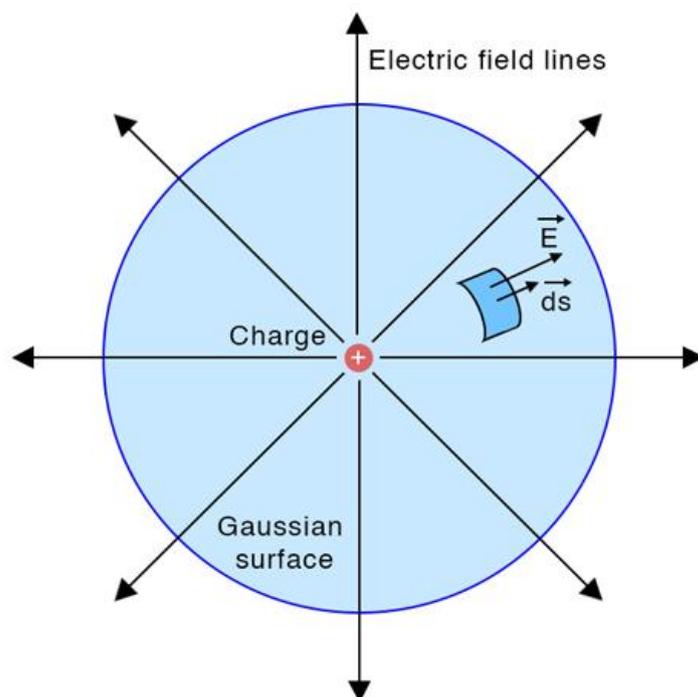


**1ST YEAR LMD-MI**  
**ELECTRICITY COURSE**

## Chapter II: Gauss' theorem

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### Gauss's Law



$$\phi \propto Q_{enc}$$

$$\phi = \oint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$$

$\phi$  : Electric flux

$\vec{E}$  : Electric field

$d\vec{s}$  : Infinitesimal surface area

$Q_{enc}$  : Charge enclosed

$\epsilon_0$  : Permittivity of air

## Summary

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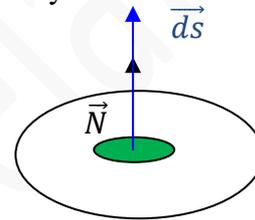
### 1. Introduction

Gauss's law is a mathematical model that can be used to obtain the electric fields of certain charge distributions with a high degree of symmetry, such as cylinders, spheres and infinite wires.

It is therefore a specialized method, but it is very useful for this class of problems to which it can be applied. At this stage, Gauss's law will help us to better understand the shapes of electric fields due to continuous charge distributions.

### 2. Definitions

**A- Surface vector:** The surface vector  $\vec{ds}$  is a vector carried by the unit vector normal to the surface.

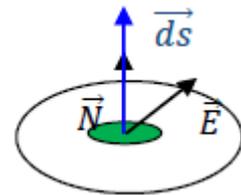


**B- Flux of a vector field:** The elementary flux  $d\Phi$  is,

$$d\Phi = \vec{E} \cdot \vec{ds} \Rightarrow \Phi = \iint \vec{E} \cdot \vec{ds} = \iint \vec{E} \cdot ds \cdot \vec{N}$$

With  $\vec{ds} = ds \cdot \vec{N}$

The unit of flow is the Weber (Wb).



### 3. Electric field flow through a closed surface

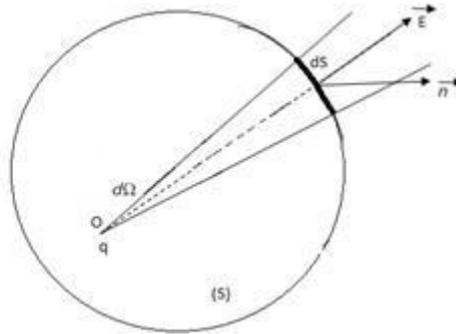
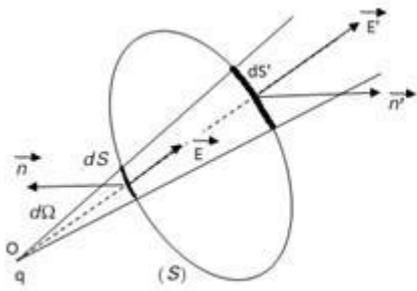
Let S be an arbitrary closed surface and q be the charge enclosed within the surface S. The elementary electric field flux created by the charge q across the closed surface S is given by:

$$d\Phi = \vec{E} \cdot \vec{ds} = E \cdot ds \cdot \cos \alpha$$

$\alpha$  : the angle between  $\vec{E}$  and  $\vec{N}$  ( $\vec{ds}$ ) The electric field  $\vec{E} = \frac{kq}{r^2} \vec{u}$  and  $d\Phi = \frac{kq}{r^2} \vec{u} \cdot \vec{ds} = kq \frac{\vec{u} \cdot \vec{ds}}{r^2}$

$$\vec{u} \cdot \vec{ds} = ds |\vec{u}| \cos \alpha$$

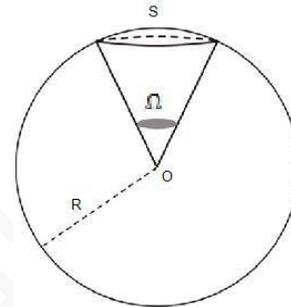
## Chapter II: Gauss' theorem



The electric field will be :

$$\vec{E} = \frac{kq}{r^2} \vec{u} \text{ and } \phi = \oiint \frac{kq}{r^2} \vec{u} \cdot \vec{ds} = \oiint kq \frac{ds \cdot \cos\alpha}{r^2}$$

with  $\frac{ds \cdot \cos\alpha}{r^2} = d\Omega = \text{solid angle}$



**Note:** The unit of the solid angle is the steradian.

Since the area of a sphere of radius  $R$  is  $S = 4\pi R^2$ , we deduce that the largest measurable solid angle, which corresponds to an object covering the entire sphere, is  $4\pi$  steradians.

$\Omega = 4\pi =$  the solid angle to see all of space

$$\text{So } \phi = \frac{q}{4\pi\epsilon_0} \cdot 4\pi = \frac{q}{\epsilon_0}$$

In the case of several point charges, the flux is written as:

$$\phi = \oiint \vec{E} \cdot \vec{ds} = \frac{\sum Q_{\text{int}}}{\epsilon_0}$$

### 4. Gauss Theorem

#### a- Statement of Gauss Theorem

**« The flux inside a closed surface called a Gauss surface is equal to the sum of the net charges  $Q_{\text{int}}$  inside this surface divided by the dielectric permittivity in vacuum  $\epsilon_0$  »**

$$\phi = \oiint \vec{E} \cdot \vec{ds} = \frac{\sum Q_{\text{int}}}{\epsilon_0}$$

## Chapter II: Gauss' theorem

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### b. The steps involved in applying Gauss's theorem

- Choosing a coordinate system
- Study the invariance of the system
- Study symmetry
- Choice of Gaussian surface; the table shows the different cases where a cylinder is chosen as the SG and the cases where a sphere is chosen as the SG:

	Possible cases	Possible cases	Possible cases	Possible cases
<b>GS is a cylinder</b>	An infinite wire	An infinite plane	A surface or volume charged cylinder	Two or more cylinders
<b>GS is a sphere</b>	A surface- or volume- charged sphere			

### c. GT for different continuous charge distribution :

- **Linear distribution ( $dq = \lambda dl$ )**  $\phi = \oint \vec{E} \cdot \vec{ds} = \frac{\int \lambda dl}{\epsilon_0}$
- **Surface distribution ( $dq = \sigma ds$ )**  $\phi = \oint \vec{E} \cdot \vec{ds} = \frac{\int \sigma ds}{\epsilon_0}$
- **Volume distribution ( $dq = \rho dv$ )**  $\phi = \oint \vec{E} \cdot \vec{ds} = \frac{\int \rho dv}{\epsilon_0}$

## 5. Application examples

### 5.1. Case of an infinite wire

#### - Choice of coordinate system:

If we zoom in on the wire, we'll have a cylinder with an infinitely small radius, so we use cylindrical coordinates.

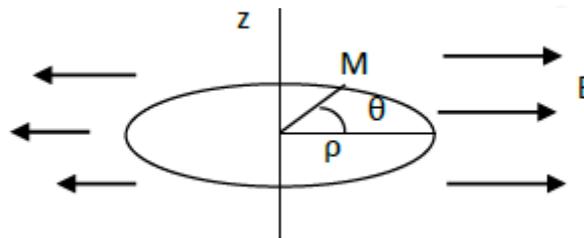


## Chapter II: Gauss' theorem

### - Study of invariance

We study invariance with respect to  $\rho$ ,  $\theta$  and  $z$  (cylindrical coordinates).

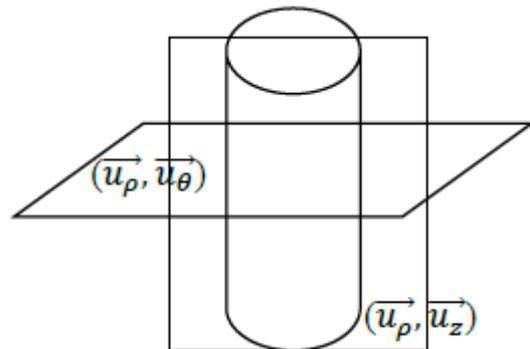
- If we change the angle  $\theta$ , M rotates around the wire, but the electric field does not change.
- If we change  $z$ , M translates along (Oz) and since the wire is infinite, we still have the same wire, so  $\vec{E}$  remains invariant.
- If we change  $\rho$ , M can move away from or towards the wire, so  $\vec{E}$  does not remain the same. So  $\vec{E}$  depends only on  $\rho$ .



### - Study of symmetry

In this case, we have two planes of symmetry:

1. The plane intersecting the infinite wire horizontally ( $\vec{u}_\rho, \vec{u}_\theta$ )
2. The plane intersecting the infinite wire vertically ( $\vec{u}_\rho, \vec{u}_z$ )



So the axis of symmetry is the intersection of the two planes, this is the axis following  $\vec{u}_\rho$  so the electric field is following  $\vec{u}_\rho$ .

### - Choice of Gauss surface

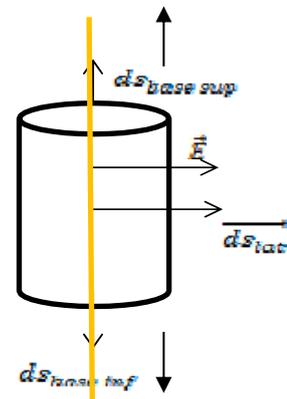
The Gaussian surface is a cylinder of radius  $r$  and height  $h$ . Because of symmetry, the field follows the radius  $\rho$ , so we say the field is said to be "radial" and constant in the Gaussian surface ( $\vec{E}$  depends only on  $\rho$ ).

$$\text{According to Gauss's Theorem: } \phi = \iint \vec{E} \cdot \vec{ds} = \frac{\Sigma Q_{int}}{\epsilon_0}$$

$$\phi = \iint \vec{E} \cdot \vec{ds} = \iint \vec{E} \cdot \vec{ds}_{base\ 1} + \iint \vec{E} \cdot \vec{ds}_{lat} + \iint \vec{E} \cdot \vec{ds}_{base\ 2}$$

$$\vec{E} \perp \vec{ds}_{base} \Rightarrow \iint \vec{E} \cdot \vec{ds}_{lat} = 0$$

$$\vec{E} \parallel \vec{ds}_{lat} \text{ so : } \phi = \iint \vec{E} \cdot \vec{ds}_{lat} = \iint E \cdot ds_{lat} = E \cdot \int ds_{lat} = E \cdot S_{lat}$$



## Chapter II: Gauss' theorem

so  $\Phi = E 2\pi r h$

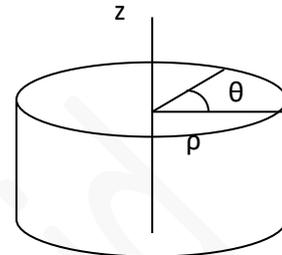
Let's find  $Q_{int}$ , the elementary charge is :  $dq = \lambda dl \Rightarrow Q = \lambda \int_0^h dl = \lambda h$

$$E 2\pi r h = \frac{\lambda h}{\epsilon_0} \Rightarrow E = \frac{\lambda}{2\pi r \epsilon_0}$$

### 5.1. Case of an infinite cylinder

#### - Choice of coordinate system

Since we're studying a cylinder, we use cylindrical coordinates.



#### - Study of invariance

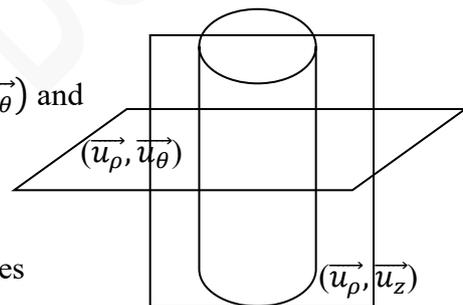
It's the same as the wire. The electric field does not change by varying  $\theta$  and  $z$ , however,  $\vec{E}$  depends on  $\rho$ .

#### - Study of symmetry

In this case, too, we have two planes of symmetry:

The plane intersecting the infinite wire horizontally ( $\vec{u}_\rho, \vec{u}_\theta$ ) and

The plane intersecting the infinite wire vertically ( $\vec{u}_\rho, \vec{u}_z$ )



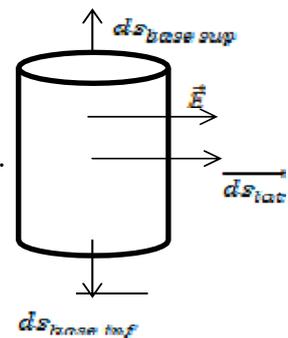
So the axis of symmetry is the intersection of the two planes

It's the axis along  $\vec{u}_\rho$  so the electric field is along  $\vec{u}_\rho$ .

#### - Choice of Gauss surface

The Gaussian surface is a cylinder of radius  $r$  and height  $h$ .

Because of symmetry, the radial field is constant in the Gaussian surface.



According to Gauss's Theorem:  $\Phi = \iint \vec{E} \cdot \vec{ds} = \frac{\Sigma Q_{int}}{\epsilon_0}$

$$\Phi = \iint \vec{E} \cdot \vec{ds} = \iint \vec{E} \cdot \vec{ds}_{base\ 1} + \iint \vec{E} \cdot \vec{ds}_{lat} + \iint \vec{E} \cdot \vec{ds}_{base\ 2}$$

$$\vec{E} \perp \vec{ds}_{base} \Rightarrow \iint \vec{E} \cdot \vec{ds}_{base} = 0$$

$$\vec{E} \parallel \vec{ds}_{lat} \text{ so : } \Phi = \iint \vec{E} \cdot \vec{ds}_{lat} = \iint E \cdot ds_{lat} = E \cdot \int ds_{lat} = E \cdot S_{lat}$$

Then  $\Phi = E 2\pi r h = Q_{int}/\epsilon_0$

## Chapter II: Gauss' theorem

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The cylinder can be either surface or volume charged.

### Important note:

The choice of Gaussian surface for a cylinder charged either on the surface or in volume, or two cylinders (one charged in volume and the other charged on the surface, or both charged on the surface...) is always a cylinder of radius  $r$  and height  $h$ . The flux calculation will be the same, only the  $Q_{int}$  charge will vary according to the given distribution.

### a- For a surface-charged cylinder

#### - The electric field

We have two cases ;

**1<sup>st</sup> case**  $r < R$  we take the Gauss surface inside the charged cylinder to calculate the internal field. Then, in a surface distribution, we have :

$$Q_{int} = 0 \Rightarrow \mathbf{E}_1 = \mathbf{E}_{ins} = \mathbf{0}$$

**2<sup>nd</sup> case**  $r \geq R$  we take the Gauss surface outside the charged cylinder to calculate the field outside.

$$dq = \sigma ds \Rightarrow Q_{int} = \sigma S = \sigma 2\pi R h$$

$$\text{So } E_2 2\pi r h = \frac{\sigma 2\pi R h}{\epsilon_0} \Rightarrow \mathbf{E}_2 = \mathbf{E}_{out} = \frac{\sigma R}{\epsilon_0 r}$$

#### - The potential

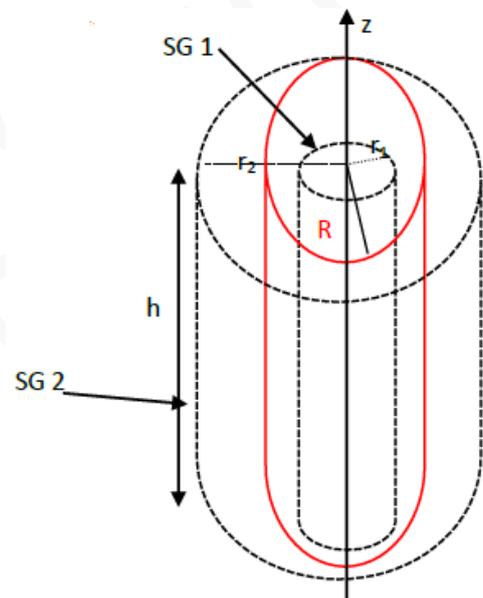
$$\vec{E} = -\overrightarrow{\text{grad}} V \text{ with } E = E(r) \Rightarrow E = -\frac{dV}{dr} \text{ so } V = -\int E \cdot dr$$

$$\text{1<sup>st</sup> case } r < R \text{ we have } E_1 = 0 \Rightarrow V_1 = C_1$$

#### 2<sup>nd</sup> case $r \geq R$

$$\mathbf{E}_2 = \frac{\rho R}{\epsilon_0 r} \Rightarrow V_2 = -\frac{\sigma R}{\epsilon_0} \int \frac{1}{r} \cdot dr = -\frac{\sigma R}{\epsilon_0} \ln r + C_2$$

**Note:** In the case of a cylinder, the constants  $C_1$  and  $C_2$  cannot be calculated because the potential at infinity is non-zero.



### b. Volume-charged cylinder

We have two cases.

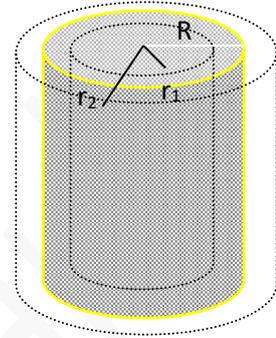
#### - The electric field

**1<sup>st</sup> case**  $r < R$  we take the Gauss surface inside the charged cylinder to calculate the internal field. Then, in a volume distribution  $dq = \rho dV$ , we have :

$$Q_{int} = \iiint \rho dv = \rho \int_0^r 2\pi hr dr = \rho\pi hr^2 \Rightarrow E2\pi rh = \frac{\rho\pi hr^2}{\epsilon_0}$$

$$\text{Because } V = \pi hr^2 \Rightarrow dV = 2\pi hr dr$$

$$\Rightarrow \mathbf{E}_1 = \mathbf{E}_{ins} = \frac{\rho}{2\epsilon_0} \mathbf{r}$$



#### 2<sup>nd</sup> case $r \geq R$

we take the Gauss surface outside the loaded cylinder to calculate  $E_2 = E_{ins}$ , so we integrate between 0 and R (because  $Q_{int}$  lies on the cylinder of radius R).

$$Q_{int} = \iiint \rho dv = \rho \int_0^R 2\pi hr dr = \rho\pi hR^2 \Rightarrow E2\pi rh = \frac{\rho\pi hR^2}{\epsilon_0}$$

$$\Rightarrow \mathbf{E}_2 = \frac{\rho R^2}{2\epsilon_0} \frac{1}{r}$$

#### - The potential

$$\vec{E} = -\overrightarrow{grad} V \text{ with } E = E(r)$$

$$\Rightarrow E = -\frac{dV}{dr} \text{ so } V = -\int E \cdot dr \text{ (this calculation is valid for any cylinder).}$$

$$V_1 = -\int E_1 \cdot dr \Rightarrow V_1 = \frac{\rho}{2\epsilon_0} \int r dr = -\frac{\rho}{4\epsilon_0} r^2 + C_1$$

$$V_2 = -\int E_2 \cdot dr = -\frac{\rho R^2}{2\epsilon_0} \int \frac{1}{r} \cdot dr = -\frac{\rho R^2}{2\epsilon_0} \ln r + C_2$$

### 5.3. Case of a sphere

**- Choice of coordinate system**

Since we're studying a cylinder, we use spherical coordinates.

**- Study of invariance**

If we change the angle  $\theta$  or the angle  $\varphi$ , the electric field  $\vec{E}$  does not change, but by varying  $r$  the electric field  $\vec{E}$  varies  $\vec{E}$ .

**- Study of symmetry**

We have two planes of symmetry:

The plane intersecting the infinite wire horizontally ( $\vec{u}_r, \vec{u}_\theta$ ) and the plane intersecting the infinite wire vertically ( $\vec{u}_r, \vec{u}_\varphi$ ).

So the axis of symmetry is the intersection of the two planes,

This is the axis along  $\vec{u}_r$  so the electric field is along  $\vec{u}_r$

The field is then said to be radial

**- Choosing the Gaussian surface**

The Gaussian surface is a sphere with center O and radius  $r$ . Due to symmetry, the field is radial and constant in the Gaussian surface.

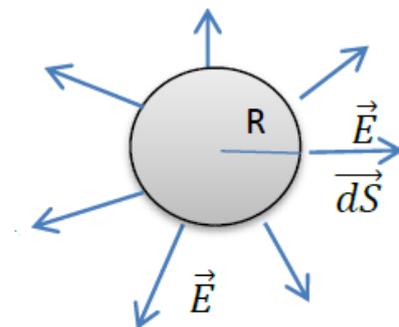
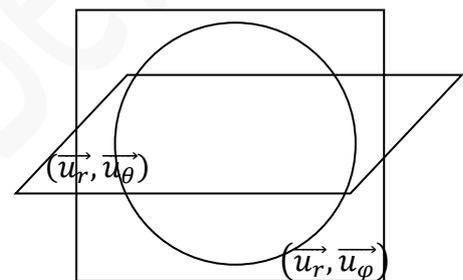
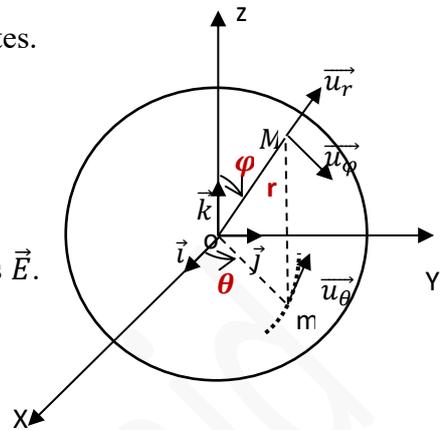
$$\phi = \oiint \vec{E} \cdot \vec{ds} = \frac{\Sigma Q_{int}}{\epsilon_0}$$

$\vec{E} \parallel \vec{ds}$  :

$$\text{So : } \oiint \vec{E} \cdot \vec{ds} = \iint E \cdot ds = E \iint ds = E \cdot S = E4\pi r^2 \Rightarrow E4\pi r^2 = \frac{\Sigma Q_{int}}{\epsilon_0}$$

**Important note:**

The choice of Gaussian surface for a sphere charged either on the surface or in volume, or two spheres (one charged in volume and the other charged on the surface, or both charged on the surface...) is always a sphere of radius  $r$  and center O. And the flux calculation will be the same, only the  $Q_{int}$  charge will vary according to the distribution.



## Chapter II: Gauss' theorem

### a- Surface-charged sphere

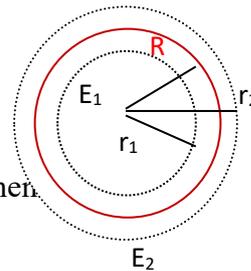
- The electrostatic field  $E(r)$  at any point in space.

We have 2 cases :

#### 1<sup>st</sup> case $r < R$

The Gaussian surface is inside the sphere to calculate  $E_1 = E_{int}$  then

$$Q_{int} = 0 \Rightarrow E_1 = 0$$



#### 2<sup>nd</sup> case $r \geq R$

The Gauss surface is outside the sphere to calculate  $E_2 = E_{outs}$

$$dq = \sigma ds \Rightarrow Q_{int} = \sigma 4\pi R^2$$

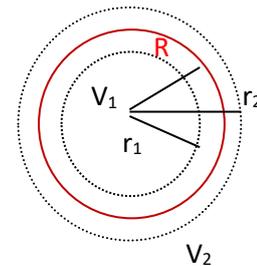
$$\text{So } E_2 4\pi r^2 = \frac{\sigma 4\pi R^2}{\epsilon_0} \Rightarrow E_2 = \frac{\sigma R^2}{\epsilon_0 r^2}$$

- The electrostatic potential  $V(r)$  at any point in space.

$$\vec{E} = -\overrightarrow{grad}v \Rightarrow E = -\frac{dv}{dr} \quad \text{so} \quad v = -\int E dr$$

1<sup>st</sup> case  $r < R$   $E_1 = 0 \Rightarrow v_1 = C_1$

2<sup>nd</sup> case  $r \geq R$   $E_2 = \frac{\sigma R^2}{\epsilon_0 r^2} \Rightarrow v_2 = -\sigma \frac{\sigma R^2}{\epsilon_0} \int \frac{dr}{r^2} = \frac{\sigma R^2}{\epsilon_0 r} + C_2$

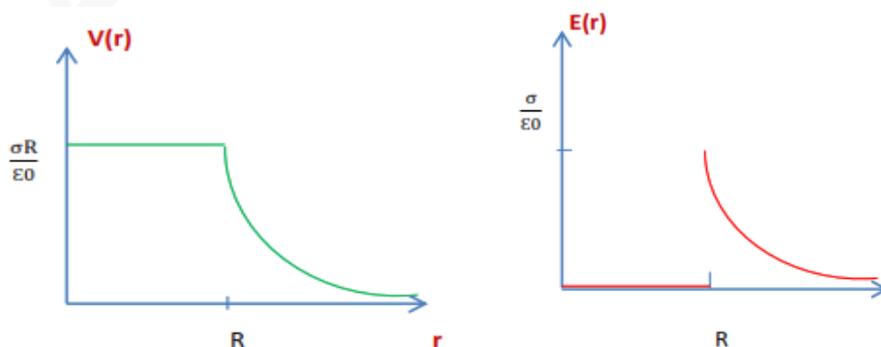


#### Calculating constants :

- The potential at infinity is zero ( $v_\infty=0$ ) so  $\lim_{r \rightarrow \infty} v = 0$  so  $C_2=0$  then  $v_2 = \frac{\sigma R^2}{\epsilon_0 r}$
- The potential is a continuous function in  $R$  so  $v_1(R) = v_2(R)$

then  $v_1 = C_1 = \frac{\sigma R^2}{\epsilon_0 R}$  so  $v_1 = \frac{\sigma R}{\epsilon_0}$

- Plot the graphs  $E(r)$  and  $V(r)$  as a function of  $r$  :



## Chapter II: Gauss' theorem

### b- Sphere charged in volume

1- The electrostatic field  $E(r)$  at any point in space.

We have 2 cases:

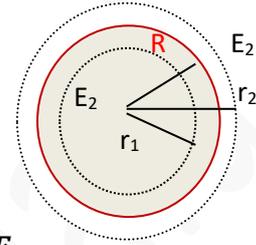
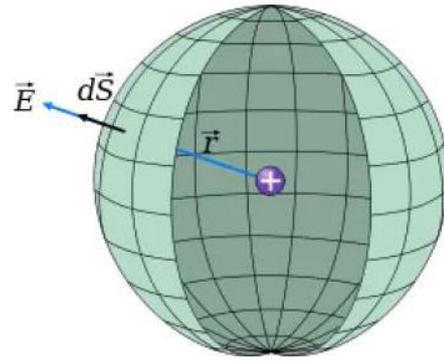
#### 1<sup>st</sup> case $r < R$

$$dq = \rho dv = \rho 4\pi r^2 dr \Rightarrow Q_{int} =$$

$$= \rho \iiint dv = \rho 4\pi \int_0^r r^2 dr$$

We integrate between **0 and r** because the  $Q_{int}$  charge is located in the volume of the Gauss sphere of **radius r**.

$$\Rightarrow Q_{int} = \rho \frac{4}{3}\pi r^3 \quad \text{so } (*) \Rightarrow E_1 = \frac{\rho \frac{4}{3}\pi r^3}{4\pi r^2 \epsilon_0} \quad \text{then } \mathbf{E}_1 = \frac{\rho}{3\epsilon_0} \mathbf{r} = \mathbf{E}_{ins}$$



#### 2<sup>nd</sup> case $r \geq R$

$$dq = \rho dv = \rho 4\pi r^2 dr \Rightarrow Q_{int} = \rho \iiint dv = \rho 4\pi \int_0^R r^2 dr$$

We integrate between **0 and R** because the  $Q_{int}$  charge is located in the volume of the sphere of **radius R**.

$$\text{So } Q_{int} = \rho \frac{4}{3}\pi R^3$$

$$(*) \Rightarrow E_2 = \frac{\rho \frac{4}{3}\pi R^3}{4\pi r^2 \epsilon_0} \quad \text{so } \mathbf{E}_2 = \frac{\rho R^3}{3\epsilon_0 r^2} = \mathbf{E}_{out}$$

2- The electric potential  $v(r)$  at any point in space.

$$\vec{E} = -\overrightarrow{grad}v \Rightarrow E = -\frac{dv}{dr} \quad \text{So } v = -\int E dr$$

#### 1<sup>st</sup> case $r < R$ :

$$E_1 = \frac{\rho}{3\epsilon_0} r \Rightarrow v_1 = -\frac{\rho}{3\epsilon_0} \int r dr \quad \text{so } v_1 = -\frac{\rho}{6\epsilon_0} r^2 + C_1$$

#### 2<sup>nd</sup> case $r \geq R$ :

$$E_2 = \frac{\rho R^3}{3\epsilon_0 r^2} \Rightarrow v_2 = -\frac{\rho R^3}{3\epsilon_0} \int \frac{1}{r^2} dr \quad \text{so } v_2 = \frac{\rho R^3}{3\epsilon_0} \frac{1}{r} + C_2$$

### Calculating constants :

The potential at infinity is zero ( $v_{\infty}=0$ ) so  $\lim_{r \rightarrow \infty} v = 0$  then  $v_2 = \frac{\rho R^3}{3\epsilon_0} \frac{1}{r}$

The potential is a continuous function in R, so  $v_1(R) = v_2(R)$

$$\frac{\rho R^3}{3\epsilon_0} \frac{1}{R} = -\frac{\rho}{6\epsilon_0} R^2 + C_1 \Rightarrow C_1 = \frac{\rho R^2}{\epsilon_0} \left(\frac{1}{3} + \frac{1}{6}\right) = \frac{\rho R^2}{2\epsilon_0}$$

$$\text{so } v_1 = -\frac{\rho r^2}{6\epsilon_0} + \frac{\rho R^2}{2\epsilon_0}$$

### 5.4. Case of an infinite plan

To find the electric field in an infinite plane,

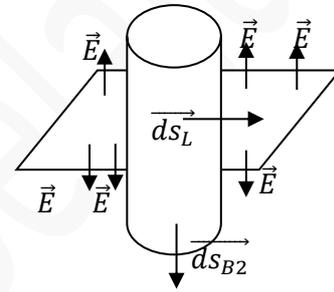
we use Gauss's theorem.

The Gaussian surface is a cylinder intersecting the plane.

The cylinder has radius r and height h.

For reasons of symmetry, the field is radial and constant at any point on the Gaussian surface.

In the Gaussian surface.



$$\phi = \oiint \vec{E} \cdot \vec{dS} = \frac{\sum Q_{int}}{\epsilon_0}$$

$$\phi = \phi_{sbase1} + \phi_{stat} + \phi_{sbase2} = \iint \vec{E} \cdot \vec{dS}_{B1} + \iint \vec{E} \cdot \vec{dS}_{B2} + \iint \vec{E} \cdot \vec{dS}_L$$

$$= 2 \iint E \cdot dS_{base} = 2E \cdot S_{base}$$

$$\Rightarrow \phi = 2E \cdot S_{base} = \frac{\sum Q_{int}}{\epsilon_0} \quad (1)$$

$$dq = \sigma ds \Rightarrow Q_{int} = \sigma \iint ds = \sigma S_{base}$$

$$(1) \Rightarrow 2E \cdot S_{base} = \frac{\sigma S_{base}}{\epsilon_0} \text{ so } \mathbf{E} = \frac{\sigma}{2\epsilon_0}$$

Then  $\lim_{R \rightarrow \infty} |E| = \frac{\sigma}{2\epsilon_0}$  and the field for an infinite plane is identical.