Chapter 2: Combinatorial Analysis

I. Definition Combinatorial analysis is the mathematical theory of counting.

II. Fundamental Counting Principle Let $A_1, A_2, \dots, A_k$ be k *set s* sets of cardinals $n_1, n_2, \dots, n_k$ respectively. Then, the number of ways to choose one element from each set is: $n_1 \times n_2 \times \dots \times n_k$

Example: If we want to buy a computer system consisting of three components: a computer, a monitor, and a printer, and we have three brands of computers, two brands of monitors, and four brands of printers, then the total number of different systems we can buy is: $3 \times 2 \times 4 = 24$

Thus, we have 24 possible configurations for the three components.

Notably, we distinguish three types of configurations: arrangements, permutations, and combinations.

I. Arrangements

a) Arrangements without repetition: An arrangement without repetition is any ordered selection of k elements chosen from n elements without replacement. The number of such arrangements is given by

$$A_n^k = \frac{n!}{(n-k)!}$$

Examples:

1) The number of five-letter words (with or without meaning) that can be formed using the 26 letters of the alphabet corresponds to the number of arrangements without repetition with $k=5, n=26$.
2) The number of committees (president, secretary, treasurer) of three members that can be formed from a group of eight people corresponds to the number of arrangements without repetition with $k=3, n=8$.
3) A DNA sequence consists of a chain of four nucleotides [Adenine, Cytosine, Guanine and Thymine]. The number of possible arrangements of two nucleotides (dinucleotides) with $k=2, n=4$.

b) Arrangements with repetition: An arrangement with repetition is an ordered selection of k elements chosen from n elements with replacement. The number of such arrangements is:

$$\widetilde{A}_n^k = n^k$$

Examples:

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1) In the DNA sequence example, if we assume that a nucleotide can appear multiple times in the sequence (which is the reality), the number of possible dinucleotides is: $4^2=16$ possible dinucléotides.
2) How many seven-digit phone numbers exist? 10^7
3) In how many ways can 10 people be assigned to three service counters? 3^{10}

Remark: In the case of arrangements with repetition, it is possible to have $k \geq n$

II. Permutations

a) Permutations without repetition: A permutation without repetition is an arrangement without replacement where all the n elements are used. The number of such permutations is: $P_n = n!$

Examples:

1) In a train with 10 different wagons, there are 10! ways to arrange them, assuming the locomotive is always at the front.
2) There are 7!7 ways to arrange a group of seven people in a row of seven chairs.

Permutations with repetition: A permutation with repetition occurs when n elements are divided into k classes of sizes $n_1, n_2, \dots, n_k$, respectively. The number of such permutations is:

$$\widetilde{P}_n = \frac{n!}{n_1! n_2! \cdots n_k!}$$

Examples

1) A class has 12 students. In how many ways can these 12 students take three different exams, knowing that four students take the same exam? $\widetilde{P}_{12} = \frac{12!}{4!4!4!}$
2) How many anagrams can be formed with the letters of the word "excellence"? $\widetilde{P}_{10} = \frac{10!}{4!2!2!1!1!}$

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III. Combinations

a) *Combinations withoiut repetition:* A combination of k elements chosen from n is an unordered selection of these k elements where each appears at most once. The number of such combinations is:

$$C_n^k = \frac{n!}{k!(n-k)!}$$

Examples:

1) The selection of a five-member delegation from a group of 50 is a combination with $k=5$ and $n=50$; C_{50}^5
2) Given the numbers 1, 2, 3, and 4, selecting two numbers results in six combinations: C_4^2.

b) *Combinations with repetition:* A combination with repetition of k elements is a grouping of n elements in any order, where elements are not necessarily distinct and k is not necessarily less than or equal to n. The number of such combinations is:

$$\widetilde{C}_n^k = \frac{(n+k-1)!}{n!\,(k-1)!}$$

Examples:

1) Twelve delegates of an association must elect a representative to the board. Two candidates are running, and all delegates must vote for one or the other. The number of possible voting outcomes is a combination with repetition where $n=2$ and $k=12$. $\widetilde{C}_2^{12} = \frac{(2+12-1)!}{2!(12-1)!}$
2) The number of groups of nine letters (with repetition) that can be formed using the 03 letters a, b and c is: $\widetilde{C}_3^9 = \frac{(3+9-1)!}{3!(9-1)!}$