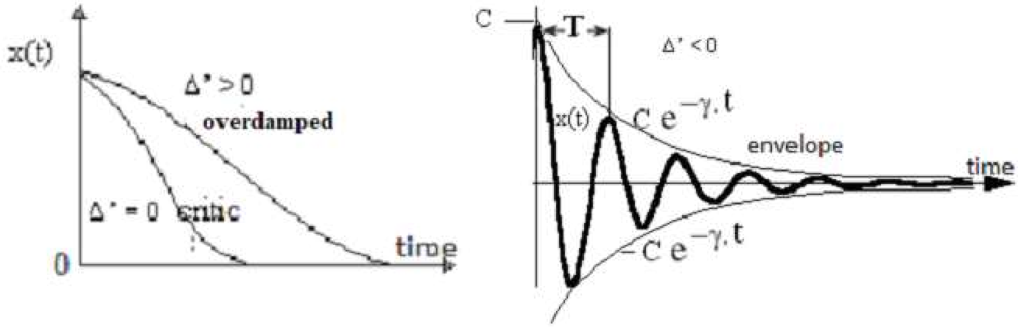


Figure IV.2 : Transient state for X_{SSM} solution.

The regime X_p will remain as long as $F(t)$ is imposed on the system (with the excitation frequency Ω). X_p is a steady state of the same type as $F(t)$.
If the excitation function is harmonic of the type $F\exp^{j\Omega t} = \dfrac{F_0}{m}\exp^{j\Omega t}$, so the response must be harmonic : $X_p(t) = A.\exp^{j(\Omega t-\varphi)}$,
with φ phase shift between $F(t)$ and $X_p(t)$.
If X_p is a solution of the differential equation, it must therefore verify the differential equation, which leads us to replace $X_p$in the differential equation.

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F(t) \tag{4.6}$$

$$\ddot{x_p} + 2\gamma\dot{x_p} + \omega_0^2 x_p = F(t) \tag{4.7}$$

with :

$$\dot{x_p} = jA\Omega\exp^{j(\Omega t-\varphi)} = j\Omega X_p(t) \tag{4.8}$$

$$\ddot{x_p} = -A\Omega^2\exp^{j(\Omega t-\varphi)} = -\Omega^2 X_p(t) \tag{4.9}$$

We replace expressions (4.8) and (4.9) in equation (4.7), we can write :

$$-\Omega^2 X_p(t) + 2\gamma j\Omega X_p(t) + \omega_0^2 X_p = \frac{F_0}{m}\exp^{j\Omega t} \tag{4.10}$$

$$X_p(t)\left[-\Omega^2 + 2\gamma j\Omega + \omega_0^2\right] = \frac{F_0}{m}\exp^{j\Omega t} \tag{4.11}$$

We replace X_p with its expression :

$$A.\exp^{j(\Omega t-\varphi)}\left[-\Omega^2+2\gamma j\Omega+\omega_0^2\right]=\frac{F_0}{m}\exp^{j\Omega t} \tag{4.12}$$

$$A\left[-\Omega^2+2\gamma j\Omega+\omega_0^2\right]=\frac{F_0}{m}\exp^{-j\varphi} \tag{4.13}$$

We know that

$$\exp^{j\theta}=\cos\theta+j\sin\theta$$

$$A\left[-\Omega^2+2\gamma j\Omega+\omega_0^2\right]=\frac{F_0}{m}\left(\cos\varphi+j\sin\varphi\right). \tag{4.14}$$

So :

$$\frac{F_0}{m}\cos\varphi = A\left(\omega_0^2-\Omega^2\right) \tag{4.15}$$

$$\frac{F_0}{m} = A2\gamma\Omega \tag{4.16}$$

$$\frac{(4.16)}{(4.15)}\Rightarrow\tan\varphi=\frac{2\gamma\Omega}{\omega_0^2-\Omega^2} \tag{4.17}$$

$$(4.15)^2+(4.16)^2\Rightarrow\frac{\left[A\left(\omega_0^2-\Omega^2\right)\right]^2}{\left(\dfrac{F_0}{m}\right)^2}+\frac{\left(A2\gamma\Omega\right)^2}{\left(\dfrac{F_0}{m}\right)^2}=1 \tag{4.18}$$

$$A^2\left[\left(\omega_0^2-\Omega^2\right)^2+\left(2\gamma\Omega\right)^2\right]=\left(\frac{F_0}{m}\right)^2$$

From where :

$$A\left(\Omega\right)=\frac{F_0/m}{\sqrt{\left(\omega_0^2-\Omega^2\right)^2+4\gamma^2\Omega^2}}. \tag{4.19}$$

We note that the amplitude A depends on the excitation strength, the damping, and the natural pulsation of the system.

Study of $A\left(\Omega\right)$ (resonance phenomenon)

At resonance, the amplitude $A\left(\Omega\right)$ has reached its maximum value, so :

$$\frac{dA(\Omega)}{d\Omega} = 0 \Longrightarrow \frac{d}{d\Omega}\left[\frac{F_0/m}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2}}\right] = 0 \tag{4.20}$$

By posing $A_0 = F_0/m$, equation (4.20) reduces to :

$$A_0\left(-\frac{1/2\left[8\gamma^2\Omega - 4\Omega\left(\omega_0^2 - \Omega^2\right)\right]\left[4\gamma^2\Omega^2 + \left(\omega_0^2 - \Omega^2\right)^2\right]^{-1/2}}{\left[\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2\right]^{1/2}}\right) = 0 \tag{4.21}$$

$$A_0\frac{2\Omega\left(\omega_0^2 - \Omega^2\right) - 4\gamma^2\Omega}{\left[\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2\right]^{3/2}} = 0 \tag{4.22}$$

$$2\Omega\left(\omega_0^2 - \Omega^2\right) - 4\gamma^2\Omega = 0$$

$$2\Omega\left[\left(\omega_0^2 - \Omega^2\right) - 2\gamma^2\right] = 0 \rightarrow 2 \text{ solutions.} \tag{4.23}$$

Either $\Omega = 0$ physically unacceptable because of non-oscillation.

or else :

$$\omega_0^2 = \Omega^2 + 2\gamma^2 \Rightarrow \Omega^2 = \omega_0^2 - 2\gamma^2$$

$$\Omega = \sqrt{\omega_0^2 - 2\gamma^2} = \Omega_r \text{ with existence condition } \omega_0^2 - 2\gamma^2 > 0 \tag{4.24}$$

The amplitude $A(\Omega)$ is maximum for $\Omega = \Omega_r = \sqrt{\omega_0^2 - 2\gamma^2} = \Omega_{resonance}$

$$A_{\max} = A(\Omega_r) = \frac{F_0/m}{\sqrt{\left(\omega_0^2 - \Omega_r^2\right)^2 + 4\gamma^2\Omega_r^2}} = \frac{F_0/m}{\sqrt{-4\gamma^2 + 4\gamma^2\omega_0^2}} = \frac{F_0/m}{2\gamma\sqrt{\left(\omega_0^2 - \gamma^2\right)}} \tag{4.25}$$

We define the resonant angular frequency $\Omega_r^2 = \omega_0^2 - 2\gamma^2 <$ damped angular frequency $\omega^2 = \omega_0^2 - \gamma^2 <$ natural angular frequency $\omega_0^2 = k/m$.
$\gamma = \beta/2m \rightarrow$ (damping).

Remark 5 *In the case of a system designed without damping then* $\beta = 0 \rightarrow \gamma = 0 \Rightarrow A(\Omega_r) \rightarrow \infty$ *(Figure IV.2).*

Danger of resonance *: An infinite amplitude imposed on an oscillating mechanical* *system causes a* ***rupture (breakage)*** *of the system (Figure IV.4).*

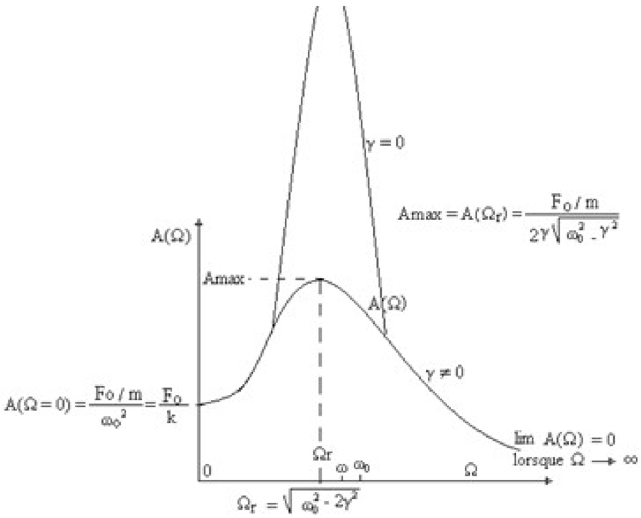


Figure IV.3 : Diagram of $A(\Omega)$ as a function of Ω.

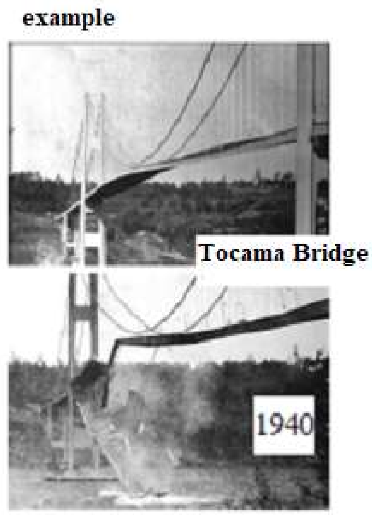


Figure IV.4 : example rupture system.

To avoid this situation, it is necessary to design damping devices for all systems subject to vibration $\beta \neq 0 \rightarrow$ that is, the existence of γ.
As soon as $\gamma \neq 0$, $A_{\max}$ decreases when γ increases, we obtain for different γ in unit of ω_0, a plot of resonance curves (see Figure IV.5).

Phase shift study

We have :

$$\begin{aligned} \tan\varphi(\Omega) &= \frac{2\gamma\Omega}{\Omega^2-\omega_0^2} \Rightarrow [\tan\varphi(\Omega)]' = \frac{2\gamma(\Omega^2-\omega_0^2)-4\gamma\Omega^2}{(\Omega^2-\omega_0^2)^2} \\ &= 2\gamma\frac{\Omega^2+\omega_0^2}{(\Omega^2-\omega_0^2)^2} < 0 \end{aligned}$$

So $\tan\varphi(\Omega)$ decreasing, we obtain a plot of curves according to the damping (Figure IV.6) :

$\lim_{\Omega \to 0} \tan\varphi = 0 \rightarrow \varphi = 0 \rightarrow x(t)$ and $F(t)$in phase.

$\lim_{\Omega \to \Omega_r} \tan\varphi = -\infty \rightarrow \varphi = -\pi/2 \rightarrow x(t)$ and $F(t)$in quadrature phase.

$\lim_{\Omega \to \infty} \tan\varphi = 0 \rightarrow \varphi = -\pi \rightarrow x(t)$ and F(t) in phase opposition.

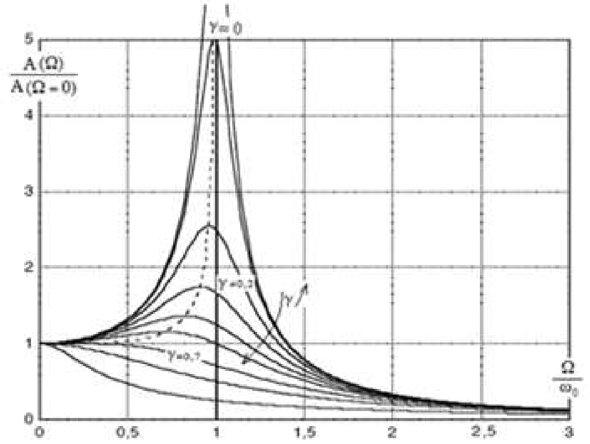


Figure IV.5 : Evolution of amplitude as a function of pulsation.

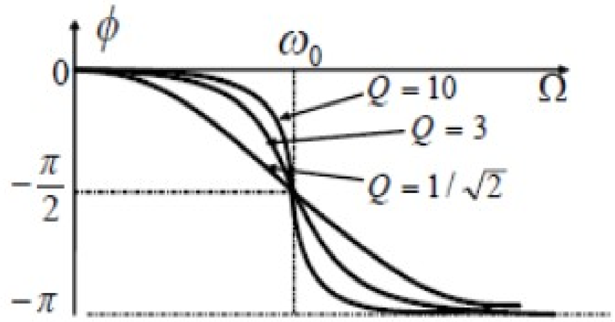


Figure IV.6 : Evolution of the phase as a function of the pulsation.

4.2.2 Electromechanical analogy

Electromechanical analogies establish a correspondence between mechanical and electrical quantities, allowing both types of systems to be studied using the same methods. The major advantage is that they work both ways : a mechanical system can be modeled using an electrical equivalent and vice versa, thus facilitating the analysis, simulation, and understanding of dynamic phenomena..

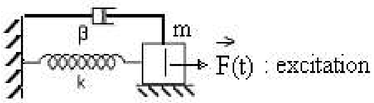
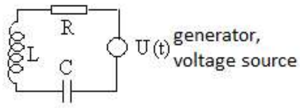
Forced mechanical system	powered Electrical system
(mass + spring+damping)	(oscillating RLC circuit)
	
Excitation force $= F(t)$	Supply voltage $= U(t)$
Damping coefficient β	Resistance R (energy dissipation)
$f =$ Friction force, $= -\beta v$, $v = dx/dt$	$U_R =$ Effect Joule $= Ri$, $i = dq/dt$
Spring, Restoring Force : $F = kx$	Capacitor, Voltage across C : $Uc = q/C$
Fundamental relationship of dynamics	Law in a mesh
$\sum \overrightarrow{F} = m\overrightarrow{\gamma}$	$V_L + V_R + V_C = U(t)$
$m\ddot{x} + \beta\dot{x} + kx = F(t)$	$\frac{Ldi}{dt} + Ri + 1/C \int idt$ $= L\ddot{q} + R\dot{q} + \frac{1}{C}q = U(t)$
$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\Omega t$	$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = \frac{U_0}{L}\cos\Omega t$
with $\gamma = \beta/2m \qquad \omega_0^2 = k/m$	with $\gamma = R/2L \qquad \omega_0^2 = 1/LC$

Table IV.1 : Analogy of electromechanical parameters.

We have seen the amplitude of the steady state of a mechanical system :

$$A(\Omega) = \frac{F_0/m}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2}} \qquad A_{\max} = A(\Omega_r) = \frac{F_0/m}{2\gamma\sqrt{(\omega_0^2 - \gamma^2)}},$$

with :

$$\gamma = \frac{\beta}{2m}, \text{ and } \omega_0^2 = \frac{k}{m}$$

Then, by analogy for electrical oscillations :

$$A(\Omega) = \frac{U_0/L}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2}} \qquad A_{\max} = A(\Omega_r) = \frac{U_0/L}{2\gamma\sqrt{(\omega_0^2 - \gamma^2)}},$$

with :

$$\gamma = \frac{R}{2L}, \quad \text{and } \omega_0^2 = \frac{1}{LC}$$

4.2.3 Electromechanical Analogies in Terms of Impedance

The impedance representation allows mechanical systems to be modeled as equivalent electrical circuits, where the equations of motion take the form of generalized Ohm's laws. Conversely, electrical behavior can also be interpreted in mechanical terms, providing a deeper understanding of the similarities and interactions between the two domains.

Consider a series RLC circuit excited by a sinusoidal voltage :

$$U(t) = U_0 \sin \Omega t \tag{4.26}$$

$$I(t) = I_0 \sin(\Omega t + \varphi) \tag{4.27}$$

Using complex notation :

$$\overline{U} = U_0 \exp^{j\Omega t} \tag{4.28}$$

$$\overline{I} = I_0 \exp^{j(\Omega t - \varphi)} \tag{4.29}$$

The relation between excitation and response is :

$$\overline{U} = \overline{ZI} \tag{4.30}$$

where U is the exitation, I is the response and the electrical impedance is defined as :

$$\overline{Z} = R + j(L\Omega - \frac{1}{C\Omega}) \tag{4.31}$$

So, the magnitude of the current is obtained from :

$$I_0 = \frac{|\overline{U}|}{|\overline{Z}|} = \frac{U_0}{\sqrt{R^2 + \left(L\Omega - \frac{1}{C\Omega}\right)^2}} \tag{4.32}$$

The phase angle between the current and the voltage is given by :

$$\varphi = \arg(I) = \arctan \frac{L\Omega - \dfrac{1}{C\Omega}}{R} \tag{4.33}$$

The voltage is imposed constant, the current is max for $\Omega = \omega_0$ such as $LC\omega_0^2 = 1$
We can then consider that, by analogy, there exists a mechanical impedance :

$$\overline{Z}_m = \beta + j(m\Omega - \frac{k}{\Omega}) = \frac{\overline{F}}{\overline{v}} \tag{4.34}$$

Thus, studying sinusoidal steady state in electrical circuits helps in understanding vibration of mechanical systems. This is the essence of electromechanical analogies.

4.2.4 Bandwidth

The $3dB$ bandwidth (Figure IV.7) is by definition the set of pulsations for which the amplitude $A \geq A_{\max}/\sqrt{2}$. The bandwidth width is noted $\Delta\Omega = \Omega_2 - \Omega_1$, with Ω_2, and Ω_1 correspond to the $A(\Omega_1) = A(\Omega_2) = A_{\max}/\sqrt{2}$.

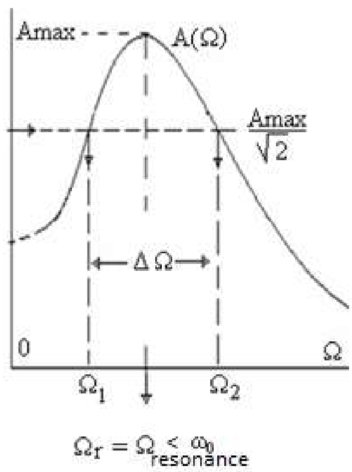


Figure IV.7 : A as a function of Ω (bandwidth).

In practice, in oscillating electrical circuits, and unlike mechanical oscillatory systems where the resonance phenomenon is a danger, we seek to have a strong resonance with **low damping** (see plot of the curves of $A(\Omega)$:

$$\gamma^2 << \omega_0^2 \rightarrow \left\{ \begin{array}{c} A_{wax} = \dfrac{U_0/L}{2\gamma\sqrt{\omega_0^2 - \gamma^2}} \sim \dfrac{U_0/L}{2\gamma\omega_0} \\ \Omega_r = \sqrt{\omega_0^2 - 2\gamma^2} \sim \omega_0 \end{array} \right\} \quad (4.35)$$

The lower γ, the narrower the resonance peak (smaller $\Delta\Omega$), so we can consider in $\Delta\Omega$ that Ω is near to ω_0.
We then show that $\Omega_1 \approx \omega_0\ (1 - \gamma/\omega_0)$; $\Omega_2 \approx \omega_0\ (1 + \gamma/\omega_0)$,hence the bandwidth of the oscillator :

$$\Delta\Omega = \Omega 2 - \Omega 1 = 2\gamma = R/L = \beta/m;$$

depending on whether it is electrical or mechanical

4.2.5 Quality factor

We define a quality factor of the oscillator which reflects the sharpness and amplitude of the peak :

$$Q = \frac{\Omega_r}{\Delta\Omega} \sim \frac{\omega_0}{2\gamma} \quad (4.36)$$

Example 2 *Example : The quality factor of an electronic filter represents the selectivity of the bandpass filter : it only lets through frequencies around fo included in Δf.*

$Q = \dfrac{f_0}{\Delta f}$ with center frequency f_0 and the filter bandwidth Δf for a gain of -3 dB.

Remark 6 *Unlike mechanical systems, where significant dampers must be designed to avoid $\gamma = 0$, in oscillating electrical systems, the goal is to have γ as small as possible (thin peak).*

Indeed, a low γ corresponds to light damping, which results in a frequency response with a sharp and well-defined resonance peak. This means that the system responds very strongly at a specific frequency while effectively attenuating other frequencies. This characteristic is especially desirable in applications such as electronic filters, tuned **circuits**, and **sensing devices**, where frequency selectivity is essential.

4.3 Conclusion

Forced oscillations in a single-degree-of-freedom (SDOF) system occur when the system is subjected to an external periodic force. The system responds with oscillatory motion whose amplitude depends on the excitation frequency, the system's natural frequency, and the damping factor.

Resonance arises when the excitation frequency is close to the natural frequency, resulting in a maximum response, especially in lightly damped systems. The bandwidth ($\Delta\Omega$), which defines the width of the resonance peak, is directly proportional to the damping coefficient γ. A small damping value leads to high frequency selectivity, a desirable feature in many engineering applications.

Understanding these dynamics is essential for designing or analyzing mechanical and electrical systems under periodic excitation, enabling better control of their stability, accuracy, and efficiency.

4.4 Corrected exercises

4.4.1 Exercise 1

An oscillator has the equation of motion :

$$2\ddot{x} + 16\dot{x} + 50x = 102\cos t \tag{4.37}$$

1) Determine, in this case, the natural period T_0, the coefficient, γ and the excitation pulsation Ω.

2) Show that the solution of the transient state is a damped oscillatory motion, Deduce its pseudo pulsation ω. Write the solution to this transient state.

3) Determine the steady-state solution.

Solution

The equation of motion is given by :

$$\left\{ \begin{array}{c} 2\ddot{x} + 16\dot{x} + 50x = 102\cos t \\ (4.29) \Longleftrightarrow \ddot{x} + 8\dot{x} + 25x = 51\cos t \end{array} \right\} \tag{4.38}$$

of the same type as :

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_0 \cos \Omega t$$

$$\omega_0^2 = 25 \Longrightarrow T_0 = \frac{2\pi}{\omega_0} = 1.25s$$

$$2\gamma = 8 \Longrightarrow \gamma = 4, \qquad \Omega = 1rad/s$$

2) The solution of the transient state

$$X_g = X_{SSm} + X_p$$

X_{SSm} = solution $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$

$\Delta' = \gamma^2 - \omega_0^2 = 16 - 25 = -9 < 0 \longrightarrow$ low damping, the solution is a damped oscillatory regime.

$$X(t) = c\exp^{(-\gamma t)}\cos(\omega t + \varphi)$$

$$\omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{25 - 16} = 3rad/s.$$

$$\Longrightarrow X(t) = \exp^{(-4t)}\cos(3t + \varphi)$$

3) The solution of the steady state

$$X_p = x\cos(\Omega t + \phi) = x\cos(t + \phi)$$

complex writing : $\cos t \rightarrow \exp^{it}$

$X_p = X\cos(t + \phi) \longrightarrow \overline{X}_p = X\exp^{i(t+\phi)} = \overline{X}\exp^{it}$ with $\overline{X} = X\exp^{i\phi}$.

X_p is a solution to the differential equation, therefore, it verifies the equation :

$$\ddot{x_p} + 8\dot{x_p} + 25x_p = 51\cos t$$

$$\overline{X}(-1 + i8 + 25) = 51$$

$$X = \frac{51}{\sqrt{(24)^2 + 64}} = 2$$

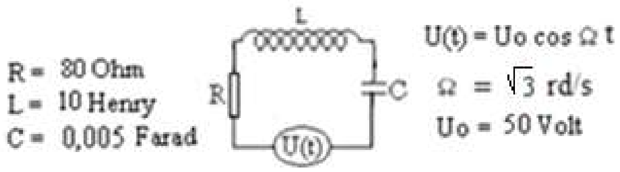
$$\phi = -\arctan\frac{8}{24} = -18.4^0$$

from where

$$X_p = 2\cos(t - 0.32)$$

4.4.2 Exercise 2

Establish the differential equation first in terms of current, then in terms of charge, for the oscillatory electrical circuit shown in the figure.



1) Calculate the natural period T_0 and the damping coefficient γ.

2) Determine the transient solution, and deduce the pseudo-angular frequency ω.

3) Determine the steady-state solution.

Solution

1) By applying Kirchhoff's voltage law (KVL) to a closed loop in an electrical circuit, the sum of the voltage sources must be equal to the sum of the voltage drops across the elements. Therefore,

$$\begin{aligned} U(t) &= U_R + U_L + U_C \\ &= RI + L\frac{di}{dt} + \frac{1}{C}\int idt \end{aligned}$$

$$\left\{\begin{array}{c} i = \dfrac{dq}{dt} = \dot{q} \\ \dfrac{di}{dt} = \dfrac{d^2q}{dt^2} = \ddot{q} \\ \int idt = q \end{array}\right\} \Longrightarrow \begin{array}{c} R\dot{q} + L\ddot{q} + \dfrac{q}{c} = U_0 \cos \Omega t \\ \ddot{q} + \dfrac{R}{L}\dot{q} + \dfrac{1}{LC}q = \dfrac{U_0}{L}\cos \Omega t \end{array}$$

Of the same as $\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = \dfrac{U_0}{L}\cos \Omega t$

$\omega_0^2 = \dfrac{1}{LC} \Longrightarrow \omega_0 = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{10 \times 0.05}} = 20 rad/s$

$T_0 = \dfrac{2\pi}{20} \Longrightarrow T_0 = 1.4s$

$2\gamma = \dfrac{R}{L} \Longrightarrow \gamma = \dfrac{R}{2L} = 4$

2) Solution of the transient state

$$X_g = X_{SSm} + X_p$$

X_{SSm} = solution $\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = 0$

$\Delta' = \gamma^2 - \omega_0^2 = 16 - 20 = -4 < 0 \longrightarrow$ **low damping**, the solution is a damped oscillatory regime.

$$X(t) = c\exp^{(-4t)}\cos(\omega t + \varphi)$$

$$\omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{20 - 16} = 2 rad/s.$$

$$\Longrightarrow X(t) = \exp^{(-4t)}\cos(2t + \varphi)$$

3) The solution of the steady state

$$X_p = A\cos(\sqrt{3}t + \varphi)$$

Complex writing : $\cos t \rightarrow \exp^{it}$

$X_p = A\cos(\sqrt{3}t + \phi) \longrightarrow \overline{X}_p = X\exp^{i(t+\phi)} = \overline{X}\exp^{it}$ with $\overline{X} = X\exp^{i\phi}$.

X_p is a solution to the differential equation, therefore, it verifies the equation :

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = U_0/L\cos\sqrt{3}t$$

$$\Longrightarrow \ddot{X}_p + 2\gamma\dot{X}_p + \omega_0^2 X_p = 5\cos\sqrt{3}t$$

In complex notation, we can write :

$$X_p = A\exp^{j(\sqrt{3}t+\varphi)} = A\exp^{j\varphi}\exp^{j\sqrt{3}t} = \overline{A}\exp^{j\sqrt{3}t}$$
$$\dot{X}_p = A\exp^{j\varphi} j\sqrt{3}\exp^{j\sqrt{3}t}$$
$$\ddot{X}_p = -3A\exp^{j\sqrt{3}t}\exp^{j\varphi}$$

$$\overline{A}(-3 + j8\sqrt{3} + 20) = 5 \Longrightarrow \overline{A}\left(17 + j8\sqrt{3}\right) = 5$$

$$\Longrightarrow \sqrt{(17)^2 + \left(8\sqrt{3}\right)^2}\,|\overline{A}| = 5 \Longrightarrow A = \frac{5}{\sqrt{(17)^2 + \left(8\sqrt{3}\right)^2}} = 0.2$$

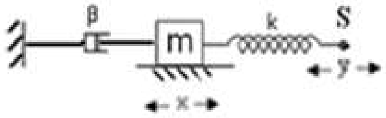
$$\phi = -\arctan\frac{8\sqrt{3}}{17} = -0.7rad$$

from where :

$$X_p = 0.2\cos\left(\sqrt{3}t - 0.7\right)$$

4.4.3 Exercise 3

We consider the mechanical oscillatory system shown in Figure below :



Point S undergoes a horizontal sinusoidal motion with amplitude $y(t) = acos(\Omega t)$.

1) Using the fundamental principle of dynamics, derive the differential equation of motion for the mass m.

2) Given : $m = 0.3kg; k = 1.2N/m; \beta = 0.6kg/s$.

Show that the transient motion of mass m is a damped oscillatory motion. Write the corresponding solution.

3) Given : $a = 1.3kg; \Omega = 2rd/s$. Determine the steady-state solution.

Solution

1) **Differential equation of motion**

$$\textbf{FLD} \Longrightarrow \sum \overrightarrow{F} = m\gamma \Rightarrow -\beta v - kx - ky = m\ddot{x}$$

$$\Longrightarrow \ddot{x} + \frac{\beta}{m}\dot{x} + \frac{k}{m}x = \frac{k}{m}y$$

Therefore :

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{k}{m}acos(\Omega t) \qquad (4.39)$$

2) Given : $m = 0.3\ kg$; $k = 1.2\ N/m$; $\beta = 0.6\ kg/s$, we have :

$$\omega_0^2 = \frac{k}{m} = \frac{1.2}{0.3} = 4rad/s$$

$$\gamma = \frac{\beta}{2m} = \frac{0.6}{2 \times 0.3} = 1$$

Then, to solve the differential equation, we perform a change of variable, hence, the characteristic equation :

$$r^2 + 2\gamma r + \omega_0^2 = 0$$

$\Delta' = \gamma^2 - \omega_0^2 = 1 - 4 = -3 < 0 \longrightarrow$ low damping, the solution is a damped oscillatory regime.

$$X(t) = c\exp^{(-\gamma t)}\cos(\omega t + \varphi)$$

$$\omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{4-1} = \sqrt{3} rad/s.$$

$$\Longrightarrow x(t) = C\exp^{(-t)}\cos(\sqrt{3}t + \varphi)$$

3) The solution of the steady state

Given : $a = 1.3kg; \Omega = 2rd/s.$

X_p is a solution to the differential equation, therefore, it verifies the equation :

$$\Longrightarrow \ddot{X}_p + 2\dot{X}_p + 4X_p = 4\cos 2t$$

In complex notation, we can write :

$$\left\{\begin{array}{l} y(t) = \cos 2t \rightarrow \exp^{j2t} \\ X_p = a\cos(2t+\phi) \longrightarrow \exp^{j(2t+\phi)} \end{array}\right\}$$

$$\begin{aligned} X_p &= A\exp^{j(2t+\varphi)} = A\exp^{j\varphi}\exp^{j2t} = \overline{A}\exp^{j2t} \\ \dot{X}_p &= 2A\exp^{j\varphi} j\exp^{j2t} \\ \ddot{X}_p &= -4A\exp^{j2t}\exp^{j\varphi} \end{aligned}$$

$$-4A\exp^{j2t}\exp^{j\varphi} + 4A\exp^{j\varphi} j\exp^{j2t} + 4A\exp^{j\varphi}\exp^{j2t} = 4\exp^{j2t}$$

$$A\exp^{j\varphi}(-4 + 4j + 4) = 4$$

$$\overline{A}(4j) = 4$$

$$\Longrightarrow \sqrt{(4)^2}\,|\overline{A}| = 4 \Longrightarrow A = \frac{4}{\sqrt{(4)^2}} = 1$$

$$\phi + \underbrace{\arctan\frac{4}{0}}_{\infty} = \underbrace{\arctan 0}_{0} \Longrightarrow \phi = -\frac{\pi}{2}$$

From where :

$$X_p = \cos\left(2t - \frac{\pi}{2}\right)$$

Chapter V

Oscillations of systems with two degrees of freedom

Chapter 5

Oscillations of systems with two degrees of freedom

5.1 Introduction

In the study of dynamic systems, many real physical systems cannot be satisfactorily modeled by a single-degree-of-freedom model. It is then necessary to introduce several independent coordinates to fully describe the system's motion. In this case, we speak of **n degree** of freedom systems.

5.1.1 Free systems with two degrees of freedom

Systems that require two independent coordinates to specify their positions are called two-degree-of-freedom systems. **Two-degree**-of-freedom systems consist of two coupled one-degree-of-freedom systems (Figure V.1).
But often there are (mathematical) relationships or (physical) links between the coordinates (Figure V.2), (Figure V.3).

n is dimension of the system equal to number of **degrees of freedom**. **A degree of freedom** represents an independent mode of motion (translation or rotation). Thus, a system with n degrees of freedom can exhibit n different forms of simultaneous motion.

Common examples include :

* a multi-story structure subjected to an earthquake,

* a system of masses connected by springs,

* an articulated robotic arm,

* a vibrating beam modeled using finite elements.

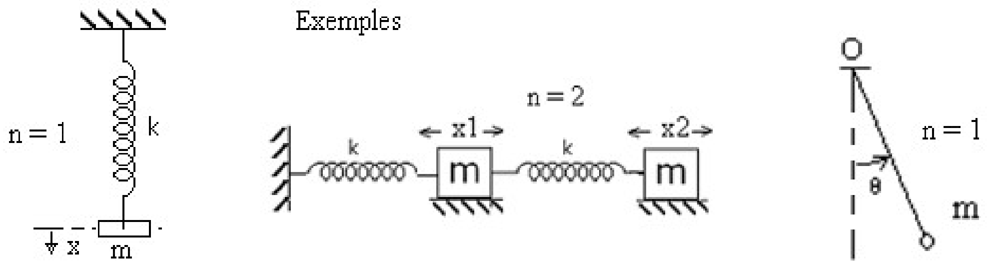


Figure V.1 : Examples of two-degree-of-freedom systems.

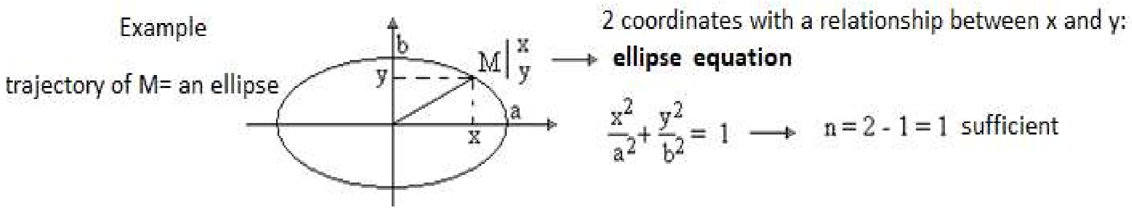


Figure V.2 : Example of relation ship between the coordinates (case of ellipse).

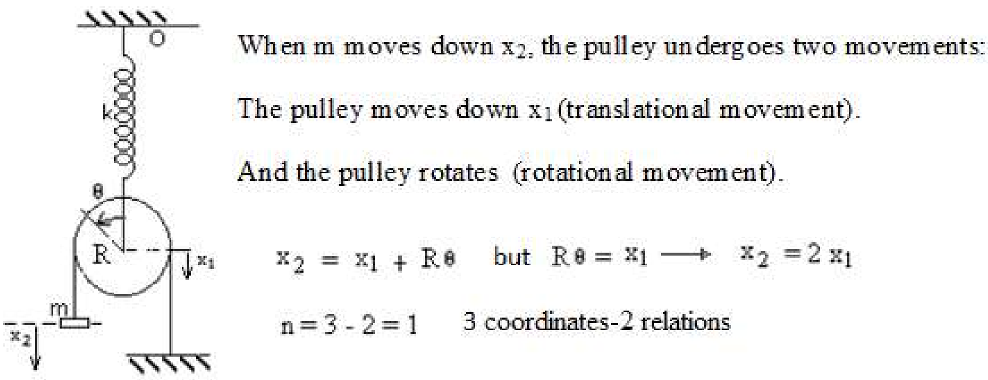


Figure V.3 : Example of relation ship between the coordinates (case of pulley).

In this chapter, we will develop a comprehensive understanding of the dynamic behavior of systems with multiple degrees of freedom. We begin by formulating the general equations of motion for 2-DOF systems using matrix notation, accounting for mass, stiffness, damping, and external forces. We will then examine the free vibrations (in the absence of external forces). A key focus will be the concept of normal modes of vibration — the natural patterns in which a system oscillates — and the use of modal analysis to decuple and simplify the equations governing motion. The role of damping will also be investigated to understand how energy dissipation influences amplitude, stability, and resonance behavior. Finally, these theoretical tools will be applied to a range of engineering problems, enabling the analysis and design of practical systems such as multi-storey structures, mechanical linkages, and vibrating machinery.

5.1.2 Normal Modes

In the context of oscillatory systems, **a normal mode** refers to a specific pattern of motion in which all parts of the system oscillate at the same frequency and with fixed amplitude ratios. This frequency is known as a natural frequency, and the corresponding motion is called a mode of vibration.

When the equations of motion are linear and there is no external forcing or damping, the system's response can be analyzed by solving the associated homogeneous differential equations. The solution reveals that the system behaves as a superposition of n independent simple harmonic oscillators, each oscillating at its own natural frequency. These independent oscillators are referred to as the system's **normal modes**.

In a system with n degrees of freedom, there are exactly n such normal modes, each characterized by :
* A natural frequency ω_i
* A mode shape (or eigenvector) ϕ_i which describes the relative motion of the system's components during that mode.

Each normal mode describes a very specific way in which the system can vibrate naturally, without the other modes intervening. If the system is excited in exactly one mode, it will vibrate only in that mode (ideally).

In practice, the overall motion of the system is a linear combination of all **normal modes**, according to the principle of modal superposition.

5.1.3 couplings

A system with n degrees of freedom is the union of n subsystems with one degree of freedom. The motion of each subsystem influences the other subsystems.
The subsystems are said to be **coupled**.

Elastic Coupling

Coupling in mechanical systems is provided by elasticity (a spring). In electrical systems, circuits are coupled by capacitance, which is equivalent to elastic coupling (Figure V.4).

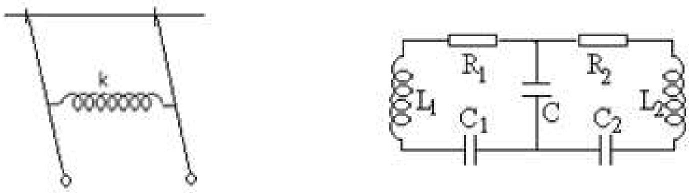


Figure V.4 : Coupling of elastic and electrically equivalent systems.

viscous coupling

Coupling in mechanical systems is evident by damper coupling. In electrical systems, we see resistance-coupled circuits, equivalent to damper coupling (Figure V.5).

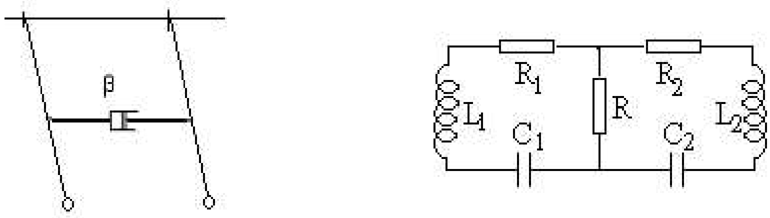


Figure V.5 : Coupling of viscous and electrically equivalent systems.

Inertial coupling

Coupling in mechanical systems is provided by inertia. In electrical systems, we find inductance-coupled circuits, equivalent to inertial coupling (Figure V.6).

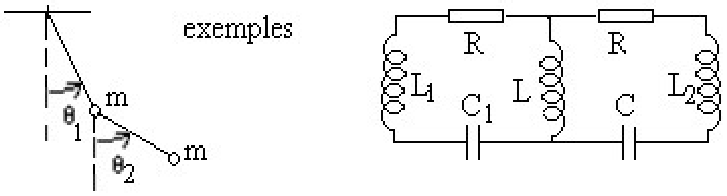


Figure V.6 : Coupling of inertial and electrically equivalent systems.

5.1.4 Example of oscillations of a system with two degrees of freedom

Complex system (mass-springs)

Consider the mechanical system represented in Figure V.7 and composed of two harmonic oscillators (m_1, k_1), and (m_2, k_2) **coupled** by a spring of stiffness constant k. Two masses are assumed to move without friction on a horizontal plane and their elongations relative to their equilibrium positions are identified by x_1, and x_2.

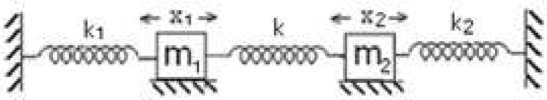


Figure V.7 : Oscillatory motion of a coupled system
with two degrees of freedom.

Differential equations of motion

1) **Newton's Dynamic Principle**

$\sum \overrightarrow{F} = m\overrightarrow{\gamma}$

$$\Longrightarrow \left\{ \begin{array}{l} m_1\ddot{x}_1 = -k_1x_1 - k\left(x_1 - x_2\right) \\ m_2\ddot{x}_2 = -k_2x_2 - k\left(x_2 - x_1\right) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \ddot{x}_1 + \dfrac{k_1}{m_1}x_1 + \dfrac{k}{m_1}x_1 - \dfrac{k}{m_1}x_2 = 0 \\ \ddot{x}_2 + \dfrac{k_2}{m_2}x_2 + \dfrac{k}{m_2}x_2 - \dfrac{k}{m_2}x_1 = 0 \end{array} \right\}$$

$$\Longrightarrow \left\{ \begin{array}{l} \ddot{x}_1 + \left(\dfrac{k_1 + k}{m_1}\right)x_1 - \dfrac{k}{m_1}x_2 = 0 \\ \ddot{x}_2 + \left(\dfrac{k_2 + k}{m_2}\right)x_2 - \dfrac{k}{m_2}x_1 = 0 \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \ddot{x}_1 + a_{11}x_1 + a_{12}x_2 = 0 \\ \ddot{x}_2 + a_{21}x_2 + a_{22}x_1 = 0 \end{array} \right\}$$

with $\left\{ \begin{array}{ll} a_{11} = \dfrac{k_1 + k}{m_1}, & a_{12} = -\dfrac{k}{m_1} \\ a_{21} = \dfrac{k_2 + k}{m_2}, & a_{22} = -\dfrac{k}{m_2} \end{array} \right\}$

2) Lagrange method for a free system

$$\text{Lagrange equation} : \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad (i = 2) \tag{5.1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0 \tag{5.2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0$$

$$L = E_c - E_p = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \left(\frac{1}{2}k_1x_1^2 + \frac{1}{2}k(x_2 - x_1)\right)^2 + \frac{1}{2}k_2x_2^2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = m_1\ddot{x}_1, \qquad \frac{\partial L}{\partial x_1} = -k_1x_1 - k\left(x_1 - x_2\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = m_2x_2, \quad {}^{\cdot\cdot} \qquad \frac{\partial L}{\partial x_2} = -k_2x_2 - k\left(x_2 - x_1\right)$$

$$\Longrightarrow \left\{ \begin{array}{l} m_1\ddot{x}_1 = -k_1x_1 - k\left(x_1 - x_2\right) \\ m_2\ddot{x}_2 = -k_2x_2 - k\left(x_2 - x_1\right) \end{array} \right\} \tag{5.3}$$

The system of equations may be expressed in the following form :

$$\left\{ \begin{array}{l} \ddot{x}_1 + a_{11}x_1 + a_{12}x_2 = 0 \\ \ddot{x}_2 + a_{21}x_2 + a_{22}x_1 = 0 \end{array} \right\} \tag{5.4}$$

Resolution

To solve the system of linear differential equations, we can state that the free oscillatory subsystems are **MHS** so the individual solutions are of the type :

$$\begin{aligned} x_1 &= A_1 \cos(\omega_0 t + \varphi_1) \\ x_2 &= A_2 \cos(\omega_0 t + \varphi_2) \end{aligned}$$

these solutions verify the differential equations, so we replace :

$$x_{1,2} = A_{1,2} \cos(\omega_0 t + \varphi_{1,2}) \tag{5.5}$$

Or, in complex notation :

$$x_{1,2} = A_{1,2} \exp^{j(\omega_0 t + \varphi_{1,2})} = A_{1,2} \exp^{j\varphi_{1,2}} \exp^{j\omega_0 t}$$

$$= \overline{X}_{1,2} \exp^{j\omega_0 t}, \quad \text{avec } \overline{X}_{1,2} = A_{1,2} \exp^{j\varphi_{1,2}} \tag{5.6}$$

So

$$x_1 = \overline{X}_1 \exp^{j\omega_0 t} \rightarrow \dot{x}_1 = j\omega_0 \overline{X}_1 \exp^{j\omega_0 t} \rightarrow \ddot{x}_1 = -\omega_0^2 \overline{X}_1 \exp^{j\omega_0 t}$$

$$x_2 = \overline{X}_2 \exp^{j\omega_0 t} \rightarrow \dot{x}_2 = j\omega_0 \overline{X}_2 \exp^{j\omega_0 t} \rightarrow \ddot{x}_2 = -\omega_0^2 \overline{X}_2 \exp^{j\omega_0 t} \tag{5.7}$$

The substitution of assumed solutions into the differential model leads to a system of linear equations :

$$\left\{ \begin{array}{l} -\omega_0^2 \overline{X}_1 \exp^{j\omega_0 t} + a_{11} \overline{X}_1 \exp^{j\omega_0 t} + a_{12} \overline{X}_2 \exp^{j\omega_0 t} = 0 \\ -\omega_0^2 \overline{X}_2 \exp^{j\omega_0 t} + a_{21} \overline{X}_2 \exp^{j\omega_0 t} + a_{22} \overline{X}_1 \exp^{j\omega_0 t} = 0 \end{array} \right\} \tag{5.8}$$

$$\left\{ \begin{array}{l} (-\omega_0^2 + a_{11}) \overline{X}_1 + a_{12} \overline{X}_2 = 0 \\ (-\omega_0^2 + a_{21}) \overline{X}_2 + a_{22} \overline{X}_1 = 0 \end{array} \right\} \tag{5.9}$$

This system lends itself to a matrix representation :

$$\begin{pmatrix} -\omega_0^2 + a_{11} & a_{12} \\ a_{22} & -\omega_0^2 + a_{21} \end{pmatrix} \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5.10}$$

We obtain a system of two equations with two unknowns and no right-hand side (homogeneous system).

A necessary and sufficient condition for the existence of non-trivial solutions is that the determinant equals zero.

$$\det(\omega) = \begin{pmatrix} -\omega_0^2 + a_{11} & a_{12} \\ a_{22} & -\omega_0^2 + a_{21} \end{pmatrix} = 0 \tag{5.11}$$

$$\det = \left(-\omega_0^2 + a_{11}\right)\left(-\omega_0^2 + a_{21}\right) - a_{12}a_{22} = 0 \tag{5.12}$$

$$\omega_0^4 - (a_{21} + a_{11})\,\omega_0^2 + a_{11}a_{21} - a_{12}a_{22} = 0 \tag{5.13}$$

$$\omega_0^4 - (a_{21} + a_{11})\,\omega_0^2 + a_{11}a_{21}(1 - \frac{a_{12}a_{22}}{a_{11}a_{21}}) = 0 \tag{5.14}$$

knowing that :

$$\frac{a_{12}a_{22}}{a_{11}a_{21}} = \frac{\left(\frac{-k}{m_1}\right)\left(\frac{-k}{m_2}\right)}{\left(\frac{k_1 + k}{m_1}\right)\left(\frac{k_2 + k}{m_2}\right)} = \frac{k^2}{(k_1 + k)(k_2 + k)} = K^2,$$

K is called the coupling coefficient

This leads to the characteristic equation for the natural frequencies :

$$\omega_0^4 - (a_{21} + a_{11})\,\omega_0^2 + a_{11}a_{21}(1 - K^2) = 0 \tag{5.15}$$

Coupling Coefficient

The spring k in the middle represents the elastic coupling between the simple harmonic motions (SHMs) of the masses m_1 and m_2.

* When $k = 0$, i.e., when there is no spring, there is no coupling between the masses $\rightarrow K = 0$. In this case, the oscillations of m_1 and m_2 are independent, and the coupling is said to be very weak (**loose coupling**).

* When k is infinitely large, that is, much greater than k_1 and $k_2, \rightarrow K = 1$. In this case, the middle spring behaves like a rigid bar, resulting in strong coupling between the two masses.

m_1 and m_2 oscillate as a single rigid body ; this situation is referred to as **tight coupling**.

* When $k \neq 0$ and is small, we have $0 \leq K \leq 1$. The situation is referred to as **loose (or weak) coupling**.

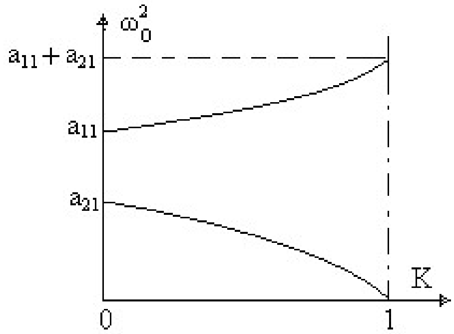
5.1.5 Resolution of the characteristic equation for the Eigen-frequencies

$$\omega_0^4 - (a_{21} + a_{11})\,\omega_0^2 + a_{11}a_{21}(1 - K^2) = 0 \tag{5.16}$$

$$\begin{aligned} \text{Discriminant } \Delta &= (a_{11} + a_{21})^2 - 4a_{11}a_{21}(1 - K_2) \\ &= a_{11}^2 + a_{21}^2 + 2a_{11}a_{21} - 4a_{11}a_{21} + 4a_{11}a_{21}K^2 \end{aligned}$$

$$\Delta = (a_{11} + a_{21})^2 + 4a_{11}a_{21}K^2 \text{ always} > 0$$

$$\longrightarrow \omega_{0\pm}^2 = \frac{(a_{11} + a_{21})}{2} \pm \sqrt{\frac{(a_{11} - a_{21})^2}{2} + a_{11}a_{21}K^2}$$



For $\begin{array}{ll} K=0 \longrightarrow \omega_{0+}^2 = a_{11} & \text{or } \omega_{0-}^2 = a_{21} \\ K=1 \longrightarrow \omega_{0+}^2 = a_{11}+a_{21} & \text{or } \omega_{0-}^2 = 0 \end{array}$

Let us assume that $a_{11} > a_{21}$

The effect of the coupling $(0 \leq K \leq 1)$ is to increase the separation between the natural frequencies.

Discussions

* Very loose coupling $K = 0$, there are two cases :

If $\omega_0^2 = \omega_+^2 = a_{11}$

$$\longrightarrow \left(-\omega_0^2 + a_{11}\right)\overline{X}_1 + a_{12}\overline{X}_2 = 0 \longrightarrow \overline{X}_2 = \frac{\omega_0^2 - a_{11}}{a_{22}}\overline{X}_1 = 0, \quad m_1 \text{ oscillates alone.} \tag{5.17}$$

If $\omega_0^2 = \omega_-^2 = a_{21}$

$$\longrightarrow \left(-\omega_0^2 + a_{21}\right)\overline{X}_2 + a_{22}\overline{X}_1 = 0 \longrightarrow \overline{X}_1 = \frac{\omega_0^2 - a_{11}}{a_{22}}\overline{X}_2 = 0, \quad m_2 \text{ oscillates alone.} \tag{5.18}$$

In both cases, the natural periods of the oscillatory system are those of the two individual oscillations taken separately.

* **Strong coupling**

$K = 1$, There is one possible case.

If $\omega_{0+}^2 = a_{11} + a_{21} \longrightarrow$ An oscillation is given by the expression $T = 2\pi/\left(a_{11} + a_{21}\right)$

If $\omega_{0-}^2 = 0 \rightarrow$ Infinite period T, no oscillations.

* **Weak coupling**

$0 \leq K \leq 1$, It can be shown that the possible periods of $m_1$and m_2 differ only slightly from their natural periods.

The individual solutions are :

$$x_1 = X_1.\cos(\omega_{01}t + \varphi_1), \tag{5.19}$$

$$x_2 = X_2.\cos(\omega_{02}t + \varphi 2), \tag{5.20}$$

Thus, when there are two solutions to a problem, the general solution is a linear combination of these two solutions. In other words, the most general motion is the superposition of the two modes :

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = p \begin{bmatrix} X_1.\cos(\omega_{01}t + \varphi_1) \\ X_2.\cos(\omega_{02}t + \varphi 2) \end{bmatrix} \tag{5.21}$$

Thus $p =$ Modal matrix= $\left(\vec{V}1\vec{V}2\right)$.
$\vec{V}1$, and $\vec{V}2$ are the eigenvectors associated with the vibration modes.

Remark 7 *If there is no damping, mode 1 or mode 2 will continue indefinitely. Since the general solution is a combination of the two solutions with different frequencies, the resulting motion will evolve into a beating phenomenon. The constants X and φ are determined by the initial conditions applied to the general solution. A system with n degrees of freedom has n modes.*

5.2 Conclusion

This chapter introduced the fundamental principles of systems with two degrees of freedom. We derived their equations of motion, analyzed free vibrations, and explored the concepts of natural frequencies and normal modes. These systems exhibit coupled behavior, where energy can transfer between components.

By solving the eigenvalue problem, we identified how such systems respond dynamically and how their modes of vibration are determined. The tools and methods presented here are essential for understanding more complex vibrating systems and for performing modal analysis in engineering applications.

5.3 Corrected exercises

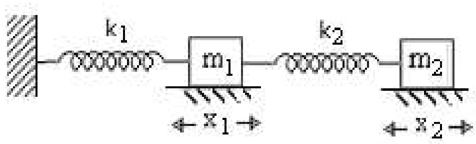
5.3.1 Exercise 1

We consider the free oscillations of the following two-degree-of-freedom system :

Given : $k_1 = k$, $k_2 = 2k$, $m_1 = m$, $m_2 = 2m$

1) Establish the differential equations of motion.

2) Find the natural frequencies of the system.



Solution

1) **Differential equations of motion**

Using Lagrange's equations Using Lagrange's equations for a system of two masses and two springs, we can write :

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) \tag{5.22}$$

$$m_1\ddot{x}_1 = -k_2(x_2 - x_1) \tag{5.23}$$

knowing that $k_1 = k$, $k_2 = 2k$, $m_1 = m$, $m_2 = 2m$, the system of equations reduces to :

$$\begin{aligned} \ddot{x}_1 + \frac{3k}{m}x_1 - 2\frac{k}{m}x_2 &= 0 \\ \ddot{x}_2 + \frac{k}{m}x_2 - \frac{k}{m}x_1 &= 0 \end{aligned}$$

we have a system of second-order differential equations without forcing term. The solutions are sinusoidal harmonic motions :

$$x_{1,2} = X_{1,2}\cos(\Omega t + \varphi_{1,2});$$

Complex representation of harmonic motion :

$$\overline{x}_{1,2} = X_{1,2} \exp j(\Omega t + \varphi_{1,2}) = \overline{X}_{1,2} \exp \exp j\Omega t, \text{ avec } \overline{X}_{1,2} = X_{1,2} \exp j\varphi_{1,2}$$

$$\longrightarrow \left\{ \begin{array}{c} \dot{x}_{1,2} = j\Omega \overline{X}_{1,2} \exp \exp j\Omega t \\ \ddot{x}_{1,2} = -\Omega^2 \overline{X}_{1,2} \exp \exp j\Omega t \end{array} \right\} \tag{5.24}$$

This system lends itself to a matrix representation :

$$\begin{Bmatrix} -\Omega^2 + \frac{3k}{m} & -2\frac{k}{m} \\ -\frac{k}{m} & -\Omega^2 + \frac{k}{m} \end{Bmatrix} \begin{Bmatrix} \overline{X}_1 \\ \overline{X}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5.25}$$

We obtain a system of two equations with two unknowns and no forcing term, has non-trivial solutions (other than $\overline{X}_{1,2} = 0$) when the determinant is zero.

$$\det = \left(-\Omega^2 + \frac{3k}{m} \right) \left(-\Omega^2 + \frac{k}{m} \right) - 2\frac{k^2}{m^2} = 0 \tag{5.26}$$

$$\Omega^4 - \Omega^2 4\frac{k}{m} + \frac{k^2}{m^2} = 0 \tag{5.27}$$

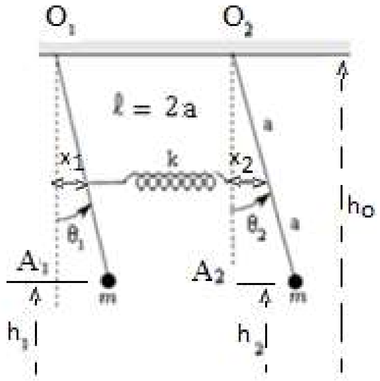
$$\Delta = 16\frac{k^2}{m^2} - 4\frac{k^2}{m^2} = 12\frac{k^2}{m^2}$$

As a result, the system exhibits two natural angular frequencies, denoted by Ω_1^2 and Ω_2^2, each associated with a specific mode of vibration, with :

$$\begin{aligned} \Omega_1^2 &= \frac{k}{m}\left(2 + \sqrt{3}\right) \\ \Omega_2^2 &= \frac{k}{m}\left(2 - \sqrt{3}\right) \end{aligned}$$

5.3.2 Exercise 2

The oscillatory system depicted in the figure is considered. Two identical pendulums of length $l = 2a$ each carry a point mass m at their ends. A spring with stiffness k provides the elastic coupling between them. The angular displacements of the two pendulums at any given time are denoted by θ_1, and θ_2.



1) Determine the kinetic and potential energy of the system. Write the Lagrangian as a function of θ_1 and θ_2.

2) Deduce the two differential equations of motion for small oscillations.

3) Calculate the two natural pulsations of the system corresponding to the possible vibration modes.

4) Deduce the transition matrix and write the general solutions.

Solution

1) Kinetic and potential energy of the system

$$E_c = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2$$

$$E_p = mgh_1 + mgh_2 + \frac{1}{2}k(x_2 - x_1)^2$$

$$\sin\theta_{1,2} = \frac{x_{1,2}}{a}$$

$$\cos\theta_{1,2} = \frac{O_{1,2}A_{1,2}}{l} \qquad \longrightarrow n = \text{dimensions}=6 \text{ coordinates } (x_{1,2}, \theta_{1,2}, h_{1,2})\text{-4 relations.}$$

$$h_{1,2} = h_0 - O_{1,2}A_{1,2}$$

$$\left\{ \begin{array}{l} x_{1,2} = a\sin\theta_{1,2} \\ h_{1,2} = h_0 - l\cos\theta_{1,2} \end{array} \right\} \longrightarrow n = 2 \text{ degrees of freedom.}$$

$$E_c = \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) \tag{5.28}$$

$$E_p = mg(h_0 - l\cos\theta_1) + mg(h_0 - l\cos\theta_2) + \frac{1}{2}k(x_2 - x_1)^2$$

$$E_p = -mgl\cos\theta_1 - mgl\cos\theta_2 + \frac{1}{2}ka^2(\sin\theta_2 - \sin\theta_1)^2 \tag{5.29}$$

$$L = E_c - E_p = \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) + mgl\cos\theta_1 + mgl\cos\theta_2 - \frac{1}{2}ka^2(\sin\theta_2 - \sin\theta_1)^2$$

2) Differential equations of motion for small oscillations

$$\left\{\begin{array}{l} \dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\theta}_1} - \dfrac{\partial L}{\partial\theta_1} = 0 \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\theta}_2} - \dfrac{\partial L}{\partial\theta_2} = 0 \end{array}\right\} \tag{5.30}$$

$$\left\{\begin{array}{l} 4ma^2\ddot{\theta}_1 + 2mga\sin\theta_1 - ka^2(\sin\theta_2 - \sin\theta_1)\cos\theta_1 = 0 \\ 4ma^2\ddot{\theta}_2 + 2mga\sin\theta_2 + ka^2(\sin\theta_2 - \sin\theta_1)\cos\theta_2 = 0 \end{array}\right\} \tag{5.31}$$

In the case of small oscillations, θ_1 and θ_2 small, we can make the approximations : $\sin\theta \sim \theta$ and $\cos\theta \sim 1$

$$\left\{\begin{array}{l} 4ma^2\ddot{\theta}_1 + (2mga + ka^2)\,\theta_1 - ka^2\theta_2 = 0 \\ 4ma^2\ddot{\theta}_1 + (2mga + ka^2)\,\theta_1 - ka^2\theta_2 = 0 \end{array}\right\} \tag{5.32}$$

$$\left\{\begin{array}{l} \ddot{\theta}_1 + \left(\dfrac{g}{2a} + \frac{k}{4m}\right)\theta_1 - \dfrac{k}{4m}\theta_2 = 0 \\ \ddot{\theta}_2 + \left(\dfrac{g}{2a} + \frac{k}{4m}\right)\theta_2 - \dfrac{k}{4m}\theta_2 = 0 \end{array}\right\} \tag{5.33}$$

We obtain a system of two coupled linear second-order differential equations without a right-hand side. The subsystems (simple pendulums) taken separately have MHS

3) Natural pulsations of the system corresponding to the possible vibration modes.

we can state that the free oscillatory subsystems are **MHS** so the individual solutions are of the type :

$$\theta_{1,2} = A_{1,2}\cos(\omega_0 t + \varphi_{1,2}) \tag{5.34}$$

Or, in complex notation :

$$\overline{\theta}_{1,2} = X_{1,2}\exp^{j(\omega_0 t+\varphi_{1,2})} = X_{1,2}\exp^{j\varphi_{1,2}}\exp^{j\omega_0 t}$$

$$\theta_{1,2} = \overline{X}_{1,2}\exp^{j\omega_0 t}, \quad \text{avec } \overline{X}_{1,2} = X_{1,2}\exp^{j\varphi_{1,2}} \tag{5.35}$$

So

$$\overline{\theta}_{1,2} = \overline{X}_{1,2}\exp^{j\omega_0 t} \rightarrow \dot{\theta}_{1,2} = j\omega_0\overline{X}_{1,2}\exp^{j\omega_0 t} \rightarrow \ddot{\theta}_{1,2} = -\omega_0^2\overline{X}_{1,2}\exp^{j\omega_0 t}$$

The substitution of assumed solutions into the differential model leads to a system of linear equations :

$$\left\{\begin{array}{l} \ddot{\theta}_1 + (\frac{g}{2a} + \frac{k}{4m})\theta_1 - \frac{k}{4m}\theta_2 = 0 \\ \ddot{\theta}_2 + (\frac{g}{2a} + \frac{k}{4m})\theta_1 - \frac{k}{4m}\theta_2 = 0 \end{array}\right\} \tag{5.36}$$

$$\longrightarrow \left\{\begin{array}{c} -\omega_0^2\overline{X}_1\exp^{j\omega_0 t} + (\frac{g}{2a} + \frac{k}{4m})\overline{X}_1\exp^{j\omega_0 t} - \frac{k}{4m}\overline{X}_2\exp^{j\omega_0 t} = 0 \\ -\omega_0^2\overline{X}_2\exp^{j\omega_0 t} + (\frac{g}{2a} + \frac{k}{4m})\overline{X}_2\exp^{j\omega_0 t} + -\frac{k}{4m}\overline{X}_1\exp^{j\omega_0 t} = 0 \end{array}\right\} \tag{5.37}$$

We obtain a system of two equations in two unknowns without constant terms :

$$\left\{\begin{array}{c} (-\omega_0^2 + \frac{g}{2a} + \frac{k}{4m})\overline{X}_1 - \frac{k}{4m}\overline{X}_2 = 0 \\ (-\omega_0^2\overline{X}_2 + \frac{g}{2a} + \frac{k}{4m})\overline{X}_2 + -\frac{k}{4m}\overline{X}_1 = 0 \end{array}\right\}$$

This system lends itself to a matrix representation :

$$\begin{pmatrix} -\omega_0^2 + \frac{g}{2a} + \frac{k}{4m} & -\frac{k}{4m} \\ -\frac{k}{4m} & -\omega_0^2 + \frac{g}{2a} + \frac{k}{4m} \end{pmatrix} \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5.38}$$

The obvious solution is $\overline{X}_{1,2} = 0 \longrightarrow$ no oscillations, so it's not acceptable, so necessary and sufficient condition for the existence of non-trivial solutions is that the determinant equals zero.

$$\det = \left(-\omega_0^2 + \frac{g}{2a} + \frac{k}{4m}\right)^2 - \left(\frac{k}{4m}\right)^2 = 0, \text{ natural frequency equation} \tag{5.39}$$

$$\left(-\omega_0^2 + \frac{g}{2a} + \frac{k}{2m}\right)\left(-\omega_0^2 + \frac{g}{2a}\right) = 0 \tag{5.40}$$

$$\left\{ \begin{array}{c} -\omega_0^2 + \frac{g}{2a} + \frac{k}{2m} = 0 \\ -\omega_0^2 + \frac{g}{2a} = 0 \end{array} \right\}$$

As a result, the system exhibits two natural angular frequencies, denoted by ω_{01}^2 and ω_{02}^2, each associated with a specific mode of vibration, with :

$$\begin{aligned} \omega_{01}^2 &= \frac{g}{2a} + \frac{k}{2m} \longrightarrow \omega_{01} = \sqrt{\frac{g}{2a} + \frac{k}{2m}} \\ \omega_{02}^2 &= \frac{g}{2a} \longrightarrow \omega_{02} = \sqrt{\frac{g}{2a}} \end{aligned}$$

two modes of vibration :

$$\theta_{1,2} = X_{1,2}\cos(\omega_{01,2} + \varphi_{1,2})$$

In the case of two solutions, the general solution is a linear combination of both; in other words, the most general motion results from the superposition of the two vibration modes.

The general solutions of this system are :

$$\begin{Bmatrix} \theta_1(t) \\ \theta_1(t) \end{Bmatrix} = p \begin{Bmatrix} X_{1,2}\cos(\omega_{01} + \varphi_1) \\ X_{1,2}\cos(\omega_{02} + \varphi_2) \end{Bmatrix}$$

With :

$$p = \left\{\overrightarrow{V}_1 \overrightarrow{V}_2\right\} \text{ transition matrix.}$$

4) General solutions

* First mode $\omega_{01}^2 = \dfrac{g}{2a} + \dfrac{k}{2m}$, which can be substituted into

$$\begin{pmatrix} -\omega_0^2 + \dfrac{g}{2a} + \dfrac{k}{4m} & -\dfrac{k}{4m} \\ -\dfrac{k}{4m} & -\omega_0^2 + \dfrac{g}{2a} + \dfrac{k}{4m} \end{pmatrix} \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\frac{k}{4m} \begin{Bmatrix} -1 & -1 \\ -1 & -1 \end{Bmatrix} \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \longrightarrow \begin{Bmatrix} -\overline{X}_1 - \overline{X}_2 = 0 \\ -\overline{X}_1 - \overline{X}_2 = 0 \end{Bmatrix} \longrightarrow -\overline{X}_1 = \overline{X}_2$$

$$\longrightarrow \text{eigenvector} \overrightarrow{V}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

* second mode $\omega_{02}^2 = \dfrac{g}{2a}$, which can be substituted into

$$\begin{pmatrix} -\omega_0^2 + \dfrac{g}{2a} + \dfrac{k}{4m} & -\dfrac{k}{4m} \\ -\dfrac{k}{4m} & -\omega_0^2 + \dfrac{g}{2a} + \dfrac{k}{4m} \end{pmatrix} \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\frac{k}{4m} \begin{Bmatrix} 1 & -1 \\ -1 & 1 \end{Bmatrix} \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \longrightarrow \begin{Bmatrix} \overline{X}_1 - \overline{X}_2 = 0 \\ -\overline{X}_1 + \overline{X}_2 = 0 \end{Bmatrix} \longrightarrow \overline{X}_1 = \overline{X}_2$$

$$\longrightarrow \text{eigenvector} \overrightarrow{V}_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Thus, the transition matrix :

$$p = \left\{ \overrightarrow{V}_1 \overrightarrow{V}_2 \right\} = \begin{Bmatrix} 1 & 1 \\ -1 & 1 \end{Bmatrix} \tag{5.41}$$

general solutions :

$$\begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \end{Bmatrix} = p \begin{Bmatrix} X_1 \cos(\omega_{01} t + \varphi_1) \\ X_2 \cos(\omega_{02} t + \varphi_2) \end{Bmatrix} = \begin{Bmatrix} 1 & 1 \\ -1 & 1 \end{Bmatrix} \begin{Bmatrix} X_1 \cos(\omega_{01} t + \varphi_1) \\ X_2 \cos(\omega_{02} t + \varphi_2) \end{Bmatrix}$$

$$\begin{aligned} \theta_1(t) &= X_1 \cos(\omega_{01} t + \varphi_1) + X_2 \cos(\omega_{02} t + \varphi_2) \\ \theta_2(t) &= -X_1 \cos(\omega_{01} t + \varphi_1) + X_2 \cos(\omega_{02} t + \varphi_2) \end{aligned}$$

The four constants X_1, X_2, φ_1, φ_2 are determined by the initial conditions.

General Conclusion

General Conclusion

The study of vibratory systems is a fundamental step in understanding the dynamic behavior of mechanical, electromechanical, and even electronic systems. Throughout this course manual, we have progressively explored the basics of oscillatory motion, starting from the simplest single-degree-of-freedom systems and advancing toward more complex models with two degrees of freedom.

Our approach focused on introducing essential theoretical tools such as Lagrange's equations, differential equations of motion, and the concepts of natural frequencies and normal modes, while illustrating their application through concrete examples and solved exercises. Special attention was given to the effects of damping and forced response, which play a central role in designing systems capable of withstanding or responding effectively to external excitations.

The concepts covered in this manual provide a solid foundation for further studies in specialized fields such as mechatronics, control systems, embedded systems, or acoustics. While this document does not replace in-person instruction, it offers a structured resource for students to review, deepen, and apply key concepts related to vibrations.

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