Chapter 2

Chapter 3

Probability calculation

3.1 Introduction

We can classify the experiences into two large groups;
those whose outcome is predicted with certainty depending on established physical laws and those whose outcome depends on chance, for which rigorous predictions cannot be made, these are random experiments.
Historically, the notion of probability emerged from simple examples generally taken from games of chance.

3.2 Concept of events, universe

A random experiment (trial) can present a finite or infinite number of outcomes (results).
Each of these outcomes is an elementary event E_i....
The set of these elementary events constitutes the universe Ω it is the set of all possible outcomes.
Examples:
1) Rolling an ordinary fair die n times $\Omega = \{E_1, ..., E_n\}$.
A non-elementary event is a set of several outcomes (part of Ω).
2) For the same random die-roll experiment with $n = 3$, we consider the events:
"obtaining even number", $A = \{2, 4, 6\}$.
"obtaining odd number"; $B = \{1, 3, 5\}$.
A et B are non-elementary event.

3.3 Algebra and Event Tribe

An event being an element of $\mathcal{P}(\Omega)$ obeys the laws of set theory.
$(\Omega, \mathcal{P}(\Omega))$ is called probability space.
If Ω is infinite, we can consider $\mathcal{A}$ a restricted family of $\mathcal{P}(\Omega)$. Thus $\mathcal{A}$, *Algebra* of parts of Ω which verifies:
$C1$- for every A in $\mathcal{A}$, $\bar{A} \in \mathcal{A}$.
$C2$- for every A, B in $\mathcal{A}$, $A \cup B \in \mathcal{A}$.
the condition $C2$ is equivalente to the condition $\acute{C}2$;
$\acute{C}2$- for every A, B in $\mathcal{A}$, $A \cap B \in \mathcal{A}$.
Examples:
1) The most basic algebra is $\{\Omega, \emptyset\}$.
Let $\Omega = \{a, b, c, d\}$, from the partition $\{a, b\}, \{c\}, \{d\}$ we construct the algebra $\mathcal{A}$ such that;
$\mathcal{A} = \{\emptyset, \{a, b\}, \{c\}, \{d\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}, \Omega\}$.
Properties of the algebra: P1- $\emptyset \in \mathcal{A}$ and $\Omega \in \mathcal{A}$.
P2- If $A_j \in \mathcal{A}$, for $1 \leq j \leq n$, we can demonstrate by induction that $\cup_{j=1}^{n} A_j \in \mathcal{A}$.

P3- If $A_j \in \mathcal{A}$, pour $1 \leq j \leq n$, we can demonstrate by passing to the complement that $\cap_{j=1}^{n} A_j \in \mathcal{A}$.
Some experiments can take place indefinitely, for example the random experiment "roll a die" then the event, A_n:"get 3 on the n^{th} thrown".
The event A: "get 3" will be then $A = \cup_{n=1}^{\infty} A_n$,hence the need to move from a finite union in P2 to a countable union; then we get:
If $A_n \in \mathcal{A}$ for every n,then $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$.
In thie case, we say that $\mathcal{A}$ is $\sigma - algebra$ or a *tribe*.
thus we can associate a probability to the space $(\Omega,\ \mathcal{A})$.

3.4 Event logic

*Certain event (contains all the results of the experiment);$E = \Omega$.
*Impossible event (never happens); $E = \emptyset$.
*The union of events, $A \cup B$; (A is achieved or B is achieved).
*The intersection of Event, $A \cap B$; (A is achieved and B is achieved).
*Two events that cannot be carried out at the same time are said to be incompatible.

$$\text{A and B are incompatibles (mutually exclusive)} \Leftrightarrow A \cap B = \emptyset.$$

*Complementarity, $\bar{A}$ complementary to A; A is not achieved.
A et $\bar{A}$ are incompatibles.

3.5 Probability

Once we have defined all the events in which we are interested, we will try to translate their chances of happening into numbers.

Definitions 1 *We call probability* $\mathbb{P}$ *on* $(\Omega,\ \mathcal{A})$
a map $\mathbb{P}$*:* $\mathcal{A} \to [0,\ 1]$*, such that:*
(i) $\mathbb{P}(\Omega) = 1$.
(ii) For any sequence A_n *of incompatible events;* A_n *and* A_m *in* $\mathcal{A}$*,*
$A_n \cap A_m = \emptyset$*, for* $m \neq n$*:*

$$\mathbb{P}\left(\cup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mathbb{P}\left(A_n\right).$$

Property called σ *additivity.*

Remark 1 *For any non-disjoint sequence of* (A_n)*, we then have the Boolean inequality;*

$$\mathbb{P}\left(\cup_{n=0}^{\infty} A_n\right) \leq \sum_{n=0}^{\infty} \mathbb{P}\left(A_n\right).$$

$(\Omega, \mathcal{A}, \mathbb{P})$ is called *probability space*.

properties 1 *P1)* $\mathbb{P}(\emptyset) = 0$.
P2) $\mathbb{P}(\bar{A}) = 1 - \mathbb{P}(A)$.
P3) $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$.
P4) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Examples:
1)) Let's throw a coin in the air, this game constitutes a test, that is to say an experience whose result is uncertain. There are two possible eventualities; heads or tails. If we consider the getting heads eventuality,

among the two equally probable outcomes, there is only one that is favorable to getting heads. The probability of getting heads is equal to $\frac{1}{2}$.
Generally;

$$\mathbb{P}(A) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

2) We roll a die, $\mathbb{P}(\text{get an odd number}) = 3/6$.
3) In an urn, there are 30 red balls, 20 blue balls, 10 green balls indistinguishable to the touch and arranged randomly. We draw a ball.
$\mathbb{P}(\text{drawing a green ball}) = 10/60$.
$\mathbb{P}(\text{drawing a blue ball}) = 20/60$.
$\mathbb{P}(\text{drawing a red ball}) = 30/60$.

3.5.1 Random selection and equiprobable events

The purpose of selecting objects at random is to ensure that each has the same chance of being selected. This method of selection is called fair or unbiased, and the selection of any particular object is said to be equally likely or equiprobable. When one object is randomly selected from n objects,

$$p_i = \frac{1}{n}, \text{ avec } 1 \leq i \leq n, \ n = card(\Omega).$$

Exemple: If we roll a well-balanced die, then the probability of a number between 1 and 6 occuring is:

$$p_i = \frac{1}{6}$$

avec $\Omega = \{1, 2, ..., 6\}$.
2) An urn contains 12 balls numbered from 1 to 12, we draw one at random, calculate the probability of drawing an even number or a multiple of three.
Solution:
**Express the events considered;
Note that the "or" between these events.
State the total probability theorem to use.
Ask the question about mutually exclusive.
A : « drawing an even number»$= \{2, 4, 6, 8, 10, 12\}$
B : « drawing a multiple of three » $= \{3, 6, 9, 12\}$
$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
$A \cap B = \{6, 12\}$.
$\mathbb{P}(A \cup B) = 7/12 + 4/12 - 2/12 = 2/3$.

3.6 Conditional Probability

The word conditional is used to describe a probability that is dependent on some additional information given about an outcome or event. let A et B be two avents we read $\mathbb{P}(A|B)$ as "the probability that A occurs, given that B occurs".
Introductory Example: :The composition of an amphitheater of 200 students in a university is as follows:
130 students are girls.
100 students live with their families.
Among these 100 students who live with their families,80 are girls.
We choose a student at random and we are interested in the three events:
A: "the student lives with his family".
B: "the student is a girl".
$A \cap B$: "the studentis a girl who lives with her family".
$\mathbb{P}(A) = 100/200$, $\mathbb{P}(B) = 130/200$,

$\mathbb{P}(A \cap B) = 80/200$.(number of girls who live with their parents/number of students)
If we know beforehand that the student is a girl then the reference set is made up of 130 student girls and the probability that she lives with her parents knowing that she is a girl becomes 80/130 (number of girls who live with their parents/number of student girls= $\mathbb{P}(A|B)$.
Indeed: this probability is conditioned by the additional information, the event B: "the student is a girl" is realized. Notice that $80/130 = (80/200)/(130/200)$, this is a general result.
$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.
The probability that the girl does not live with her parents among all student girls is 50/130, hence $\mathbb{P}(\bar{A}|B) = 1 - \mathbb{P}(A|B)$.

Definitions 2 *Soit* $(\Omega, \mathcal{A}, \mathbb{P})$ *espace probabilisé, A évènement tel que* $\mathbb{P}(A) \neq 0$, *pour tout évènement B, on pose*

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$$

Properties:
1) $\mathbb{P}(\bar{B}|C) = 1 - \mathbb{P}(B|C)$.
2) $\mathbb{P}(B \setminus C|A) = \mathbb{P}(B|A) - \mathbb{P}(B \cap C|A)$.
3) $\mathbb{P}(B_1 \cup B_2|A) = \mathbb{P}(B_1|A) + \mathbb{P}(B_2|A) - \mathbb{P}(B_1 \cap B_2|A)$.
4) Let I be a finite or infinite part of N, $\mathbb{N}$, $(B_i)_{i \in I}$ a sequence of incompatible events. Then

$$\mathbb{P}(\cup_{i \in I} B_i|A) = \sum_{i \in I} \mathbb{P}(B_i|A).$$

Exemple 2: Consider an urn containing 30 white, 20 red and 10 black balls. We draw two balls without replacement in the urn, what is the probability that the first ball is red and the second white?
Solution: Let be the events:
R:"draw a red ball".
B:"draw black ball".
$\mathbb{P}(R \cap B) = \mathbb{P}(R) * \mathbb{P}(B|R) = \frac{20}{60} * \frac{10}{59} = \frac{10}{117}$.
$\mathbb{P}(B|R) = \frac{10}{59}$, (59 balls remain in the urn, 10 of which are black).

Formula for compound probabilities:
Let $(A_i)_{1 \leq i \leq n}$ be sequence of events in $\mathcal{A}$ such that

$$\mathbb{P}(A_1 \cap A_2 \cap ... \cap A_n) \neq 0.$$

Then

$$\mathbb{P}(\cap_{i=1}^{n} A_i) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)....\mathbb{P}(A_n|A_1 \cap A_2.... \cap A_{n-1})$$

Théorème 1

$$\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A) = \mathbb{P}(A|B) * \mathbb{P}(B) = \mathbb{P}(B|A) * \mathbb{P}(A)$$

If A and B are independents, then

$$\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A) = \mathbb{P}(A) * \mathbb{P}(B)$$

Remark 2 *If A and B are independent, then*

$$\mathbb{P}(A|B) = \mathbb{P}(A).$$

The occurrence of one event does not modify the probability of occurrence of the other.

do not confuse incompatibility $(B \cap A = \emptyset)$ and independence.

Definitions 3 *We say that* $\{A_1, ..., A_1\}$ *is a complete system if*
$*\mathbb{P}(A_i) > 0.$
$*A_i \cap A_j = \emptyset$, *pour* $i \neq j$.*(incompatibles)*
$*\cup_{i=1}^{n} A_i = \Omega$

Théorème 2 *(Bayes' Theorem) Let* $\{A_1, ..., A_1\}$ *be a complete system and* B *an event. We have (the total probability Formula)*

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\,\mathbb{P}(A_i)$$

Moreover, (the Bayes' Formula)

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\,\mathbb{P}(A_i)}{\sum_{i=1}^{n} \mathbb{P}(B|A_i)\,\mathbb{P}(A_i)}$$

Example 1: Three urns are drawn at random, the composition of which is indicated in the table below. Knowing that we have obtained a red ball, we ask ourselves what is the probability that it comes from urn U2

	Red	Blue	green
U1	3	4	1
U2	1	2	3
U3	4	3	2

$\mathbb{P}(R) = \sum_{i=1}^{3} \mathbb{P}(R \cap Ui) = \sum_{i=1}^{3} \mathbb{P}(R|Ui)\,\mathbb{P}(Ui)$
$= \frac{1}{3}(\frac{3}{8} + \frac{1}{6} + \frac{4}{9}) = \frac{71}{216}$
The probability at posteriori is than
$\mathbb{P}(U2|R) = \frac{\mathbb{P}(R|U2)\mathbb{P}(U2)}{\mathbb{P}(R)} = \frac{\frac{1}{3}*\frac{1}{6}}{\frac{1}{3}*\frac{71}{72}}$
$= \frac{12}{71} < \mathbb{P}(U2)$.
Exemple 2: Three machines A, B, C produce respectively 40%, 35% and 25% of the total number of tablets manufactured by a pharmaceutical laboratory, each of these machines produces respectively 5%, 6% and 3% of defective tablets. a) We take a tablet at random, what is the probability that it is defective? b) We take a tablet at random, we see that it is defective, what is the probability that it is produced by machine A?
Answer:
$\mathbb{P}(A) = 0.4,\ \mathbb{P}(B) = 0.35,\ \mathbb{P}(C) = 0.25.$
D : « defective tablet ».
$\mathbb{P}(D|A) = 0.05,\ \mathbb{P}(D|B) = 0.06,\ \mathbb{P}(D|C) = 0.03.$
The realization of D can be due to each of the three causes A, B, C, these three causes are incompatible;
$\mathbb{P}(D) = \mathbb{P}(A)\mathbb{P}(D|A) + \mathbb{P}(B)\mathbb{P}(D|B) + \mathbb{P}(C)\mathbb{P}(D|C) = 0.4 * 0.05 + 0.35 * 0.06 + 0.25 * 0.03 = 0.0485.$
b) We know that the event "defective tablet" has occurred, we look for the probability of the cause « machine A »;
Let calculate $\mathbb{P}(A|D)$.
$\mathbb{P}(A|D) = \frac{\mathbb{P}(D|A)\mathbb{P}(A)}{\mathbb{P}(D)} = \frac{0.4*0.05}{0.0485} = 0.41$