1. Corrigé Examen Algèbre 1

Exercice 1

① ⓑ (1pt) ② ⓐ False (1pt) ⓑ True (1pt)

ⓒ False (1pt) ⓓ True (1pt)

Exercice 2

① R is an equivalence relation $\Leftrightarrow$ $\begin{cases} R \text{ is reflexive} \\ R \text{ is symmetric} \\ R \text{ is transitive} \end{cases}$ (0,5 pt) (or conclusion)

* R is reflexive $\Leftrightarrow \forall x \in \mathbb{Z} \quad x R x$ (0,25)

$x R x \Leftrightarrow x^2 - x^2 = x - x$

$\Leftrightarrow 0 = 0$ True Then R is reflexive (0,75)

* R is symetric $\Leftrightarrow \forall x, y \in \mathbb{Z} \quad x R y \Rightarrow y R x$ (0,25)

Let $x R y \Leftrightarrow x^2 - y^2 = x - y \quad \times(-1)$

$\Leftrightarrow y^2 - x^2 = y - x$

$\Leftrightarrow y R x$ True Then R is symetric (0,75)

* R is transitive $\Leftrightarrow \forall x, y, z \in \mathbb{Z} \quad \begin{cases} x R y \\ \text{and} \\ y R z \end{cases} \Rightarrow x R z$ (0,25)

Let $\begin{cases} x R y \\ y R z \end{cases} \Leftrightarrow \begin{cases} x^2 - y^2 = x - y & ① \\ y^2 - z^2 = y - z & ② \end{cases}$ $① + ② \Rightarrow x^2 - z^2 = x - y$

$\Leftrightarrow x R z$ True Then R is Transitive (0,75)

② $cl(0) = \{x \in \mathbb{Z} \,/\, x R 0\}$ (0,5)

$x R 0 \Leftrightarrow x^2 - 0^2 = x - 0$

$\Leftrightarrow x^2 = x \Leftrightarrow x^2 - x = 0$

$\Leftrightarrow x(x-1) = 0$

$\Leftrightarrow x = 0$ or $x = 1$

$\Rightarrow cl(0) = \{0, 1\}$ (0,75)

since $1 \in cl(0) \Rightarrow cl(0) = cl(1)$ (0,25)

Exercice 3

① since Domain of $f = D_f$ then f is an application (0,5)

② f is injective $\Leftrightarrow \forall x_1, x_2 \in \mathbb{R} - \{1\} \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (0,25)

Let $f(x_1) = f(x_2) \Leftrightarrow \dfrac{x_1 + 2}{x_1 - 1} = \dfrac{x_2 + 2}{x_2 - 1}$

$\Leftrightarrow x_1 = x_2$ (0,75)

③ $f^{-1}(\{0, 1\}) = \{x \in \mathbb{R} - \{1\} \,/\, f(x) \in \{0, 1\}\}$ (0,25)

Ex 3 (suite)

$f(x) \in \{0, 1\} \Leftrightarrow f(x) = 0$ or $f(x) = 1$ (0,25)

$f(x) = 0 \Leftrightarrow \frac{x+2}{x-1} = 0 \Rightarrow x+2 = 0 \Rightarrow \boxed{x = -2}$ (0,25)

$f(x) = 1 \Leftrightarrow \frac{x+2}{x-1} = 1 \Leftrightarrow x+2 = x-1 \Rightarrow 2 = -1$ impossible (0,25)

Then $f^{-1}(\{0, 1\}) = \{-2\}$

4) since $f(x) = 1$ impossible f is not surjective (0,5)

Exercice 4

1) $z^2 + 2z + 4 = 0$ $\quad \Delta = b^2 - 4ac \quad \Delta = 4 - 16 = -12 = 12i^2$

$\delta_1 = 2i\sqrt{3}$, $\delta_2 = -2i\sqrt{3}$ (0,5)

$z_1 = \frac{-b+\delta_1}{2a}$, $z_2 = \frac{-b+\delta_2}{2a}$

$z_1 = -1 + i\sqrt{3}$ (0,5) $z_2 = -1 - i\sqrt{3}$ (0,5)

2) $z_1 = -1 - i\sqrt{3}$ $\quad z_2 = +1 + i$

(a) $z_1 \times z_2 = (-1 - i\sqrt{3})(1+i) = (-1+\sqrt{3}) + i(-1-\sqrt{3})$ (1pt)

(b) $z_1 = [r_1, \theta_1]$ $\quad r_1 = \sqrt{x^2+y^2} = 2$

$\theta_1 = \arg z_1 + 2k\pi \mid k \in \mathbb{Z} \Rightarrow \begin{cases} \cos\theta_1 = \frac{x}{r_1} = -\frac{1}{2} \\ \sin\theta_1 = \frac{y}{r_1} = -\frac{\sqrt{3}}{2} \end{cases}$

Then $\theta_1 = \pi + \frac{\pi}{3} + 2k\pi$

$\theta_1 = \frac{4\pi}{3} + 2k\pi \mid k \in \mathbb{Z}$

ie $z_1 \left[2, \frac{4\pi}{3}\right] \Rightarrow z_1 = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ (0,5)

$z_2 = \left[\sqrt{2}, \frac{\pi}{4}\right] \Rightarrow z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (0,5)

$z_1 \times z_2 = [r_1 \times r_2, \theta_1 + \theta_2] = \left[2\sqrt{2}, \frac{4\pi}{3} + \frac{\pi}{4}\right] = \left[2\sqrt{2}, \frac{19\pi}{12}\right]$

$z_1 \times z_2 = 2\sqrt{2}\left(\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right)$ (0,5)

(c) $z_1 \times z_2 = \left[2\sqrt{2}, \frac{19\pi}{12}\right] = (-1+\sqrt{3}) + i(-1-\sqrt{3})$

$\cos\frac{19\pi}{12} = \frac{x}{r} = \frac{-1+\sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2}+\sqrt{6}}{4}$ (0,5)

$\sin\frac{19\pi}{12} = \frac{y}{r} = \frac{-1-\sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2}-\sqrt{6}}{4}$ (0,5)

• Exercice 5

$$E = \{(3x, 2y, -z) \mid x, y, z \in \mathbb{R}\}$$

E is a vector subspace of $\mathbb{R}^3 \iff \begin{cases} E \neq \phi \quad 0_{\mathbb{R}^3} \in E \\ \forall u, v \in E \quad u + v \in E \\ \forall u \in E \; \forall \alpha \in E \quad \alpha . u \in E \end{cases}$

0,5

* $0_{\mathbb{R}^3} = (0,0,0) = (3 \times 0, 2 \times 0, -0) \in E \Rightarrow \cancel{E} \neq \phi$ 0,5

* $u \in E \Rightarrow u = (3x, 2y, -z) \mid x, y, z \in \mathbb{R}$

$v \in E \Rightarrow v = (3x', 2y', -z') \mid x', y', z' \in \mathbb{R}$

$$\begin{aligned} u + v &= (3x + 3x', 2y + 2y', -z - z') \\ &= (3(x + x'), 2(y + y'), -(z + z')) \\ &= (3X, 2Y, -Z) \end{aligned}$$

0,5

$X = x + x' \in \mathbb{R} \quad Y = y + y' \in \mathbb{R} \quad Z = z + z' \in \mathbb{R}$

Then $u + v \in E$

* $\alpha \in E \Rightarrow \alpha . u = \alpha(3x, 2y, -z)$

$$\begin{aligned} &= (3(\alpha x), 2(\alpha y), -(\alpha z)) \\ &= (3X', 2Y', -Z') \end{aligned}$$

$X' = \alpha x, \; Y' = \alpha y \; Z' = \alpha z$ 0,5

Then $\alpha . u \in E$

conclusion E is v.s.s of $\mathbb{R}^3$

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