Aboubakr Belkaid University of Tlemcen.

Faculty of Sciences **Master1, all tracks**
Mathematics department **English for Technical purposes 2**
Mrs. A. Rahmoun

Lesson : Elements of mathematical logic

1 Definition

A simple mathematical proposition, p, is a sentence that can be judged **"true"** or **"false"** unambiguously and timelessly; there is no third possibility.

Examples
p: "$1 \leq 2$" a: "The cat is a mammal",
q:" $0 > 3$", b: "4 is an odd number",
r:" x is a prime number".

Proposition p is true. Proposition q is a false one. Sentence r is not a proposition. we need more information to be able to judge the sentence r. What about a and b?

2 Truth Values and Truth Table

Let "true" = 1 and "false" = 0. 1 and 0 are the truth values.

The truth table contains all possible cases for one, two, or three propositions.

In the case of a single proposition, the truth table contains only two cases:

p
1
0

The number of possibilities in the truth table depends on the number of propositions in the following way:

$$\text{nb de possibilities} = (2)^{nb \text{ of propositions}}$$

So in the case of 2 propositions we have 4 rows in the table and in the case of 3 propositions we have 8 rows and so on. We will see this in more detail later.

3 Negation of a proposition

Denote by $\overline{p}$ the logical opposite of p, for example:
$\overline{a}$:"the cat is **not** a mammal", which is false.
$\overline{b}$:"4 is **not** an odd number", which is true.

The truth table gives:

p	$\overline{p}$
1	0
0	1

Note: The negation of the negation $\overline{\overline{p}}$ is p.

4 Propositional connectives

From simple clauses, we construct compound clauses, using the following connectives:
$\wedge$ and $\vee$ or, $\Rightarrow$ implies, $\Leftrightarrow$ is equivalent to .

- **The conjonction "and":**

$p \wedge q$: "$1 \leq 2$ **and** $0 > 3$". (commutative).
Rule 1 $p \wedge q$: is true if and only if both are true. Our example is therefore false.

- **The disjunction "or" :**

$p \vee q$: "$1 \leq 2$ **or** $0 > 3$". (commutative).
Rule 2 $p \vee q$: is false if and only if both are false. So our example is true.

- **The implication : "$\Rightarrow$" :** $p \Rightarrow q$:

"**If** $1 \leq 2$ **then** $0 > 3$". Here, p is the hypothesis and q is the conclusion (or consequence); the implication is not commutative.

Rule 3 $p \Rightarrow q$ is false if and only if the first is true and the second is false. Our example is therefore false.

- **The equivalence "$\Leftrightarrow$" :** $p \Leftrightarrow q$:"$1 \leq 2$ means exactly that $0 > 3$".

We can also read : "$1 \leq 2$ **if and only if** $0 > 3$".
or "$1 \leq 2$ **is equivalent to** $0 > 3$". (commutative).

Rule 4 $p \Leftrightarrow q$: is true if both have the same truth value. Our example is therefore false.

Here is a summary of the truth values of compound sentences on the truth table:

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

1/ It is important to know that $[p \Rightarrow q] \Leftrightarrow [\overline{p} \vee q]$

2/ It should be remembered that equivalence is a double implication or an implication in both directions, i.e.[1] :

$$[(p \Leftrightarrow q)] \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)]$$

Exercise

We have the two clauses p: "The dogs bark", q: "The caravan passes".
Translate the following sentences:
(a) "The dogs do not bark".
(b) "If the caravan passes, then the dogs bark".
(c) "The caravan does not pass or the dogs bark".
(d) "The dogs do not bark and the caravan does not pass".

Correction

(a): $\overline{p}$ (b): $q \Rightarrow p$ (c) $\overline{q} \vee p$ (d) $\overline{p} \wedge \overline{q}$

[1] i.e. = id est a Latin expression meaning "that is to say that" and is still commonly used in mathematics.

		(a)	(b)		(c)	(d)
p	q	$\overline{p}$	$q \Rightarrow p$	$\overline{q}$	$\overline{q} \vee p$	$\overline{p} \wedge \overline{q}$
1	1	0	1	0	1	0
1	0	0	1	1	1	0
0	1	1	0	0	0	0
0	0	1	1	1	1	1

Case of three propositions We will learn this case through an exercise:
Exercise

Let P, Q and R be three propositions, draw the truth table of : $\overline{P}, (P \wedge Q) \vee R, \overline{P} \Rightarrow R, Q \Longleftrightarrow \overline{P}$

P	Q	R	$(P \wedge Q)$	$(P \wedge Q) \vee R$	$\overline{P}$	$\overline{P} \Rightarrow R$	$Q \Longleftrightarrow \overline{P}$
1	1	1	1	1	0	1	0
1	1	0	1	1	0	1	0
1	0	1	0	1	0	1	1
1	0	0	0	0	0	1	1
0	1	1	0	1	1	1	1
0	1	0	0	0	1	0	1
0	0	1	0	1	1	1	0
0	0	0	0	0	1	0	0

5 Tautology and Contradiction

A tautology is a proposition that is always true; a contradiction is a proposition that is always false.

Example

Draw the truth table of $(p \wedge \overline{p})$ then of $(p \vee \overline{p})$, conclude.

p	$\overline{p}$	$p \wedge \overline{p}$	$p \vee \overline{p}$
1	0	0	1
0	1	0	1

$(p \vee \overline{p})$ is a tautology, $(p \wedge \overline{p})$ is a contradiction.

5.0.1 The reciprocal and contrapositive of an implication

- The reciprocal: We call the reciprocal of $p \Rightarrow q$, the implication $q \Rightarrow p$.

Example

P:"I am in Tlemcen", Q:"I am in Algeria".

$p \Rightarrow q$: If "I am in Tlemcen" then "I am in Algeria", therefore $p \Rightarrow q$ is true.

However $q \Rightarrow p$ is the proposition: if "I am in Algeria" then "I am in Tlemcen", which is false.

- The contrapositive: We call the contrapositive of $p \Rightarrow q$ the implication $\overline{q} \Rightarrow \overline{p}$.

  In the previous example, $\overline{q} \Rightarrow \overline{p}$ is the sentence: if "I am not in Algeria" then "I am not in Tlemcen" which is true.

Important note:

It should be remembered that implication is equivalent to its contrapositive and not to its converse (proove it !), i.e.:

$$(p \Rightarrow q) \Leftrightarrow (\overline{q} \Rightarrow \overline{p})$$

6 Morgan's Law

These are the laws that give us the negation of propositions composed of logical connectives.

Exercise

Use the truth table to demonstrate Morgan's following laws:

1)$(\overline{p \wedge q}) \Leftrightarrow (\overline{p} \vee \overline{q})$

2) $(\overline{p \vee q}) \Leftrightarrow (\overline{p} \wedge \overline{q})$

3) $(\overline{p \Rightarrow q}) \Leftrightarrow (p \wedge \overline{q})$

4) To find the negation of the equivalence we first show that:

$[(p \Leftrightarrow q)] \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)]$. Then we deduce $\overline{(p \Leftrightarrow q)}$.

Correction

1)$(\overline{p \wedge q}) \Leftrightarrow (\overline{p} \vee \overline{q})$

			///////			///////	
p	q	$p \wedge q$	$\overline{p \wedge q}$	$\overline{p}$	$\overline{q}$	$\overline{p} \vee \overline{q}$	$(\overline{p \wedge q}) \Leftrightarrow (\overline{p} \vee \overline{q})$
1	1	1	**0**	0	0	**0**	1
1	0	0	**1**	0	1	**1**	1
0	1	0	**1**	1	0	**1**	1
0	0	0	**1**	1	1	**1**	1

2) $(\overline{p \vee q}) \Leftrightarrow (\overline{p} \wedge \overline{q})$

			///////			///////	
p	q	$p \vee q$	$\overline{p \vee q}$	$\overline{p}$	$\overline{q}$	$\overline{p} \wedge \overline{q}$	$(\overline{p \vee q}) \Leftrightarrow (\overline{p} \wedge \overline{q})$
1	1	1	**0**	0	0	**0**	1
1	0	1	**0**	0	1	**0**	1
0	1	1	**0**	1	0	**0**	1
0	0	0	**1**	1	1	**1**	1

3) $(\overline{p \Rightarrow q}) \Leftrightarrow (p \wedge \overline{q})$

			///////		///////	
p	q	$p \Rightarrow q$	$\overline{p \Rightarrow q}$	$\overline{q}$	$p \wedge \overline{q}$	$(\overline{p \Rightarrow q}) \Leftrightarrow (p \wedge \overline{q})$
1	1	1	**0**	0	**0**	1
1	0	0	**1**	1	**1**	1
0	1	1	**0**	0	**0**	1
0	0	1	**0**	1	**0**	1

We can also do without the truth table and proceed as follows:

$$\begin{array}{rcl}
(p \Rightarrow q) & \Leftrightarrow & (\overline{p} \vee q) \\
(\overline{p \Rightarrow q}) & \Leftrightarrow & (\overline{\overline{p} \vee q}) \\
(\overline{p \Rightarrow q}) & \Leftrightarrow & (\overline{\overline{p}} \wedge \overline{q}) \\
(\overline{p \Rightarrow q}) & \Leftrightarrow & (p \wedge \overline{q})
\end{array}$$

4) $[(p \Leftrightarrow q)] \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)]$

		///////			///////	
p	q	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$[(p \Leftrightarrow q)] \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)]$
1	1	**1**	1	1	**1**	1
1	0	**0**	0	1	**0**	1
0	1	**0**	1	0	**0**	1
0	0	**1**	1	1	**1**	1

Finding the negation of "$\Leftrightarrow$ " :

$$\begin{array}{rcl}
\text{Since } (p \Leftrightarrow q) & \Leftrightarrow & [(p \Rightarrow q) \wedge (q \Rightarrow p)], \\
\text{then : } \overline{(p \Leftrightarrow q)} & \Leftrightarrow & \overline{[(p \Rightarrow q) \wedge (q \Rightarrow p)]} \\
& \Leftrightarrow & \overline{(p \Rightarrow q)} \vee \overline{(q \Rightarrow p)} \\
& \Leftrightarrow & (p \wedge \overline{q}) \vee (q \wedge \overline{p})
\end{array}$$

Finally, remember that:

$$\overline{(p \Leftrightarrow q)} \Leftrightarrow \left[(p \wedge \overline{q}) \vee (q \wedge \overline{p})\right].$$

Aboubakr Belkaid University of Tlemcen.

Faculty of Sciences **Master1 , all tracks**
Mathematics department **English for Technical purposes 2**
Mrs. A. Rahmoun

Tutorial : Elements of mathematical logic

Exercise 1

Write the following statements in compound clause form and give their negation:

1. 7 is an even and prime number.
2. The sun does not shine and the earth is round.
3. Oran is an Algerian city or $6 = 3 + 1$.
4. If Omar plays the flute, then he plays the guitar.
5. x is an even number means exactly that x divides 2.

Exercise 2

Consider the following three propositions: p: "I'm leaving," q: "You're staying," r: "There's no one there."

Translate the following logical formulas into French, then draw their corresponding truth tables:

(a) $(p \wedge \overline{q}) \Rightarrow r$. (b) $(\overline{p} \vee \overline{q}) \Rightarrow \overline{r}$.

Exercise 3

Three people, x, y, and z, are summoned before a judge for a crime committed in the village. Here are their statements:

x declares: "z is guilty and y is innocent."

y declares: "I am innocent, but one of the other two is guilty."

z declares: "If x is guilty, then y is guilty too."

We set p, q, and r as follows: (x is innocent), (y is innocent), (z is innocent).

1. Express the statements of the three accused in terms of the simple propositions p, q, and r, then give their truth tables.
2. Suppose they are all innocent. Who lied in their statement?
3. Suppose they all told the truth in their statements. Who is innocent and who is guilty?
4. Suppose the innocent told the truth and the guilty lied, then who is innocent and who is guilty?
5. Is it possible that the innocent lied and the guilty told the truth?

Aboubakr Belkaid University of Tlemcen.

Faculty of Sciences **Master1 , all tracks**
Mathematics department **English for Technical purposes 2**
Mrs. A. Rahmoun

Correction of the tutorial : Elements of mathematical logic

Exercise 1

Write the following statements as compound propositions and give their negations:

1. "7 is an even and prime number".

If we set p: "7 is an even number", q: "7 is a prime number", we find (1): p $\wedge$ q.

Therefore

$$\overline{(1)} : \overline{(p \wedge q)} \Leftrightarrow \overline{p} \vee \overline{q}.$$

$\overline{(1)}$:"7 is not an even number or 7 is not a prime number."

1. But since the opposite of even is odd and vice versa, we may say:

   $\overline{(1)}$: "7 is an odd number or 7 is not a prime number."

   Traduction : "7 est un nombre impair ou 7 n'est pas un nombre premier".

2. "The sun doesn't shine and the Earth is round".

Let p: "The sun shines," q: "the Earth is round," and we find (2): $\overline{p} \wedge q$. So,

$$\overline{(2)} : \overline{(\overline{p} \wedge q)} \Leftrightarrow p \vee \overline{q}.$$

1. $\overline{(2)}$: "The sun shines and the Earth is not round".

Traduction : "Le soleil brille ou la terre n'est pas ronde."

3. "Oran is an Algerian city or $6 = 3 + 1$."

Put p:" Oran is an Algerian city ", q:"$6 = 3 + 1$", find (3): $p \vee q$.

1. So,

$$\overline{(3)} : \overline{(p \vee q)} \Leftrightarrow \overline{p} \wedge \overline{q}.$$

   $\overline{(3)}$:"Oran is not an Algerian city and $6 \neq 3 + 1$".

Traduction : "Oran n'est pas une ville algérienne et 6≠3+1

4. "If Omar plays the flute, then he plays the guitar".

We pose p:" Omar lays the flute ", q:"he plays the guitar", we find (4): $p \Rightarrow q$.

$$\overline{(4)} : \overline{(p \Rightarrow q)} \Leftrightarrow p \wedge \overline{q},$$

1. $\overline{(4)}$:"Omar plays the flute but he doesn't play the guitar."

Traduction : " Omar joue de la flûte, mais il ne joue pas de la guitare."

5. "x is an even number means exactly that x divides 2".

Let's put p:" x is an even number ", q:"x divides 2", we find (5): $p \Longleftrightarrow q$.

$$\overline{(5) :(p \Longleftrightarrow q)} \Leftrightarrow [(p \wedge \overline{q}) \vee (q \wedge \overline{p})],$$

$\overline{(5)}$:[(x is an even number and x does not divide 2) or (x divide 2 and x is not an even number)].

Traduction : [(x est un nombre pair et x ne divise pas 2) ou (x divise 2 et x n'est pas un nombre pair)].

Exercise 2

Consider the following three propositions: p: "I'm leaving," q: "You're staying," r: "There's no one here."

Translate the following logical formulas into French, then draw their corresponding truth tables.

p: "Je m'en vais," q: "Tu restes," r: "Il n y a personne ici."

(a) $(p \wedge \overline{q}) \Rightarrow r$.

(a) If I leave and you don't stay, then there's no one here!

Traduction : Si je pars et que tu ne restes pas, alors il n'y a personne ici.

p	q	r	$\overline{q}$	$p \wedge \overline{q}$	$(p \wedge \overline{q}) \Rightarrow r$
1	1	1	0	**0**	1
1	1	0	0	**0**	1
1	0	1	1	**1**	1
1	0	0	1	**1**	0
0	1	1	0	**0**	1
0	1	0	0	**0**	1
0	0	1	1	**0**	1
0	0	0	1	**0**	1

(b) $(\overline{p} \vee \overline{q}) \Rightarrow \overline{r}$.

(b) If I don't leave or you don't stay, then someone is here!

Traduction : Si je ne pars pas ou si tu ne restes pas, alors quelqu'un ici !

p	q	r	$\overline{p}$	$\overline{q}$	$\overline{p} \vee \overline{q}$	$\overline{r}$	$(\overline{p} \vee \overline{q}) \Rightarrow \overline{r}$
1	1	1	0	0	0	0	1
1	1	0	0	0	0	1	1
1	0	1	0	1	1	0	0
1	0	0	0	1	1	1	1
0	1	1	1	0	1	0	0
0	1	0	1	0	1	1	1
0	0	1	1	1	1	0	0
0	0	0	1	1	1	1	1

Exercise 3

Three people, x, y, and z, are summoned before a judge for a crime committed in the village. Here are their statements:

x declares: "z is guilty and y is innocent."

y declares: "I am innocent, but one of the other two is guilty."

z declares: "If x is guilty, then y is guilty too."

We set p, q, and r as follows: (x is innocent), (y is innocent), (z is innocent).

1. Express the statements of the three accused in terms of the simple propositions p, q and r, then give their truth table.

$$x : \overline{r} \wedge q, \quad y : q \wedge (\overline{p} \vee \overline{r}), \qquad z : \overline{p} \Rightarrow \overline{q}$$

						x		y	z
p	q	r	$\overline{p}$	$\overline{q}$	$\overline{r}$	$\overline{r} \wedge q$	$\overline{p} \vee \overline{r}$	$q \wedge (\overline{p} \vee \overline{r})$	$\overline{p} \Rightarrow \overline{q}$
1	1	1	0	0	0	0	0	0	1
1	1	0	0	0	1	1	1	1	1
1	0	1	0	1	0	0	0	0	1
1	0	0	0	1	1	0	1	0	1
0	1	1	1	0	0	0	1	1	0
0	1	0	1	0	1	1	1	1	0
0	0	1	1	1	0	0	1	0	1
0	0	0	1	1	1	0	1	0	1

2. Suppose they are all innocent. Who lied in their statement?

   The case where they are all innocent corresponds to the first row of the table. By examining the truth values of the statements, we realize that x and y lied.

3. Suppose they all told the truth in their statements. Who is innocent and who is guilty?

   The case where they all told the truth corresponds to the case of all true statements, which is represented by the second row of the table. Here we can see that x and y are innocent, but z is guilty.

4. Suppose the innocent told the truth and the guilty lied. Then who is innocent and who is guilty?

   This case corresponds to a match between the truth values of simple propositions and their statements, which is represented by the penultimate row of the table, which gives that x and y are guilty, and z is innocent.

5. Is it possible that the innocent lied and the guilty told the truth?

   This case corresponds to a discrepancy between (all) the simple propositions and their statements, which is not represented in any case in the table. Therefore, no, it is not possible that the innocent lied and the guilty told the truth.