course:

# Financial Mathematics

**Khaldi khadidja**

| | |
|---|---|
| **Course** | **Financial Mathematics** |
| **Number Unit** | **1** |
| **Unit Subject** | **Linear Equations**<br>**Quadratic Equations** |

# We will see in this unit

1. Linear Equations in One Variable
   Formulae
   Solving a Linear Equation in One Variable
   Applications

2. Quadratic Equations in One Variable
   Formulae
   Solving a quadratic Equation in One Variable
   Applications

# LEARNING OUTCOMES

At the end of this unit, you should be able to:

1. Understand what is meant by “linear equation” and “quadratic equation”

2. Understand how to solve linear and quadratic linear equations.

3. Solve equations for real world situations in order to solve problems, especially economic and financial.

## Linear Equation:

Any equation written in the form

$$Ax + B = C$$

Is said a linear equation where A, B and C are fixed numbers and $A \neq 0$

## Examples

- x – 5 = 16
- 2y + 4 = 12
- 5n - 4 = 6
- z/2 – 6 = 4

## How to solve linear equation?

### Two Steps for Solving linear Equations

Step1- Solve for any Addition or Subtraction on the variable side of equation by "undoing" the operation from both sides of the equation.

Step2- Solve any Multiplication or Division from variable side of equation by "undoing" the operation from both sides of the equation.

# Math Review

Example : Solve $4x - 5 = 15$

$4x - 5 = 15$

- Try the above Examples:

$x - 5 = 16$ ; $2y + 4 = 12$ ; $5n - 4 = 6$ ; $z/2 - 6 = 4$

solutions: $x =$ ; $y =$ ; $n =$ ; $z =$

## Quadratic Equation:

Any equation written in the form

$$Ax^2 + Bx + C = 0$$

Is said a quadratic equation where A, B and C are fixed numbers and $A \neq 0$

**Examples**

$$4x^2 + x + 1 = 0$$

$$4x^2 - 4x + 1 = 0$$

$$x^2 + 8x - 20 = 0$$

## How to solve quadratic equations ?

Quadratic equations can be solved using Discriminant method :

**Step1** : Express the equation in a general form

$$Ax^2 + Bx + C = 0$$

**Step2** : Calculate the discriminant: $\Delta = B^2 - 4AC$

**Step 3**: Give solution

**Case1**: $\Delta < 0$ no real roots

**Case2**: $\Delta = 0$ only one real root $r = \dfrac{-B}{2A}$

**Case3**: $\Delta > 0$ two distinct real roots

$$r_1 = \frac{-B - \sqrt{\Delta}}{2A} \quad \text{and} \quad r_2 = \frac{-B + \sqrt{\Delta}}{2A}$$

# Math Review

Example1: Solve the quadratic equation if possible

$$4x^2 + x + 1 = 0$$

**Step1 :**

**Step2 :**

**Step 3:**

Example2: Solve the quadratic equation

$$4x^2 - 4x + 1 = 0$$

**Step1 :**

**Step2 :**

**Step 3:**

!!! We can rewrite the quadratic equation

# Math Review

Example3: Solve the quadratic equation

$$x^2 + 8x - 20 = 0$$

**Step1 :**

**Step2 :**

**Step 3:**

!!! We can rewrite the quadratic equation

# Math Review

Example4: Solve the quadratic equation

$$x^2 - 4x = -3$$

**Step1** **:** rewrite the equation in general form

**A=** **, B=** **and** **C=**

**Step2** **:**

**Step 3:**

!!! We can rewrite the quadratic equation

# Applications of linear and quadratic equations in economic problems:

**Example1:**

The demand and supply equations for a good are given respectively by :

$$P = 100 - 0.5\ Q_d$$

$$P = 5 + 0.5\ Q_s$$

Calculate the equilibrium quantity & price

- Step 1: Set the right-hand side of both equations to equal on another & solve for Q* ($Q^* = Q_d = Q_s$ in equilibrium)
- Step 2: Substitute Q* into either equation & solve for P* (P*=P in equilibrium)

## Example2:

The weekly demand and supply equations for a good are given respectively by :

$$P = -(Q - 40)^2 - 10(Q - 40) + 400$$

$$P = (Q-40)^2 + 20(Q - 40) + 200$$

Where P is measured in dollars and Q is measured in units of a hundred. Calculate the equilibrium quantity & price

# We will see in the next unit

1. Essential rules of differentiation
2. Second and higher derivatives
3. Critical points
4. Optimization of functions of one variable

| | |
|---|---|
| **Course** | **Financial Mathematics** |
| **Number Unit** | **2** |
| **Unit Subject** | **Derivability**<br>**Critical points**<br>**Optimization of functions** |

# We will see in this unit

1. Essential rules of differentiation
2. Second and higher derivatives
3. Critical points
4. Optimization of functions of one variable

# LEARNING OUTCOMES

At the end of this unit, you should be able to:

1. Understand what is meant by "Optimization of functions of a single variable".
2. Apply some rules of differential calculus that are especially useful for decision making.
3. Find the critical points of a function.
4. Apply derivatives to real world situations in order to optimize unconstrained problems, especially economic and finance.

# Derivative of Function

**Definition1:**

Differentiation is a method to compute the rate at which a dependent output y changes with respect to the change in the independent input x.

**Definition2:**

This rate of change is called the derivative of y (the function) with respect to x. The derivative gives the value of the slope of the tangent line to a curve at a point (rate of change).

**Definition3:**

The slope of the tangent line is very close to the slope of the line through (a, f(a)) and a nearby point on the graph, for example (a + h, f(a + h)).

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

## Some rules of differentiation

To simplify the determination of derivatives we can use the following rules.

1/ $\frac{d}{dx}k = (k)' = 0$

2/ $\frac{d}{dx}cx = (cx)' = c$

3/ $\frac{d}{dx}x^n = \left(x^n\right)' = nx^{n-1}$

4/ $\frac{d}{dx}cx^n = \left(cx^n\right)' = ncx^{n-1}$

5/ $\frac{d}{dx}\sqrt{x} = \left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$

6/ $\frac{d}{dx}\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)' = \frac{-1}{x^2}$

7/ $\frac{d}{dx}\ln(x) = (\ln(x))' = \frac{1}{x}$

8/ $\frac{d}{dx}e^x = \left(e^x\right)' = e^x$

# Second and Higher Derivatives

Given a function f we defined first derivative of the function as:

$$f^{(1)}(x) = f'(x) = \lim_{h \to 0} \frac{f\ (x+h) - f\ (x)}{h}$$

The second derivative is obtained by:

$$f^{(2)}(x) = f''(x) = \lim_{h \to 0} \frac{f^{(1)}(x+h) - f^{(1)}(x)}{h}$$

And the n-th derivative is obtained by:

$$f^{(n)}(x) = \lim_{h \to 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}$$

# Second and Higher Derivatives

**Example:**

1/ the function: $f(x) = 4x^5 - 3x^2 + 2x - 10$

2/ first derivative : $f^{(1)} = f'(x) = 20x^4 - 6x + 2$

3/ second derivative : $f^{(2)} = f''(x) = 80x^3 - 6$

4/ third derivative : $f^{(3)} = f'''(x) = 240x^2$

5/ fourth derivative : $f^{(4)} = f''''(x) = 480x$

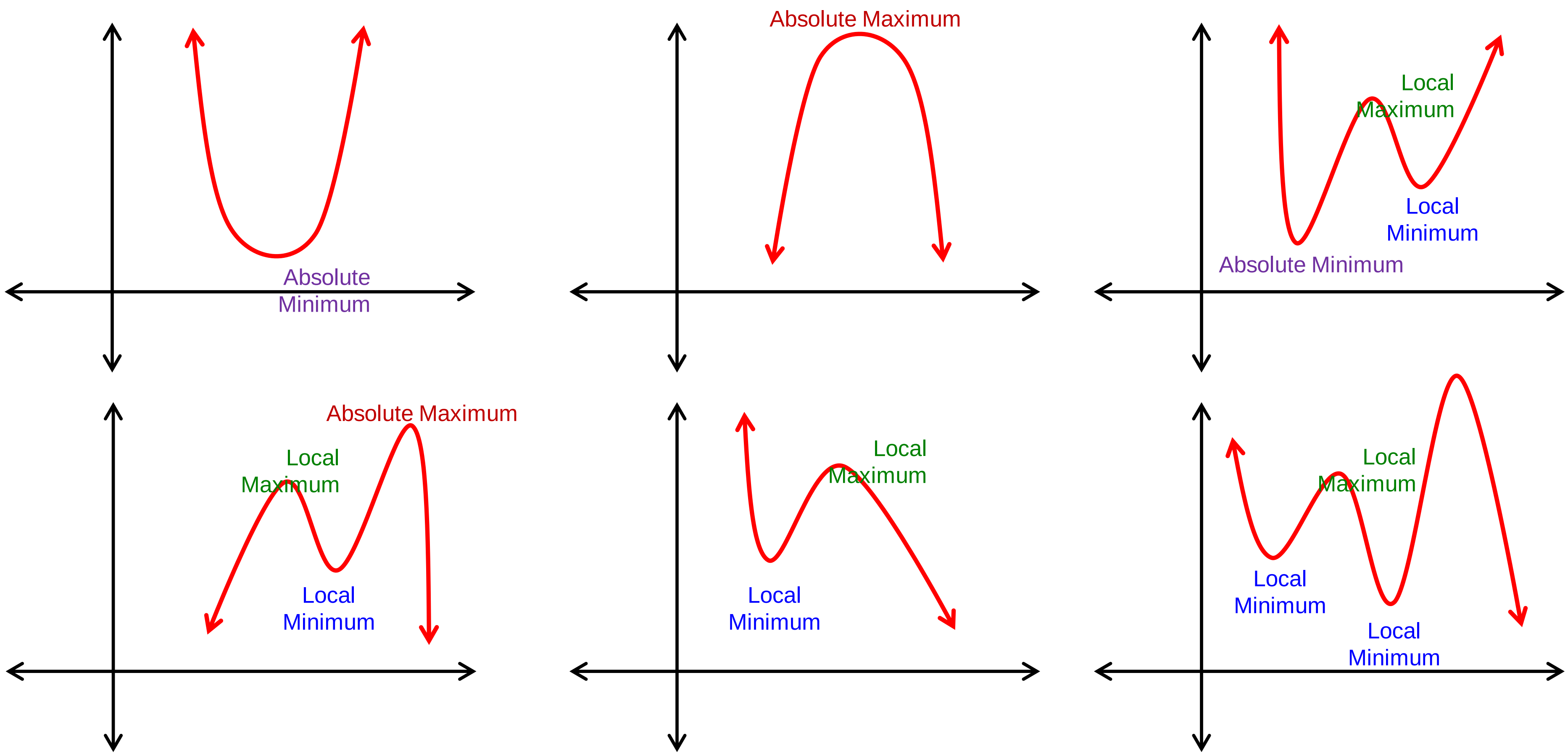
6/ fifth derivative : $f^{(5)} = f'''''(x) = 480$

# Math Review

## Classifications of Extreme Values

Absolute Minimum/Maximum – the smallest/largest function value in the domain

Local Minimum/Maximum – the smallest/largest function value in an open interval in the domain



# Critical points: local extrema

Definition :

A critical point of a function of a single real variable $f(x)$ is a value $x_0$ in the domain of $f$ where either the function is not differentiable or its derivative is 0, $f'(x) = 0$.

- ✓ The point $M = (x_0, f(x_0))$ is a local minimum if

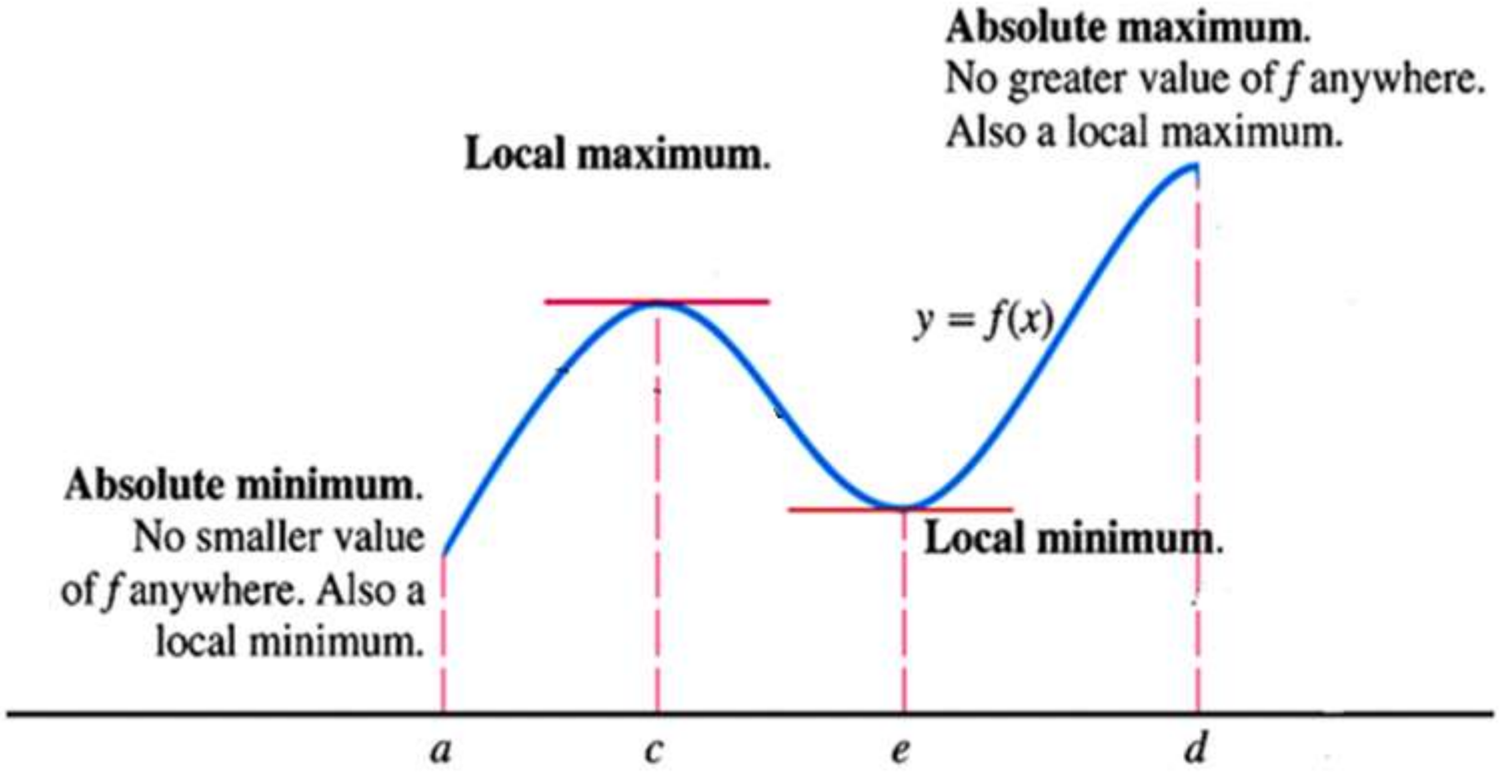
$$f'(x_0) = 0 \text{ and } f''(x_0) > 0$$

- ✓ The point $M = (x_0, f(x_0))$ is a local maximum if

$$f'(x_0) = 0 \text{ and } f''(x_0) < 0$$

•

## Maximum and Minimum points



# Example1:

Let $y = g(x) = x^2 - 3x + 2$

Find the critical point and determine its nature

$$g'(x) = 2x - 3 = 0$$

$$2x = 3$$

$$\therefore x_0 = \frac{3}{2} \quad \text{(critical value)}$$

Using the Second Derivative Test:

$$g''(x) = 2 > 0$$

The point $M = (\frac{3}{2}, -\frac{1}{4})$ is an absolute minimum

# Math Review

Example2: Let $y = f(x) = x^3 - 3x^2 + 2$

Find the critical points and determine their nature

$$f'(x) = 3x^2 - 6x = 0$$

$$3x(x-2) = 0$$

$$\therefore x_0 = 0, 2 \qquad \text{(critical values)}$$

$$f'(x) = 3x^2 - 6x = 0$$

$$f''(x) = 6x - 6$$

Second Derivative Test:

$$f''(0) = 6(0) - 6 = -6 < 0$$

$M_1 = (0,2)$ is a local maximum

$$f''(2) = 6(2) - 6 = 6 > 0$$

$M_2 = (2, -2)$ is a local minimum

# Unconstrained optimization of one variable

**Introduction:**

- In economic, financial and business contexts, optimization is a frequently relied upon tool
- Optimization is used to maximize profits, minimize costs, etc.
- We break optimization down into two types:
  - Unconstrained optimization: We seek to maximize or minimize functions without any restrictions.
  - Constrained optimization: We impose limits on the values that our variables can take (for example budget constraints, level of inputs, etc.)
- On this unit we focus on unconstrained optimization.

# Mathematical optimization

**Mathematical optimization is the selection of the best element based on a particular criterion from a set of available alternatives.**

**Optimization deals with the problem of finding numerically optimums (minimums or maximums) of a function. It consists of three components:**

- **The objective or objectives, that is, what do we want to optimize?**
- **A solution, that is, how can we achieve the optimal objective?**
- **The set of all feasible solutions, that is, among which possible options may we choose to optimize?**

# Unconstrained optimization of a function of one variable

**Definition:**

**The standard form of a continuous unconstrained optimization problem can take the following expressions:**

- $\min_{x} f(x) \quad in\ case\ of\ a\ minimization\ problem$
- $\max_{x} f(x) \quad in\ case\ of\ a\ maximization\ problem$

$\mathrm{f(x)}$ **is called the objective function and which represents the output you're trying to maximize or minimize.**

**Example:**

Economic agents seek the optimal value of some objective function such as:

- Consumers maximize utility
- Firms maximize profit and minimize cost

# Maximization of a function of one variable

**First-order condition (FOC) for maximization:**

For a function of one variable to attain its maximum value at some point, the derivative at that point must be zero

$$\left(\frac{df}{dx}\right|_{x=x*} = (f'(x)|_{x=x*} = 0$$

***FOC is a necessary condition for a maximum but not sufficient. Second order condition is required***

**Second-order condition for maximization:**

**We have a local maximum if the following is true:**

$$\left(\frac{d^2f}{dx^2}\right|_{x=x*} = (f''(x)|_{x=x*} < 0$$

# Profit maximization function

**Example:**

Suppose a monopolist served a market that faced the inverse demand function of $p$ = 250 − 2$q$ and a cost of production function of $c$ = 50q.

1- What value of $q$ maximizes the monopolist's profit?

2- What is the corresponding price and profit?

**Solution:**

1- In this example, the monopolist is maximizing his profit,

$$\max_{q} \pi(q)$$

We must first determine the profit function

Profit = total revenue – total cost = pq – 50q

$$\pi(q) = (250 - 2q)q - 50q = -2q^2 + 250q$$

# Profit maximization function

**Then,**

$$\max_q \pi(q) \Leftrightarrow = \max_q(-2q^2 + 200q\ )$$

First order condition for a maximum is:

$$\frac{d\pi}{dq} = \pi'(q) = -4q + 200 = 0$$

$$q^* = 50$$

Since the second derivative is always -4 (second order condition for a maximum is satisfied), then q = 50 is the quantity that maximizes the monopolist's profit.

2- The corresponding price = 250 – 2q* = 250 – 2(50) = 150

The corresponding Profit = $\pi(q) = -2q^2 + 200q\ =\ 5000$

# Minimization of a function of one variable

**First-order condition (FOC) for minimization:**

For a function of one variable to attain its minimum value at some point, the derivative at that point must be zero

$$\left(\frac{df}{dx}\right|_{x=x*} = (f'(x)|_{x=x*} = 0$$

**FOC is a necessary condition for a minimum but not sufficient. Second order condition is required.**

**Second-order condition for minimization:**

**We have a local minimum if the following is true:**

$$\left(\frac{d^2f}{dx^2}\right|_{x=x*} = (f''(x)|_{x=x*} > 0$$

# Cost minimization function

**Example:**

A manufacturer of POS systems/credit card readers finds that the cost (in dollars) generated by manufacturing q units per week is given by the function $C(q) = 0.15q^2 - 39q + 4500$.

1- How many units should be manufactured to minimize the cost?

2- What is the corresponding cost?

**Solution:**

1- In this example, the manufacturer is minimizing his cost,

$$\min_{q} C(q) \Leftrightarrow = \min_{q}(0.15q^2 - 39q + 4500)$$

First order condition for a minimum is:

$$\frac{dC}{dq} = C'(q) = 0.3q - 39 = 0$$

$$q^* = 130\, units$$

# Cost minimization function

Since the second derivative is always 0.3 (second order condition for a minimum is satisfied), then q = 130 is the quantity that minimizes cost.

2- Minimum cost:

$$C(q) = 0.15q^2 - 39q + 4500$$

$$C(130) = 0.15(130^2) - 39(130) + 4500 = \$1965$$

# we will see in the next unit

1. The inverse process of differentiation: Integration.

2. The connection between integration and summation.

3. How to calculate area under a curve.

4. How to calculate area between two Curves.
5. How to calculate consumer surplus

| | |
|---|---|
| **Course** | **Financial Mathematics** |
| **Unit course** | **FIN 118** |
| **Number Unit** | **3** |
| **Unit Subject** | **Integral Calculus** |

# We will see in this unit

1. Integral calculus : Definition
2. Indefinite integral
3. Definite integral
4. Some rules of integral
5. Area between two curves
6. Economic Application: Consumer surplus

# Learning Outcomes

At the end of this unit, you should be able to:

1. Understand what is meant by "integral of function".

2. Find definite or indefinite integrals.

3. Calculate the Area Between Two Curves.

4. Calculate the consumer surplus.

# Integral calculus

Frequently, we know the rate of change of a function $f'(x)$ and wish to find the original function $f(x)$ . Reversing the process of differentiation and finding the original function from the derivative is called integration or anti-differentiation. The original function, $f(x)$, is called the integral or antiderivative of $f'(x)$.

Thus, we have $\int f'(x)\,dx = f(x) + c$

# Integral calculus

**Example 1:**

1/ Find the derivative of $f_1(x) = c$ , $f_2(x) = x$ , $f_3(x) = x^2$

2/ Find the antiderivative of the results of question 1.

Solution:

1/ $f_1^{'}(x) = 0$ , $f_2^{'}(x) = 1$ , $f_3^{'}(x) = 2x$

2/ $\int f_1^{'}(x)dx = \int 0dx = c$ , $\int f_2^{'}(x)dx = \int 1dx = x + c$

$\int f_3^{'}(x)dx = \int 2xdx = x^2 + c$

# Indefinite Integral

- The indefinite integral of a function is a function defined as : $\int f(x)dx = F(x) + c$
- Every antiderivative $F$ of $f$ must be of the form $F(x) = G(x) + c$, where $c$ is a constant (constant of integration)

!!!

$$\int 2x\,dx = \underbrace{x^2 + c}$$

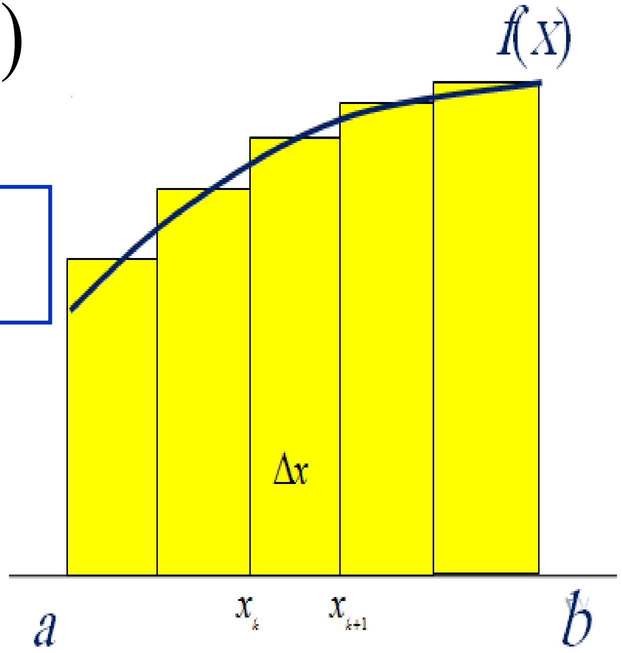
Represents every possible antiderivative of 2x.

# Definite integral

If f is a continuous function, the definite integral of f from a to b is defined as:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

An integral = Area under a curve

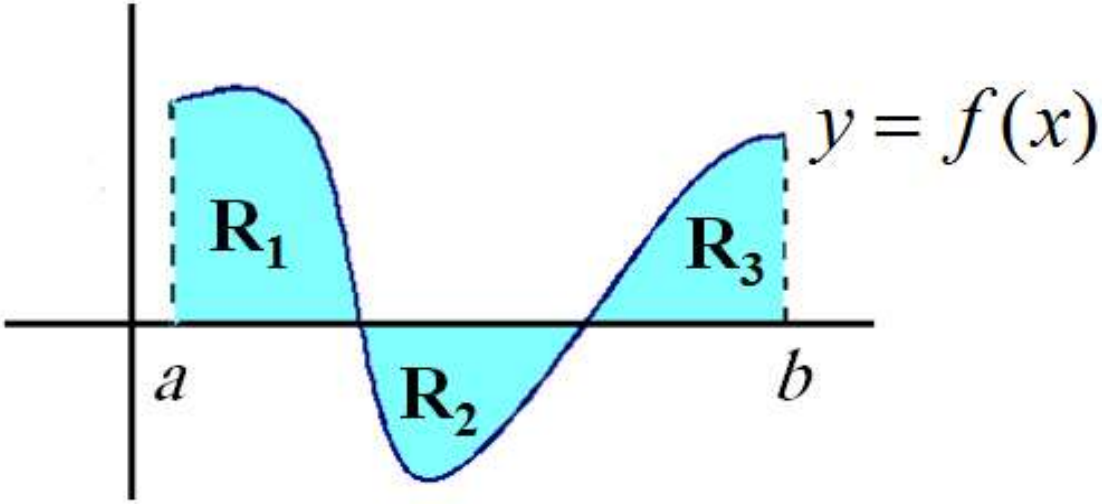



$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$$

$$\Delta x = \frac{b-a}{n} = x_{k+1} - x_k$$

# Integral Calculus

**Exemple1:**




$$\int_a^b f(x)dx = \text{Area of } R_1 - \text{Area of } R_2 + \text{Area of } R_3$$

# Some rules of integration

To simplify the determination of antiderivatives we can use the following rules.

1/ $\int dx = x + c$

2/ $\int k dx = kx + c$

3/ $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$

4/ $\int \dfrac{1}{x} dx = \ln|x| + c$

5/ $\int b^x dx = \dfrac{b^x}{\ln(b)} + c$

6/ $\int e^x dx = e^x + c$

# Some rules of integration

7/ $$\int (f \pm g)\,dx = \int f dx \pm \int g dx$$

8/ $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \quad (n \neq -1)$$

9/ $$\int (ax+b)^{-1}\,dx = \frac{1}{a}\ln|ax+b| + C$$

10/ $$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$$

11/ $$\int c^{ax+b}dx = \frac{1}{a\ln c}c^{ax+b} + C$$

# More examples

1/ $\int 2x.dx =$

2/ $\int (6y^5 + 3y)dy =$

3/ $\int \left( \frac{1}{x} - e^{2x} \right) dx =$

4/ $\int (6x-1)^2 dx =$

5/ $\int abx^3.dx =$

# More Examples

6/ $$\int_{-1}^{1} \left(x^2 - 7x + 12\right) dx =$$

7/ $$\int_{0}^{-2} \left(3x^2 - 3\right) dx =$$

8/ $$\int_{0}^{3} \left(e^x\right) dx =$$

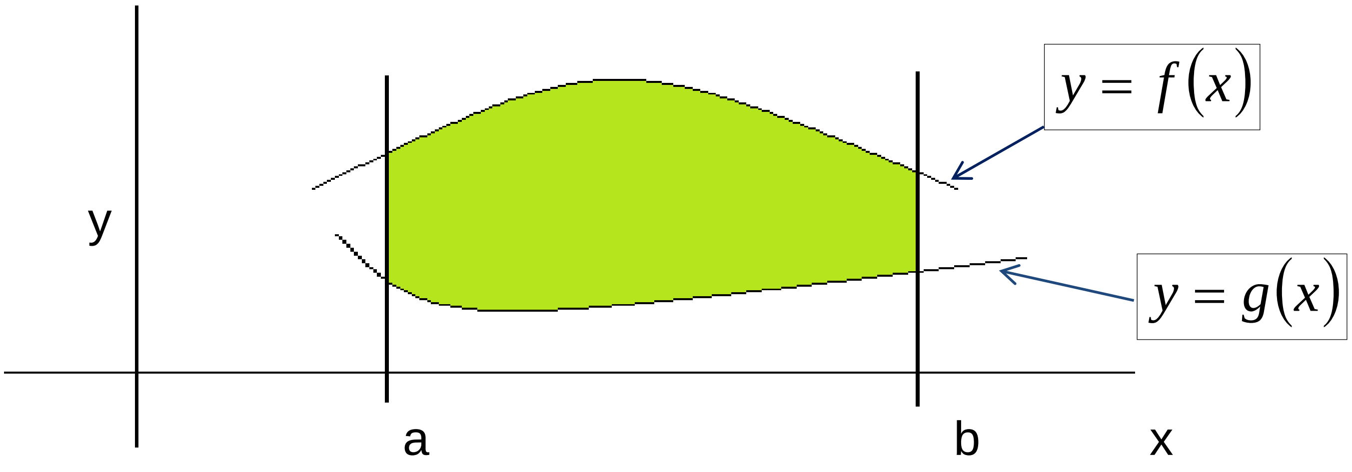
9/ $$\int_{0}^{1} \left(e^{2x+3}\right) dx =$$

10/ $$\int_{1}^{5} \left(2x - \frac{1}{x} + 1\right) dx =$$

# Area Between Two Curves

Let f and g be continuous functions, the area below f(x) and above g(x) over the interval [a, b] is:

$$R = \int_a^b \left[ f(x) - g(x) \right] dx$$



# Area Between Two Curves

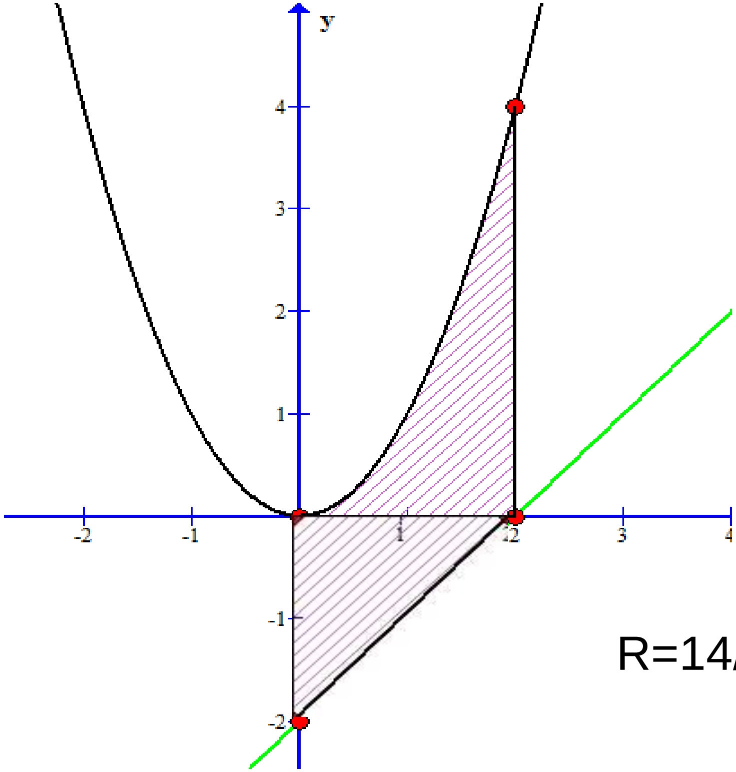
**Example:**

Find the area below f(x) and above g(x)

$$R = \int_0^2 \left[ f(x) - g(x) \right] dx$$ where

$f(x) = x^2$ and

$g(x) = x - 2$




R =

R=14/3

# Economic Application: Consumer surplus

## Definition1:

Consumer surplus is the economic gain accruing to consumers when they engage in trade. The gain is the difference between the price they are willing to pay and the actual price.

At the equilibrium level (demand and supply are equal), the consumer surplus is the difference between what consumers are willing to pay and their actual expenditure: It therefore represents the total amount saved by consumers who were willing to pay more than $p*$ per unit.

# Economic Application: Consumer surplus

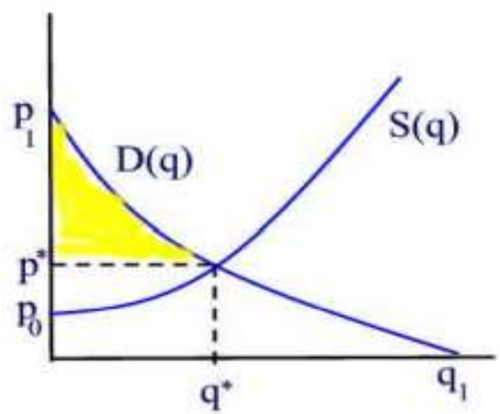
## Definition2:

Suppose that p = D(q) describes the demand function for a commodity. Then the consumer surplus is defined for the point (p*, q*) as:

$$Consumer\ surplus = \int_0^{q^*} D(q).\,dq - p^* q^*$$

$p^* q^*$ is the actual expenditure if the goods are sold at the equilibrium price.

Graphically, the consumer surplus is the area between demand curve and the horiz v)



# Economic Application: Consumer surplus

## Example:

The demand and supply functions for a given product are given by $D(q) = 60 - \frac{q^2}{10}$ and $S(q) = 30 + \frac{q^2}{5}$

Find the consumer surplus at the equilibrium price.

## Solution:

We first need to find the equilibrium price and quantity

$$D(q) = S(q) \iff 60 - \frac{q^2}{10} = 30 + \frac{q^2}{5} \iff q^* = 10$$

$$p^* = D(q^*) = 50$$

$$Consumer\ surplus = \int_0^{q^*} D(q).dq - p^*q^*$$

$$= \int_0^{10} D(q).dq - 500 = 66.67$$

# Time to Review !

1. By reversing the process of differentiation, we find the original function from the derivative. We call this operation integration or anti-differentiation.

2. The indefinite integral of a function is a function defined as :

$$\int f(x)\,dx = F(x) + c$$

3. If f is a continuous function, the definite integral of f from a to b is defined as:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

4. Consumer surplus $= \int_0^{q^*} D(q).dq - p^*q^*$

Where, $p^*q^*$ is the actual expenditure if the goods are sold at the equilibrium price.

# we will see in the next unit

- ✓ Function of several variables
- ✓ Partial differentiation
- ✓ Maximum and minimum of functions of several variables

| | |
|---|---|
| **Course** | **Financial Mathematics** |
| **Unit course** | **FIN 118** |
| **Number Unit** | **4** |
| **Unit Subject** | **Functions of several variables** |

# we will see in this unit

- ✓ Function of several variables
- ✓ Partial differentiation
- ✓ Maximum and minimum of function of several variables

# LEARNING OUTCOMES

At the end of this unit, you should be able to:

1. Understand what is meant by "Functions of several variables".

2. Calculate partial derivatives.

3. Determining the relative extrema of the functions of several variables.

# Functions of several variables

**Introduction:**

Suppose a firm that produces two products. It produces $x$ units for the first product at a profit of \$4 per unit and $y$ units for the second product at a profit of \$6 per unit.

Then, the total profit P of this firm is a function of the two variables $x$ and $y$, and it is given by:

$$P(x, y) = 4x + 6y$$

This function assigns to the input pair $(x,y)$ a unique output number, $4x + 6y$

For example,

$$P(25, 10) = 4(25) + 6(10) = \$160$$

This result means that by selling 25 units of the first product and 10 units of the second, the profit of the firm will be \$160.

**Definition1:**

A function of two variables assigns to each input pair, $(x, y)$, exactly one output number, $f(x, y)$.

**Definition2:**

A function of Several Variables is a function that has more than one independent variable.

**Example:** The total cost to a company of producing its goods is given by: $C(x, y, z, w) = 4x^2 + 5y + z - \ln(w + 1)$

Where, $x$ dollars are spent for labor, $y$ dollars for raw materials, $z$ dollars for advertising and $w$ dollars for machinery. This is a function of four variables.

Compute $C(3, 2, 0, 10)$?

$$C(3, 2, 0, 10) = 4(3^2) + 5(2) + 0 - ln(10 + 1) = \$43.6$$

**In this unit, we treat the case of two variables only.**



62

# Partial derivatives

**Definition:**

Suppose, we have a function $f(x, y)$, which depends on two variables $x$ and $y$, where $x$ and $y$ are independent of each other. Then we say that the function $f$ partially depends on $x$ and $y$.

Now, if we calculate the derivative of $f$, then that derivative is known as the partial derivative of $f$. If we differentiate the function $f$ with respect to $x$, then take $y$ as a constant and if we differentiate $f$ with respect to $y$, then take $x$ as a constant.

# Partial derivatives Formula

If $f(x, y)$ is a function which partially depends on $x$ and $y$ and if we differentiate $f$ with respect to $x$ and $y$, then the derivatives are called the partial derivatives of $f$.

- The formula for partial derivative of $f$ with respect to $x$taking $y$ as a constant is given by:

$$f_x = \frac{\partial f}{\partial x} = lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

- The formula for partial derivative of $f$ with respect to $y$taking$x$ as a constant is given by:

$$f_y = \frac{\partial f}{\partial y} = lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

# To get partial derivatives….

- To get $f_x$ assume *y* is a constant and differentiate with respect to *x*

**Example** $$f(x,y) = xy^2 + x^2 y$$

$$f_x(x,y) = (1)y^2 + (2x)y = y^2 + 2xy$$

- To get $f_y$ assume *x* is a constant and differentiate with respect to *y*

**Example** $$f(x,y) = xy^2 + x^2 y$$

$$f_y(x,y) = x(2y) + x^2(1) = 2xy + x^2$$

**Example:** Compute the partial derivatives of the following functions

1/ $f(x,y) = 2x + 3y - 4$

2/ $f(x, y) = 3x^2 y - 2 + y^3.$

3/ $f(x, y) = 3x^2 y + x \ln y$

# Second-Order Partial Derivatives

- The partial derivative of a partial derivative is called a second-order partial derivative.
- Four second order partial derivatives can be deduced:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left( f_x \right) \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left( f_x \right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left( f_y \right) \qquad f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left( f_y \right)$$

**Example1:** Find the second-order partial derivatives of the function

$$f(x, y) = 3x^2 y + x \ln y$$

$$f_x = 6xy + \ln y \qquad f_y = 3x^2 + x\left(\frac{1}{y}\right)$$

$$f_{xx} = 6y \qquad f_{yy} = -\frac{x}{y^2} \qquad f_{xy} = 6x + \frac{1}{y} \qquad f_{yx} = 6x + \frac{1}{y}$$

**Example2:** Find the second-order partial derivatives of the function

$$f(x, y) = 3x - 2y^2$$

$$f_x = 3 \qquad f_y = -4y$$

$$f_{xx} = 0 \qquad f_{yy} = -4 \qquad f_{xy} = 0 \qquad f_{yx} = 0$$

**Example:** Compute the second order partial derivatives of the following functions

$$f(x, y) = 2x - x^2 - y^2$$ 1/

$$f(x, y) = 3x^2y - 2 + y^3.$$ 2/

$$f(x, y) = 3x^2y + x\ln y$$ 3/

# Maximum and minimum of functions of several variables

**Definition:**

Let $f$ be a function defined on a region $R$ containing $(a, b)$.

- $f(a, b)$ is a *relative maximum* of $f$ if $f(x, y) \leq f(a,b)$ for all $(x, y)$ sufficiently close to $(a, b)$.
- $f(a, b)$ is a *relative minimum* of $f$ if $f(x, y) \geq f(a,b)$ for all $(x, y)$ sufficiently close to $(a, b)$.

# Critical Point of $f$

**Definition:**

A point $(a, b)$ in the domain of $f$ is a critical point of *a function* $f(x,y)$ if :

$$\frac{\partial f}{\partial x}(a,b)=0 \text{ and } \frac{\partial f}{\partial y}(a,b)=0$$

- If $f(x,y)$ has a relative extreme value, then it must occur at a critical point.

**Example1:** Determine critical point of the of the function

$$f(x,y) = 2x - x^2 - y^2$$

$$f_x = \frac{\partial f}{\partial x}(x,y) = 2 - 2x = 0 \text{ and } f_y = \frac{\partial f}{\partial y}(x,y) = -2y = 0$$

$$\begin{cases} 2 - 2x = 0 \\ -2y = 0 \end{cases} \Rightarrow \Bigg) \begin{cases} x = 1 \\ y = 0 \end{cases} \Bigg)$$

The point M = (1, 0) is a critical point.

**Example2:** Determine critical point of the of the function

$$f\left(x,y\right)=x^2-2xy+3y^2+4x-16y+22$$

| | |
|---|---|
| **Course** | **Financial Mathematics** |
| **Unit course** | **FIN 118** |
| **Number Unit** | **5** |
| **Unit Subject** | **Sequences & Series** |

# we will see in this unit

✓The “arithmetic sequences” and “arithmetic series”.

✓ The “Geometric sequences” and “Geometric series”.

✓ Solve some questions for real world situations in order to solve problems, especially economic and financial.

# LEARNING OUTCOMES

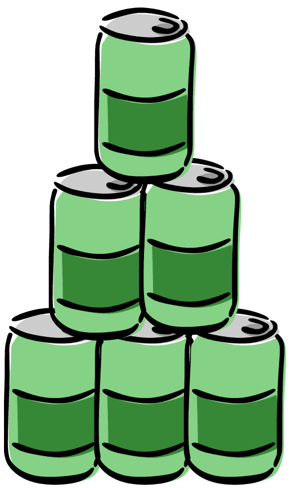
At the end of this unit, you should be able to:

1. Understand what is meant by "Arithmetic sequences" and "Arithmetic series"

2. Understand what is meant by "Geometric sequences" and "Geometric series".

3. Solve some questions for real world situations in order to solve problems, especially economic and financial.

# Arithmetic sequences and series



The can pyramid…

Q1/ How many cans are there on the bottom row in this pyramid ?

Q2/ How many cans are there in this pyramid ?

1/ There are 3 cans on the bottom row

2/ There are 6=3+2+1 cans in this pyramid

!!! How many cans are there in a pyramid with 100 cans on the bottom row?

# Arithmetic sequences and series

**Solution:**

We have S = 100+99+98+….+3+2+1 cans in a pyramid with 100 cans on the bottom row.

$$S = 100+99+98+\ldots+3+2+1$$

$$S = 1+2+3+\ldots+98+99+100$$

$$2S = 101+101+101+\ldots+101+101+101$$

100 times

Then

$$S = \frac{100 \times 101}{2} = 5050$$

# Arithmetic sequences

**Definition1:** An Arithmetic Sequence is a sequence whose consecutive terms have a common difference, r.

- In the pyramid can example the common difference r = 1.

**Example 1:**

0 2 4 6 8 10 12 14 16 18 ? ? ?

→ To find the common difference (r), just subtract any term from the term that follows it. Then

The next numbers are 20, 22 and 24 because the common difference is r = 2.

**!!! Question:**

What is the twentieth number ?

# Arithmetic sequences

## Example 1 (Continued)

If We set $u_1=0$, $u_2=2$, $u_3=4$, $u_4=6$, $u_5=8$, and so on. Then the twentieth number correspond to $u_{20}$

First Term: $u_1 = 0$

Second Term: $u_2 = u_1 + r = 0 + 2 = 2$

Third Term: $u_3 = u_2 + r = u_1 + 2r = 0 + 2\times 2 = 4$

Fourth Term: $u_4 = u_3 + r = u_1 + 3r = 0 + 3\times 2 = 6$

Fifth Term: $u_5 = u_4 + r = u_1 + 4r = 0 + 4\times 2 = 8$

And so on:

$$u_{20} = u_{19} + r = u_1 + 19r = 0 + 19\times 2 = 38$$

# Formula for the n$^{th}$ term of an ARITHMETIC sequence

$$u_n = u_1 + (n-1)r$$

- $u_n$: n$^{th}$ term
- $u_1$: 1$^{st}$ term
- $n$: # of term trying to find
- $r$: common difference

Example 2:

Given the following arithmetic sequence:

100, 120, 140, 160,... Find the 10th term

$u_1 = 100 \quad r = 20 \quad u_{10} = u_1 + 9r = 100 + 9 \times 20 = 280$

# Arithmetic sequences

Example3:

Which of the following sequences are arithmetic? Identify the common difference, and calculate $\mathbf{u}_{12}$ for arithmetic sequences.

$$-3, -1, 1, 3, 5, 7, 9, \ldots$$

$$84, 80, 74, 66, 56, 44, \ldots$$

$$15.5, 14, 12.5, 11, 9.5, 8, \ldots$$

$$-50, -44, -38, -32, -26, \ldots$$

# Aerithmetic series

**Definition:**

An Arithmetic series is the sum of the terms in an arithmetic Sequence.

If we consider the following arithmetic sequence

$$u_1, u_2, u_3, u_4, u_5, \ldots, u_n$$

then

$$S_n = \sum_{i=1}^{n} u_i$$

We can write the formula of arithmetic series as:

# of terms

$$S_n = \frac{n}{2}\left(u_1 + u_n\right)$$

1st Term Last Term

# Arithmetic series

Example 1:

Find the 20th term and the sum of 20 first terms of the sequence 2 , 5, 8, 11, 14, 17, . . .

Solution:

This is an arithmetic sequence with

•

# Arithmetic series

Example 2:

Find the sum of the terms of this arithmetic sequence.

$$151 + 147 + 143 + 139 + ..... + (-5)$$

Solution:

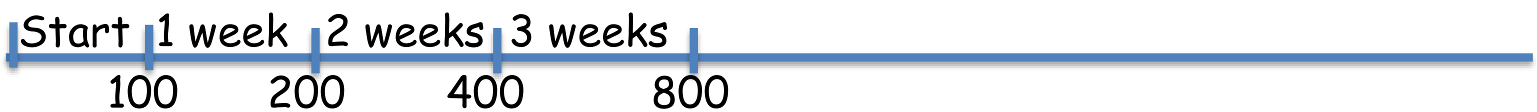
# Geometric sequences and series

Example:

What if your interest check started at \$100 a week and doubled every week. What would your interest after three weeks? What would your interest after nine weeks? What would your total interest at the end of tenth week?

Solution:

- Starting \$100.
- After one week : $2 \times \$100 = \$200$
- After two weeks : $2 \times \$200 = \$400 = 2^2 \times 100$
- After three weeks : $2 \times \$400 = \$800 = 2^3 \times 100$




!! After 3 weeks I would have \$800

# Geometric sequences and series

## Solution: continued

- Starting \$100.
- After one week : $2 \times \$100 = \$200 = 2^1 \times 100$
- After two weeks : $2 \times \$200 = \$400 = 2^2 \times 100$
- After three weeks : $2 \times \$400 = \$800 = 2^3 \times 100$

And so on

- After nine weeks : $2^9 \times 100 = \$51200$

!!! After 9 weeks I would have \$51200

- The total interest is:

$$S = 100 + 200 + 400 + 800 + .... + 51200$$

$$S = 100\left(1 + 2 + 2^2 + 2^3 + .... + 2^9\right)$$

# Geometric sequences and series

Solution: continued

$$S = 100\left(1 + 2 + 2^2 + 2^3 + .... + 2^9\right)$$

$$2S = 100\left(2 + 2^2 + 2^3 + 2^4 + ... + 2^{10}\right)$$

-

$$S - 2S = 100\left(1 - 2^{10}\right)$$

$$\left(1-2\right)S = 100\left(1 - 2^{10}\right)$$

$$S = 100\,\frac{\left(1 - 2^{10}\right)}{\left(1-2\right)} = 102300$$

!!! At the end of tenth week I would have $102300

# Geometric sequences

**Definition1:**

A Geometric Sequence is a sequence whose consecutive terms have a common ratio, q.

**Example 1:**

→ In interest example the common ratio q = 2.

suppose we have the following sequence

1 4 16 64 ? ?

→ To find the common ratio (q), just divide any term by the previous term. Then

The next numbers are 256 and 1024 because the common ratio is q = 4.

**!!! Question:**

What is the tenth number ?

# Geometric sequences

If We set $u_1=1$, $u_2=4$, $u_3=16$, $u_4=64$, and so on.
Then the tenth number correspond to $u_{10}$

First Term: $u_1 = 1$

Second Term: $u_2 = u_1 \times q = 1 \times 4 = 4$

Third Term: $u_3 = u_2 \times q = u_1 \times q^2 = 1 \times 4^2 = 16$

Fourth Term: $u_4 = u_3 \times q = u_1 \times q^3 = 1 \times 4^3 = 64$

Fifth Term: $u_5 = u_4 \times q = u_1 \times q^4 = 1 \times 4^4 = 256$

And so on:

$$u_{10} = u_9 \times q = u_1 \times q^9 = 1 \times 4^9 = 262144$$

# Formula for the $n^{th}$ term of a GEOMETRIC sequence

$$u_n = u_1 \times q^{n-1}$$

$n^{th}$ term → $u_n$

$1^{st}$ term → $u_1$

common ratio → $q$

**Example 2:**

Given the following geometric sequence:

5, 15, 45,... Find the 10th term.

# Geometric sequences

Example3:

Which of the following sequences are geometric? Identify the common ratio, and calculate $u_5$ for geometric sequences:

- 2, 6,18,… .
- 5, 15, 25, 45, ….
- 1, 5, 25, 125, ….

# Geometric series

**Definition:**

A Geometric Series is the sum of the terms in a geometric Sequence.

If we consider the following geometric sequence

$$u_1, u_2, u_3, u_4, u_5, \ldots, u_n \qquad S_n = \sum_{i=1}^{n} u_i \qquad \text{then}$$

We can write the formula of geometric series as:

The sum of n terms

# of terms

$$S_n = u_1 \times \frac{(1-q^n)}{(1-q)}$$

1st Term

The common ratio

# Geometric sequences and series

**Example:**

1/ Find the common ratio of the following sequence
2, -4, 8, -16, 32, …

2/ Find the ninth and tenth term.

3/ Find $S_{10}$.

**Solution:**

# Arithmetic and geometric sequences and series

**Time to Review!**

| Arithmetic sequences and series | Geometric sequences and series |
|---|---|
| $u_n - u_{n-1} = r$ | $\dfrac{u_n}{u_{n-1}} = q$ |
| $u_n = u_1 + (n-1) \times r$ | $u_n = u_1 \times q^{n-1}$ |
| $S_n = \dfrac{n(u_1 + u_n)}{2}$ | $S_n = u_1 \times \left[\dfrac{1-q^n}{1-q}\right]$ |

**That's All !**

# Practical Examples

**Example 1: (Arithmetic sequence)**

Abdul Aziz makes a monthly saving plan for a period of two years in a bank that doesn't give interest for saving accounts (compliant with Shariah). In the first month he deposits 1000 SAR, in the second month he deposits 1200 SAR and the third month 1400 SAR and so on.

1. What is the amount that he will deposit in the fifth month, in the 24th month?
2. What is the total amount that he will obtain at the end of the second year?

# Practical Examples

## Solution of Example1

1.

2.

# Practical Examples

## Example 2: (Geometric sequence)

Abdul Aziz makes a monthly saving plan for a period of one year in a bank that doesn't give interest for saving accounts (compliant with Shariah). In the first month he deposits 5 SAR, in the second month he deposits 10 SAR and the third month 20 SAR and so on.

1. What is the amount that he will deposit in the fourth month, in the eighth month.

2.What is the total amount that he will obtain at the end of the period.

# Practical Examples

## Solution of Example2

1.

2.

# we will see in the next unit

- ✓ The relationship between time and money.
- ✓ The simple interest rate and the interest amount
- ✓ The present value of one future cash flow
- ✓ The future value of an amount borrowed or invested.
- ✓ The relationship between Real Interest Rate, Nominal Interest Rate and Inflation.

| | |
|---|---|
| **Course** | **Financial Mathematics** |
| **Unit course** | **FIN 118** |
| **Number Unit** | **6** |
| **Unit Subject** | **Time Value of Money**<br>**Simple Interest** |

# we will see in this unit

- ✓ The relationship between time and money.
- ✓ The simple interest rate and the interest amount
- ✓ The present value of one future cash flow
- ✓ The future value of an amount borrowed or invested.
- ✓ The relationship between Real Interest Rate, Nominal Interest Rate and Inflation.

# LEARNING OUTCOMES

At the end of this unit, you should be able to:

Understand simple interest including .1 accumulating, discounting and making comparisons using the effective interest rate.

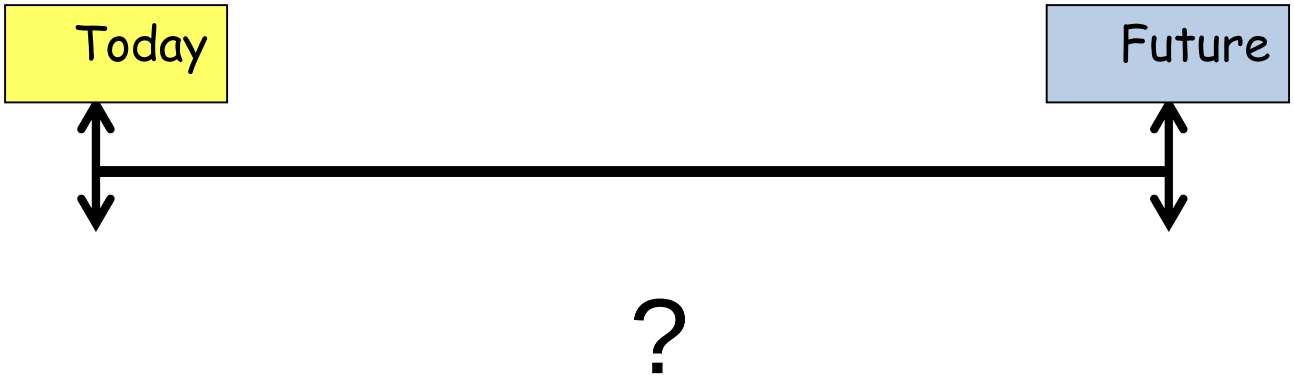
2.Identify variables fundamental to solving interest problems.

3.Solve problems including future and present value.

4.Distinguish between nominal and effective interest rates.

# Time value of Money

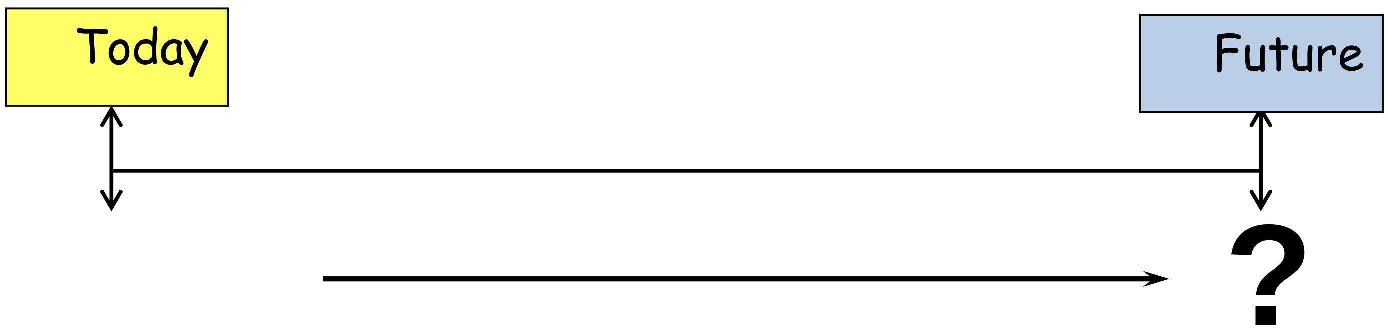
- The time value of money is the relationship between time and money.
- Receiving 1 DA today is worth more than 1 DA in the future. This is due to opportunity costs.
- TIME allows you the opportunity to postpone consumption and earn INTEREST



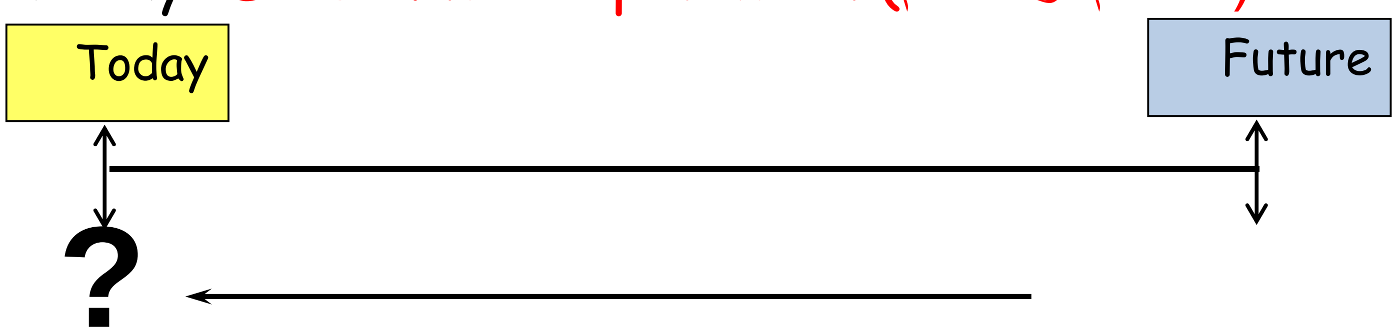
# Time value of Money

If we can measure this opportunity cost, we can:

- Translate 1 DA today into its equivalent in the future : operation of capitalization(الرسملة)




- Translate 1 DA in the future into its equivalent today: Discounted operation (الخصم أو الحسم)

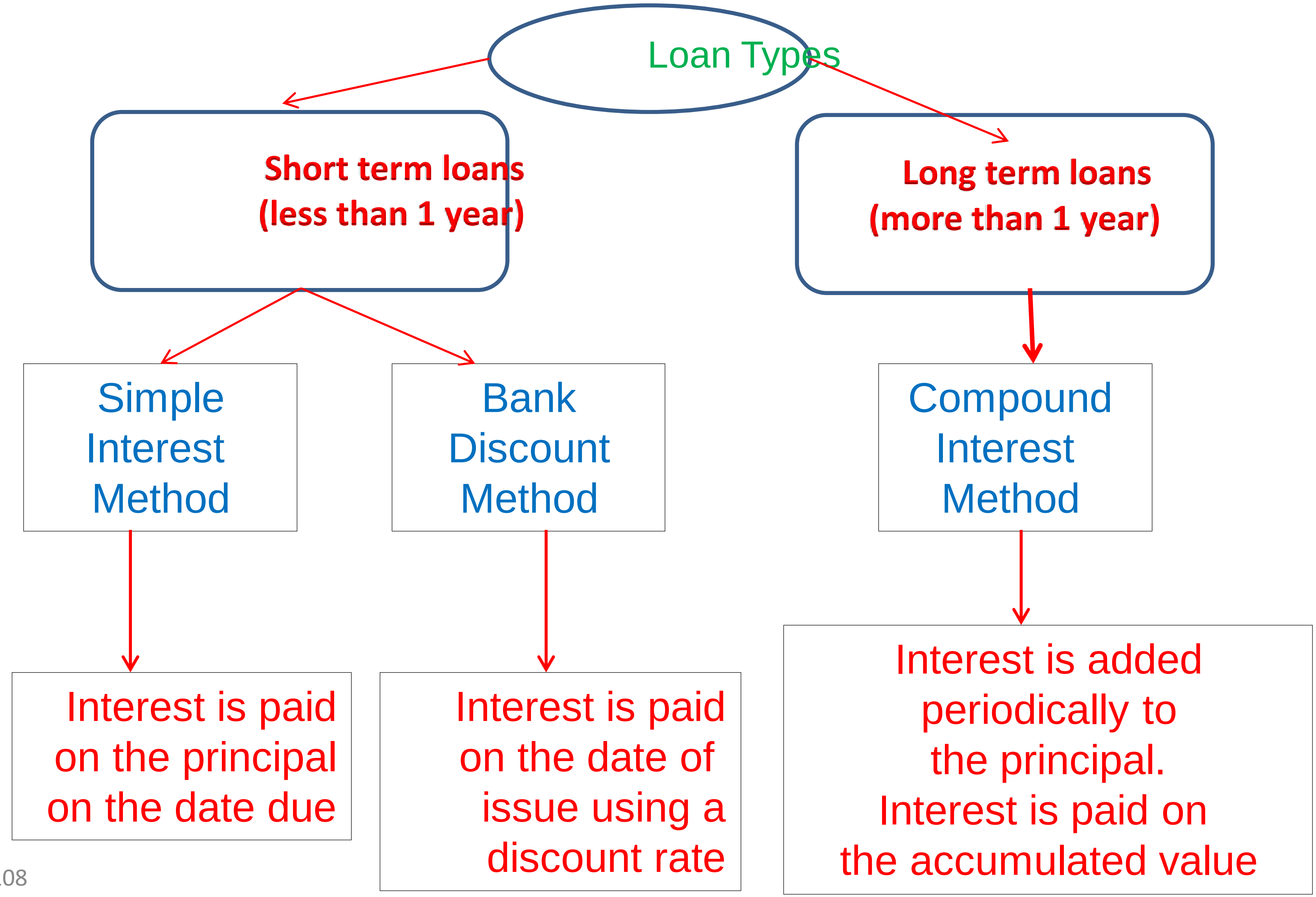


# Time value of Money

# Time value of Money Fundamental Concepts

- Principal: The amount borrowed or invested.
- Interest rate: A percentage of the outstanding principle.
- Time: The number of years or fractional portion of a year that principal is outstanding.
- A present value is the discounted value of one or more future cash flows.
- A future value is the compounded value of a present value.
- The discount factor is the present value of one riyal invested in the future.
- The compounding factor is the future value of one riyal invested today.

# Time value of Money



# The Simple Interest

**Definition1:** An interest amount in each period is computed based on a principal sum in the period.

**Interest = Principal × Interest Rate × number of periods**

$$I = PV \times i \times n$$

**Definition2:** The future value is the sum of present value and the interest amount.

**Future Value = Present Value + Interest**

$$FV_n = PV + I$$

$$FV_n = PV(1 + i \times n)$$

# Formulas of simple interest method

$$I = PV \times i \times n$$

$$i =$$

$$n =$$

$$PV =$$

$$FV_n = PV(1 + i \times n)$$

$$PV =$$

# The Simple Interest

## More Examples

Example1:

How much money would you pay in interest if you borrowed $1600 for 1 year at 16%  simple interest per annum?

Solution:

# The Simple Interest

## More Examples

Example2:

How much money would you pay in interest if you borrowed $16000 for 6 months at 12% simple interest per annum?

Solution:

# The Simple Interest

## More Examples

Example3:

How much money would you pay in interest if you borrowed $16000 for 9 months at 5% quarterly simple interest?

Solution:

# The Simple Interest

## More Examples

Example4:

You take a 40000 DA loan for 127 days. Annual simple interest rate is 12%. Calculate:

a) The interest

b) The amount that he must pay on the date due?

Solution:

a)

b)

# The Simple Interest

## More Examples

Example5:

When invested at an annual interest rate of 6% an account earned $180 of simple interest in one year. How much money was originally invested in account?

Solution:

# The Simple Interest

## More Examples

Example6:

If an investment of $7000 accumulate $910 of interest in the account after 1 year, what was the annual simple interest rate on the savings account?

Solution:

# The Simple Interest

## More Examples

Example7:

You put $600 today in an account that earns an annual simple interest rate of 8%. How many years will it take for this single investment of $600 to have a future value of $900?

Solution:

# The Simple Interest

## More Examples

Example8:

An investment earns 4.5% annual simple interest rate. If $2400 is invested, what is the total amount that can be withdrawn when the account is closed out after 2 months?

Solution:

# Nominal Interest Rates vs. Real Interest Rates

**State1:** Suppose we buy a 1 year bond for face value that pays 6% at the end of the year. We pay $100 at the beginning of the year and get $106 at the end of the year. Thus the bond pays an interest rate of 6%. This 6% is the nominal interest rate, as we have not accounted for inflation. Whenever people speak of the interest rate they're talking about the nominal interest rate, unless they state otherwise.

# Nominal Interest Rates vs. Real Interest Rates

**State2:** Now suppose the inflation rate is 3% for that year. We can buy a basket of goods today and it will cost $100, or we can buy that basket next year and it will cost $103. If we buy the bond with a 6% nominal interest rate for $100, sell it after a year and get $106, buy a basket of goods for $103, we will have $3 left over. So after factoring in inflation, our $100 bond will earn us $3 in income; a real interest rate of 3%. The relationship between the nominal interest rate, inflation, and the real interest rate is described by the Fisher Equation:



**Real Interest Rate = Nominal Interest Rate - Inflation**

# It's time to review

| Simple Interest | Compound interest |
|---|---|
| $I = PV \times i \times n$ | see Unit 9 |
| $FV_n = PV + I$ | see Unit 9 |
| $FV_n = PV(1 + i \times n)$ | see Unit 9 |
| More than one compounding periods per year | |
| See Unit 9 | |

**Real Interest Rate = Nominal Interest Rate - Inflation**

# we will see in the next unit

- ✓ The compound interest rate and the interest amount
- ✓ How to Calculate the future value of a single sum of money invested today for several periods.
- ✓ How to Calculate the interest rate or the number of periods or the principal that achieve a fixed future value.

| | |
|---|---|
| **Course** | **Financial Mathematics** |
| **Unit course** | **FIN 118** |
| **Number Unit** | **7** |
| **Unit Subject** | **Compound Interest**<br>**Non annual Compound Interest**<br>**Continuous Compound Interest** |

# !!! remember what we saw last time

- ✓ The relationship between time and money.
- ✓ The simple interest rate and the interest amount
- ✓ The present value of one future cash flow
- ✓ The future value of an amount borrowed or invested.
- ✓ The relationship between Real Interest Rate, Nominal Interest Rate and Inflation.

# we will see in this unit

- ✓ The compound interest rate and the interest amount
- ✓ How to Calculate the future value of <u>a single sum of money</u> invested today for several periods.
- ✓ How to Calculate the interest rate or the number of periods or the principal that achieve a fixed future value.

# LEARNING OUTCOMES

At the end of this unit, you should be able to:

1. Understand compound interest, including accumulating, discounting and making comparisons using the effective interest rate.

2. Distinguish between compound interest.

3. Identify variables fundamental to solving interest problems.

4. Solve problems including future and present value.

# The Compound Interest

**Definition1:** In each subsequent period, the interest amount computed is used to form a new principal sum, which is used to compute the next interest due.

* As we said, Compound Interest uses the Sum of Principal & Interest as a base on which to calculate new Interest and new Principal !

$$FV_1 = PV(1+i_1) \qquad \textit{one period}$$

$$FV_2 = FV_1(1+i_2) = PV(1+i_1)(1+i_2) \qquad \textit{two periods}$$

$$FV_3 = FV_2(1+i_3) = PV(1+i_1)(1+i_2)(1+i_3) \qquad \textit{three periods}$$

$$\ldots$$

$$FV_n = PV(1+i_1)(1+i_2)(1+i_3)\cdots(1+i_n) \qquad \textit{n periods}$$

# The Compound Interest

**Definition2:** If the interest rate is constant over different periods. we have: $i_1 = i_2 = i_3 = \dots = i_n = i$ and

**$FV_n$ = Principal ×(1+ Interest Rate)$^{\text{number of Periods}}$**

$$FV_n = PV(1+i)^n$$

$PV =$

$i =$

$n =$

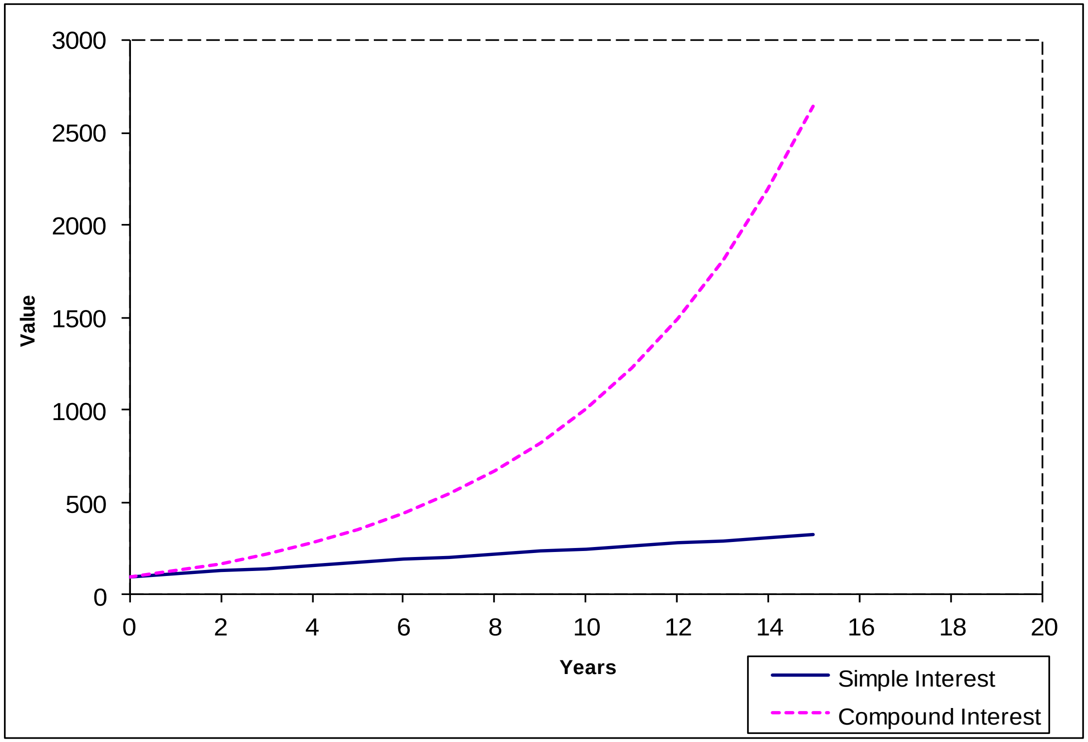
$$FV_n = PV + I$$

!!! Remember

# The Compound Interest

**Property1:**

The compound interest rate is a geometric sequences but the simple interest is an arithmetic sequences.




Simple interest: Linear growth  i = 0.15

Compound interest:  Geometric growth   i = 0.15

# The Compound Interest

## More Examples

**Example1:**

How much money would you pay in interest if you borrowed $1600 for 3 years at 16% compound interest per annum?

Solution:

# The Compound Interest

## More Examples

**Example2:**

What is the present value of $150000 to be received 5 years from today if the discount rate (annual compounded interest) is 10%?

Solution:

# The Compound Interest

## More Examples

**Example3:**

Assume that the initial amount to invest is PV = $100 and the interest rate is constant over time. What is the compound interest rate in order to have $150 after 5 years?

Solution:

# The Compound Interest

## More Examples

**Example4:**

Find the number of periods to double your investment at 6% compound interest per annum .

Solution:

**Question 1 ?** How to calculate the FV if we have more than one compounding periods per year ?

**Response:**

The table shows some common compounding periods and how many times per year interest is paid for them.

| Compounding Periods | Times per year (t) |
|---|---|
| Annually | 1 |
| Semi-annually | 2 |
| Quarterly | 4 |
| Monthly | 12 |

$$FV_{n,t} = PV\left(1+\frac{i}{t}\right)^{n\times t}$$

And

!!!

If t=1 we retrieve the old formula

# Non annual Compound Interest

**Example1 :**

You invested $1800 in a savings account that pays 4.5% interest p.a. compounded semi-annually. Find the value of the investment in 12 years.

Solution:

# Non annual Compound Interest

**Example2:**

You invested $3700 in a savings account that pays 2.5% interest p.a. compounded quarterly. Find the value of the investment in 10 years.

Solution:

# Non annual Compound Interest

**Example3:**

You invested $1700 in a savings account that pays 1.5% interest p.a. compounded monthly. Find the value of the investment in 15 years.

Solution:

# Non annual Compound Interest

**Example4:**

You expect to need $1500 in 3 years. Your bank offers 4% interest p.a. compounded semiannually. How much money must you put in the bank today (PV) to reach your goal in 3 years?

Solution:

# Non annual Compound Interest

**Example5:**

Suppose a bank quotes nominal annual interest rates on five-years of:

- 6.6% compounded annually,
- 6.5% p.a. compounded semi-annually, and
- 6.4% p.a. compounded monthly.

Which rate should an investor choose for an investment of $10000?

Solution:

# Non annual Compound Interest

**Solution : continued**

First proposition:

Second proposition:

Third proposition:

**Question 2 ?** What would happen to our money if we compounded a really large number of times?

Response:

We would have to compound not just every hour, or every minute or every second but for every millisecond. We have:

$$FV_{n,t} = PV\left(1+\frac{i}{t}\right)^{n\times t} \xrightarrow{t\to\infty} PV \times e^{n\times i}$$

- Then with Continuous compounding interest we have:

$$\boxed{FV_{n,\infty} = PV \times e^{n\times i}}$$

# Continuous Compound Interest

**Example1:**

If you invest $1000 at an annual interest rate of 5% p.a. compounded continuously, calculate the final amount you will have in the account after five years.

Solution:

# Continuous Compound Interest

**Example2:**

How long will it take an investment of \$10000 to grow to \$15000 if it is invested at 9% p.a. compounded continuously?

Solution:

# Continuous Compound Interest

**Example3:**

What is the interest rate compounded continuously of an investment of $1000 to grow to $2000 if it is invested for 7 years?

Solution:

# Compound Interest

**Example4:**

What amount will an account have after 5 years if $100 is invested at an annual nominal rate of 8% compounded annually? Semiannually? continuously?

Solution:

Compounded annually:

Compounded semi-annually:

Compounded continuously:

# Effective Interest Rate

- When analyzing a loan or an investment, it can be difficult to get a clear picture of the loan's true cost or the investment's true yield.
- The effective interest rate attempts to describe the full cost of borrowing. It takes into account the effect of compounding interest, which is left out of the nominal or "stated" interest rate.
- For example, a loan with 10% interest compounded monthly will actually carry an interest rate higher than 10%, because more interest is accumulated each month.
- The effective interest rate calculation does not take into account one-time fees like loan origination fees. These fees are considered, however, in the calculation of the annual percentage rate.

# Effective Interest Rate

- The compounding periods will generally be monthly, quarterly, annually, or continuously. This refers to how often the interest is applied.

- The effective interest rate is calculated through a simple formula:

$$r = \left(1 + \frac{i}{t}\right)^t - 1$$

r: the effective interest rate,

i: the stated interest rate,

t: the number of compounding periods per year.

# Effective Interest Rate

Example 1:

Find the effective annual rate for a stated rate of 7.5% per year compounded quarterly.

Solution:

# Effective Interest Rate

Example 2:

Determine the effective rate on the basis of the compounding period for each interest rate.

a) 9% per year, compounded yearly
b) 6% per year, compounded quarterly
c) 8% per year, compounded monthly
d) 5% per year, compounded weekly

Solution:

| Nominal interest rate (i) | Compounding period | t | Effective interest rate (r) |
|---|---|---|---|
| 9% per year | Year | | |
| 6% per year | Quarter | | |
| 8% per year | Month | | |
| 5% per year | Week | | |

# It's time to review

| Simple Interest | Compound interest |
|---|---|
| $I = PV \times i \times n$ | |
| $FV_n = PV + I$ | $FV_n = PV + I$ |
| $FV_n = PV(1 + i \times n)$ | $FV_n = PV(1+i)^n$ |
| More than one compounding periods per year $FV_{n,t} = PV\left(1 + \frac{i}{t}\right)^{n \times t}$ | Continuous Compound Interest $FV_n = PV \times e^{n \times i}$ |

| | |
|---|---|
| **Effective Interest Rate** = $r = \left(1 + \frac{i}{t}\right)^t - 1$ | |
| **Real Interest Rate** = **Nominal Interest Rate** - **Inflation** | |

# we will see in the next unit

- ✓ Meant of simple Annuity

- ✓ Simple Annuity: Ordinary Annuity, Annuity Due (unit8)

| | |
|---|---|
| **Course** | **Financial Mathematics** |
| **Unit course** | **FIN 118** |
| **Number Unit** | **8** |
| **Unit Subject** | **Ordinary Annuity**<br>**Annuity due** |

# we will see in this unit

- ✓ Meant of simple Annuity
- ✓ Ordinary Annuity
- ✓ Annuity Due
- ✓ How to Calculate present and future values of each type of annuity.

# Learning Outcomes

At the end of this unit, you should be able to:

1. Distinguish between an ordinary annuity and an annuity due.

2. Calculate present and future values of each type of annuity.

3. Apply knowledge of annuities to solve a range of problems, including problems involving principal-and-interest loan contracts.

# Introduction

In unit 7, we have seen that a single sum of money invested today for several periods will produce a higher future sum due to compounding effect. In this unit we attempt to see that the same phenomenon will occur for multiple stream of cash flow.

A multiple stream of cash flow that is made in an equal size and at a regular interval is known as simple annuity.

It exists four different forms for Simple annuity: Ordinary Annuity, Annuity Due (unit8), Deferred Annuity, and Perpetuity.

# Ordinary Annuity

**Definition:** Ordinary Annuity is a series of equal cash payments or deposits made at the end of each compounding period.

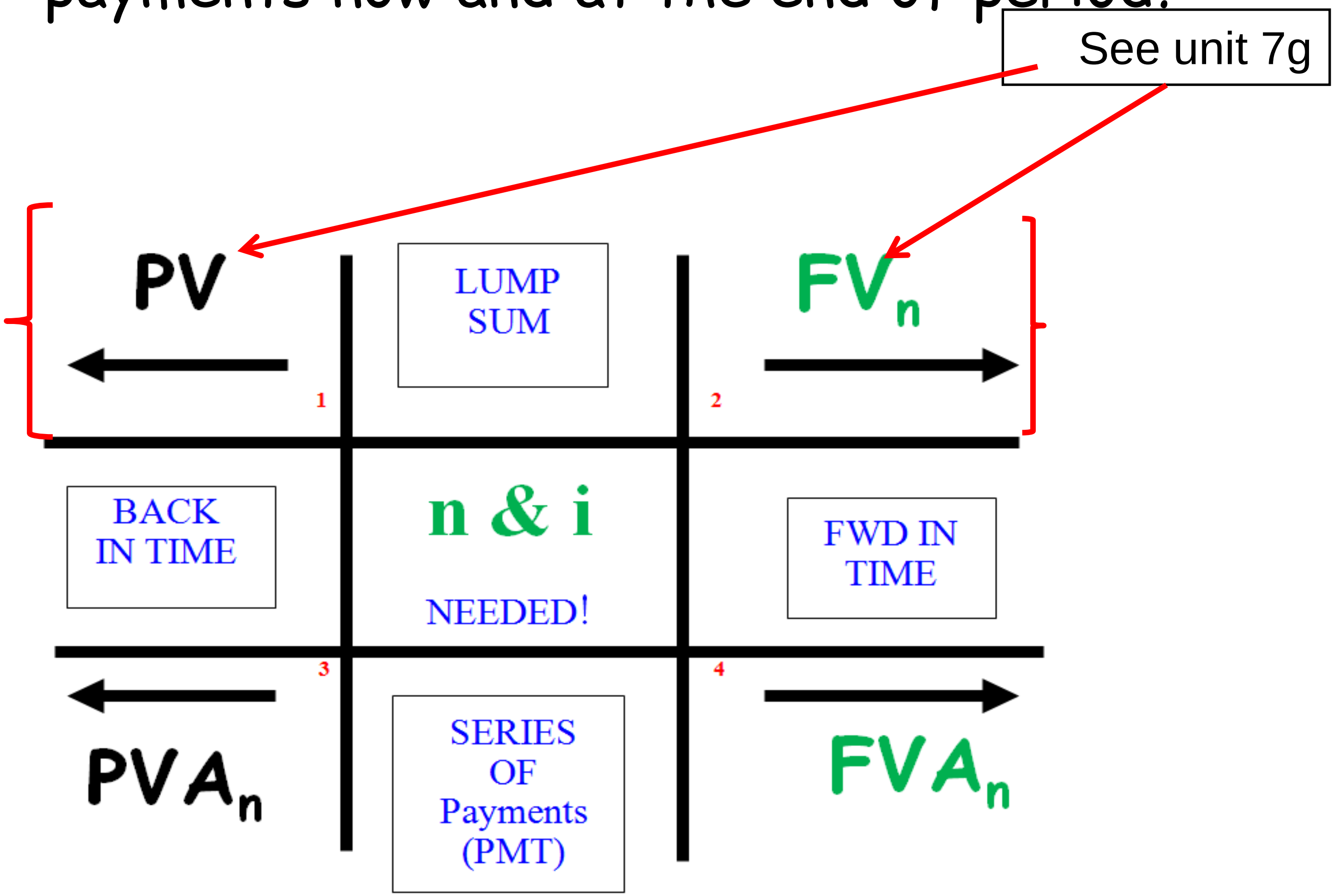
**Examples :**

i/ When a particular individual buy a bond, he will receive equal semi-annual coupon interest payments over the life of the bond.

ii/ When a particular individual borrow money to buy for example a house or a car, he will pay a stream of equal payments at the end of each period of coverage.

# Ordinary Annuity

Question: what is value of the sum of all payments now and at the end of period?



# Future Value Annuity $FVA_n$

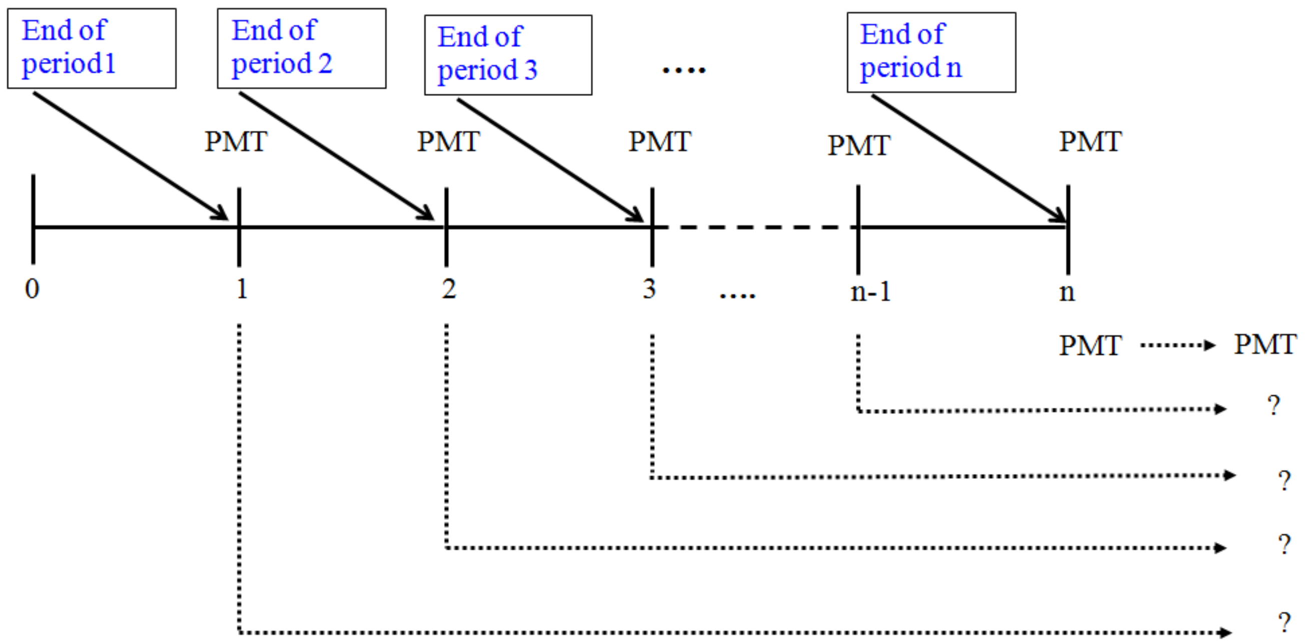
The future value annuity of an ordinary annuity is the sum of all regular equal payments and the compounded interest accumulated at the end of last period. It is determined as follow:

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

- PMT: annuity payment deposited or received at the end of each period,
- i : interest rate per period,
- n : number of payments.

# Future Value Annuity $FVA_n$

**Proof :**




$$FVA_n = PMT + PMT(1+i) + PMT(1+i)^2 + \cdots + PMT(1+i)^{n-1}$$

$$= PMT \times \left[1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{n-1}\right]$$

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

# Future Value Annuity

## More Examples

Example 1:

Suppose you plan to deposit $1000 annually into an account at the end of each of the next 7 years. If the account pays 12% annually, what is the value of the account at the end of 7 years?

Solution :

# Future Value Annuity

Example 2

What is the future value of $5000 invested at the end of each year for 10 years if money earns 6% per annum?

Solution

# Present Value Annuity $PVA_n$

The present value annuity of an ordinary annuity is the sum of all regular equal payments discounted at a certain interest rate in at the end of each period. It is determined as follow:

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]$$

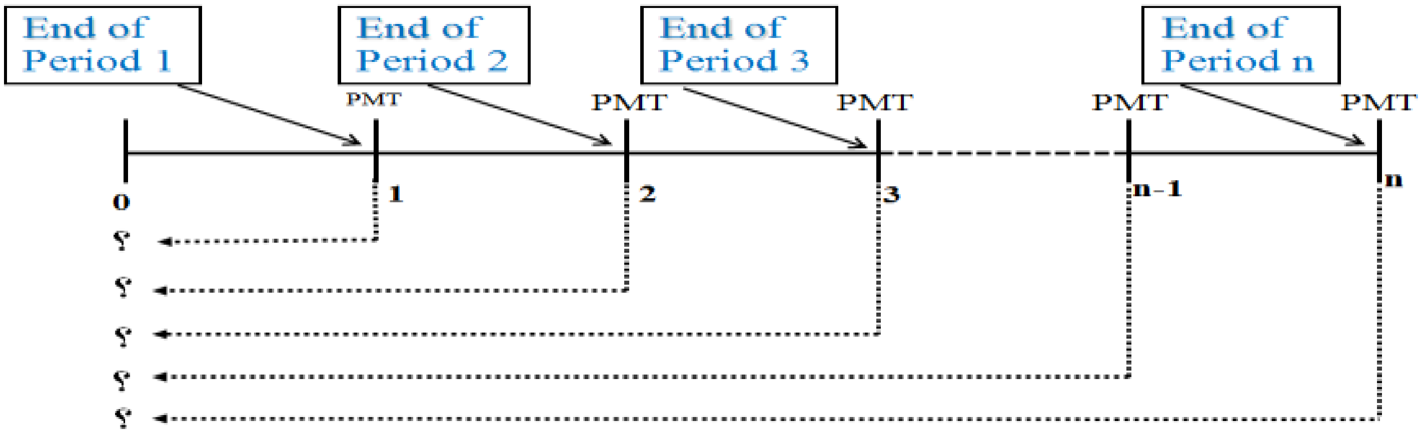
PMT: annuity payment deposited or received at the end of each period,

i : interest rate per period,

n : number of payments.

# Present Value Annuity $PVA_n$

**Proof :**




$$PVA_n = \frac{PMT}{1+i} + \frac{PMT}{(1+i)^2} + \cdots + \frac{PMT}{(1+i)^n}$$

$$= \frac{PMT}{1+i} \times \left[ 1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{n-1}} \right] =$$

$$= PMT(1+i)^{-1} \times \left[ 1 + (1+i)^{-1} + (1+i)^{-2} + \ldots + (1+i)^{-(n-1)} \right]$$

$$= PMT(1+i)^{-1} \left[ \frac{1-(1+i)^{-n}}{1-(1+i)^{-1}} \right] = PMT \left[ \frac{1-(1+i)^{-n}}{i} \right]$$

# Present Value Annuity

## More Examples

Example 1:

You plan to withdraw $1000 annually from an account at the end of each of the next 7 years. If the account pays 12% annually, what must you deposit in the account today?

Solution :

# Present Value Annuity

## More Examples

Example 2:

What is the present value of $5000 that will be invested at the end of each year for 10 years if money earns 6% per annum?

# Annuity Due

**Definition:** Annuity Due is a series of equal cash payments or deposits made at the beginning of each compounding period.

**Examples :**

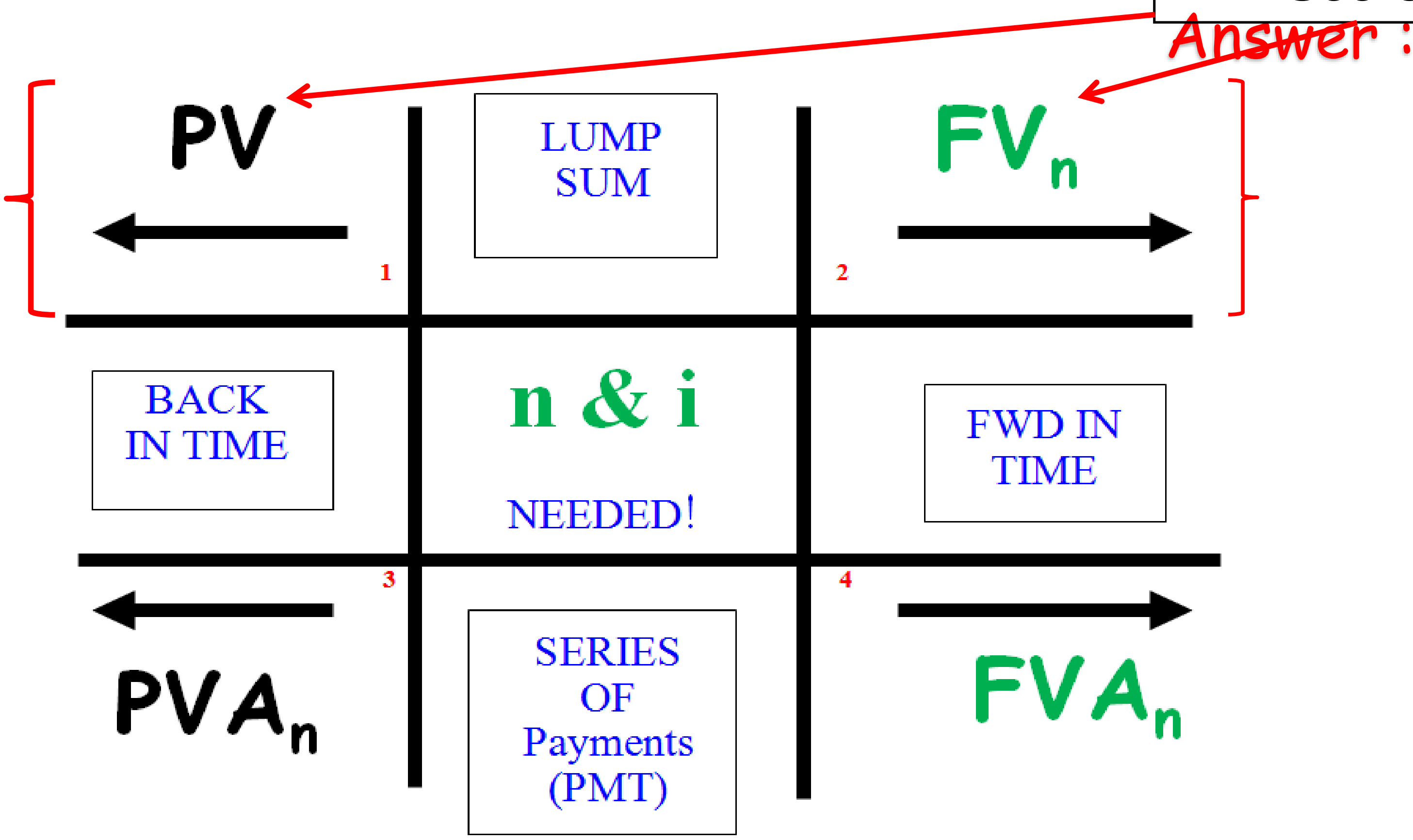
i/ When a particular individual make an apartment lease contract over a period of several years, he must paid at the beginning of each year an annual rent .

ii/ When a particular individual buy a car he must paid at the beginning of each year an annual insurance premium.

# Annuity Due

Question: what is value of the sum of all payments now and at the end of period?

See Unit 7



# Future Value Annuity $FVA_n$

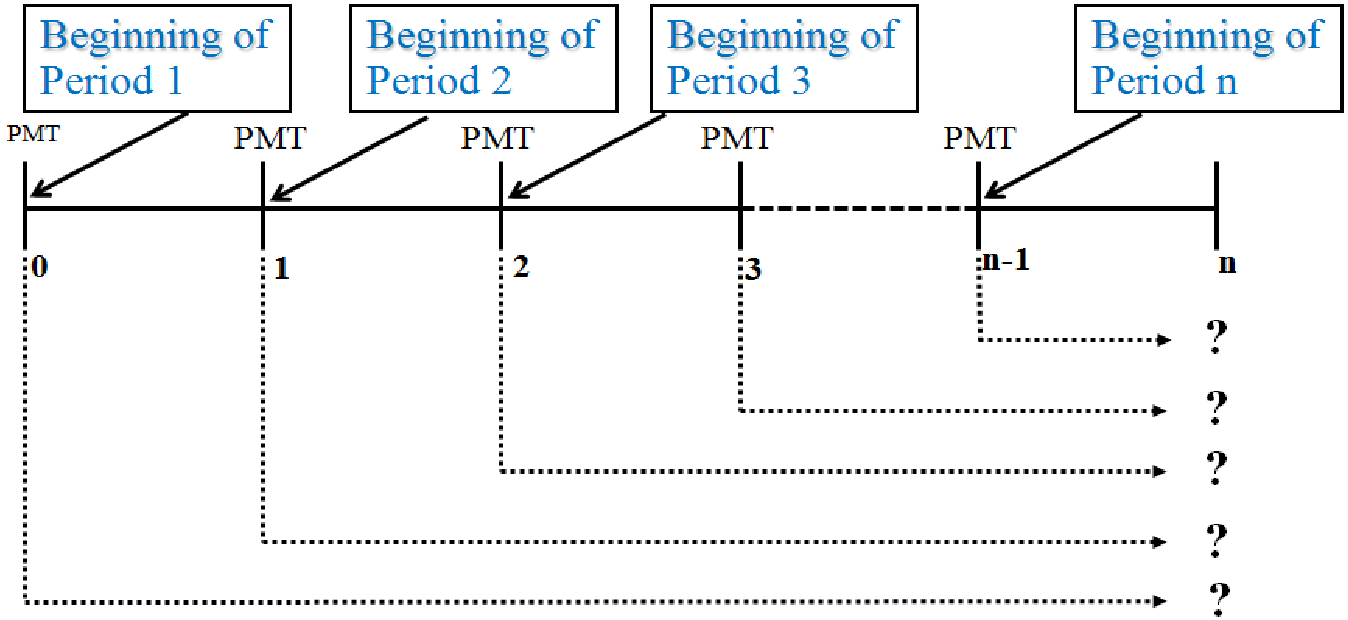
The future value annuity of an annuity due is the sum of all regular equal payments at the beginning of each period and the compounded interest accumulated at the end of last period. It is determined as follow:

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

- PMT: annuity payment deposited or received at the beginning of each period,
- i : interest rate per period,
- n : number of payments.

# Future Value Annuity $FVA_n$

**Proof :**




$$FVA_n = PMT(1+i) + PMT(1+i)^2 + PMT(1+i)^3 \cdots + PMT(1+i)^n$$

$$= PMT(1+i) \times \left[1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{n-1}\right]$$

$$FVA_n = PMT(1+i)\left[\frac{(1+i)^n - 1}{i}\right] = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

# Future Value Annuity

## More Examples

Example 1:

Suppose you plan to deposit $1000 annually into an account at the beginning of each of the next 7 years. If the account pays 12% annually, what is the value of the account at the end of 7 years?

Solution :

# Future Value Annuity

## More Examples

Example 2:

What is the future value of $5000 invested at the beginning of each year for 10 years if money earns 6% per annum?

# Present Value Annuity $PVA_n$

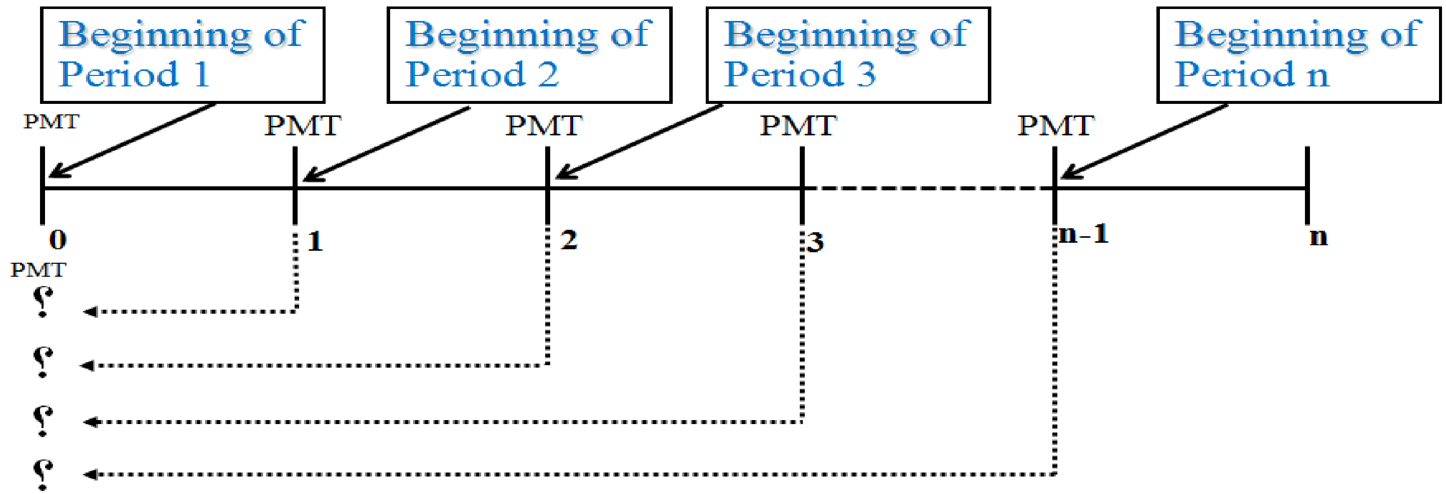
The present value annuity of an annuity due is the sum of all regular equal payments discounted at a certain interest rate in at the beginning of each period. It is determined as follow:

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

* PMT: annuity payment deposited or received at the <u>beginning</u> of each period,
* i : interest rate per period,
* n : number of payments.

# Present Value Annuity $PVA_n$

**Proof :**




$$PVA_n = PMT + \frac{PMT}{1+i} + \frac{PMT}{(1+i)^2} + \cdots + \frac{PMT}{(1+i)^{n-1}}$$

$$= PMT \times \left[1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{n-1}}\right] =$$

$$= PMT \times \left[1 + (1+i)^{-1} + (1+i)^{-2} + \ldots + (1+i)^{-(n-1)}\right]$$

$$= PMT\left[\frac{1-(1+i)^{-n}}{1-(1+i)^{-1}}\right] = PMT\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

# Present Value Annuity

## More Examples

Example 1:

You plan to withdraw $1000 annually from an account at the beginning of each of the next 7 years. If the account pays 12% annually, what must you deposit in the account today?

Solution :

# Present Value Annuity

## More Examples

Example 2:

What is the present value of $5000 that will be invested at the beginning of each year for 10 years if money earns 6% per annum?

Solution:

# Formulas

Time to Review!

**Simple Annuity**

**Ordinary Annuity**

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]$$

**Annuity Due**

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

**Deferred Annuity**

**Perpetuity**

**That's All for 50% of simple Annuity !**

# we will see in the next unit

- ✓ Long term ordinary annuity
- ✓ Long term annuity due
- ✓ Amortization & sinking Funds
- ✓ Some real life examples

| | |
|---|---|
| **Course** | **Financial Mathematics** |
| **Unit course** | **FIN 118** |
| **Number Unit** | **9** |
| **Unit Subject** | **Long Term Annuities<br>Amortization & sinking Funds** |

# we will see in this unit

- ✓ Long term ordinary annuity
- ✓ Long term annuity due
- ✓ Amortization & sinking Funds
- ✓ Some real life examples

# LEARNING OUTCOMES

At the end of this unit, you should be able to:

1. Understand what is meant by "long term annuities".

2. Calculate Present and future value long term annuities for the case of ordinary and annuity due.

3. Calculate the single payment for real life examples in the case of long term annuities ( Amortization & sinking Funds).

# Introduction

- As we say in unit 8, Annuity is a series of equal cash payments or deposits. These regular equal payments can be planned for short term, intermediate or long term periods.
- In this unit we are interested to the calculation of long term payments or deposits.
- Two basic questions can be exposed with annuities:
  - Calculate how much money will be accumulated if we consider an annuity plan for long time (30 years for example)
  - How to calculate the periodic payments or deposits in order to obtain a specific amount on a given time period (calculate monthly payments for a mortgage loan for example).

# Long Term Ordinary Annuity

- As we have seen in unit 8, the same formula is applied to the case of long term annuities.
- The future and present value of an ordinary annuity are given respectively by

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]$$

if payments are made annually.

- If payments are made non annually (more than once in year) we must introduce in the previous formula the number of times per year. So, we have the following formula

$$FVA_{n,t} = PMT\left[\frac{\left(1+\frac{i}{t}\right)^{n.t} - 1}{\frac{i}{t}}\right]$$

$$PVA_{n,t} = PMT\left[\frac{1-\left(1+\frac{i}{t}\right)^{-n.t}}{\frac{i}{t}}\right]$$

# Long Term Ordinary Annuity

## Example1

You plan to deposit in an account $5000 at the end of each year for the next 25 years. If the account pays 6% annually, what is the value of your deposits at the end of 25 years?

## Solution

# Long term Ordinary Annuity

## Example2

You plan to deposit, in an account, $100 at the end of each month for the next 25 years. If the account pays 6% annually, what is the value of your deposits at the end of 25 years?

## Solution

# Long Term Ordinary Annuity

**Example3**

You plan to withdraw at the end of each year from an account $5000 for the next 25 years. If the account pays 6% annually, what is the amount that you must deposit today in order to guarantee these withdrawals ?

**Solution**

# Long term Ordinary Annuity

**Example4**

You plan to withdraw at the end of each month $100 from an account of the next 25 years. If the account pays 6% annually, what is the amount that you must deposit today in order to guarantee these withdrawals?

**Solution**

# Long Term Annuity Due

- As we have seen in unit 8, the same formula is applied to the case of long term annuities.
- The future and present value of an annuity due are given respectively by

$$FVA_n = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

$$PVA_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)$$

if payments are made annually.

- If payments are made non annually (more than once in year) we must introduce in the previous formula the number of time per year. So, we have the following formula

$$FVA_{n,t} = PMT\left[\frac{\left(1+\frac{i}{t}\right)^{n.t} - 1}{\frac{i}{t}}\right]\left(1+\frac{i}{t}\right)$$

$$PVA_{n,t} = PMT\left[\frac{1-\left(1+\frac{i}{t}\right)^{-n.t}}{\frac{i}{t}}\right]\left(1+\frac{i}{t}\right)$$

# Long Term Annuity Due

**Example1**

You plan to deposit in an account $5000 at the beginning of each year for the next 25 years. If the account pays 6% annually, what is the value of your deposits at the end of 25 years?

**Solution**

# Long term Annuity Due

**Example2**

You plan to deposit in an account \$100 at the beginning of each month for the next 25 years. If the account pays 6% annually, what is the value of your deposits at the end of 25 years?

**Solution**

# Long Term Annuity Due

**Example3**

You plan to withdraw from an account $5000 at the beginning of each year for the next 25 years. If the account pays 6% annually, what is the amount that you must deposit today in order to guarantee these withdrawals?

**Solution**

# Long term Annuity Due

**Example4**

You plan to withdraw from an account $100 at the beginning of each month for the next 25 years. If the account pays 6% annually, what is the amount that you must deposit today in order to guarantee these withdrawals?

**Solution**

# Amortization &Sinking Funds

## إطفاء/اهلاك القروض أو استـهلاك القروض

## صندوق إطفاء الدين أو اِحْتِيَاطِيّ سداد قرض

- When a lender pays a debt (including interest) by making periodic payments at regular intervals, the debt is said to be *amortized*.

- When a payment is made to an investment fund each period at a fixed interest rate to yield a predetermined future value, the payment is called a sinking fund.

# Amortization &Sinking Funds

- Ordinary Amortization Formula

| Annual | Non-annual |
|---|---|
| $PMT = PVA\left(\frac{i}{1-(1+i)^{-n}}\right)$ | $PMT = PVA\left(\frac{\frac{i}{t}}{1-\left(1+\frac{i}{t}\right)^{-n\times t}}\right)$ |

- Ordinary Sinking Fund Payment

| Annual | Non-annual |
|---|---|
| $PMT = FVA\left[\frac{i}{(1+i)^{n}-1}\right]$ | $PMT = FVA\left[\frac{\frac{i}{t}}{\left(1+\frac{i}{t}\right)^{n\times t}-1}\right]$ |

# Real life example: Loan Amortization

**Example**

Suppose you want to borrow money to buy a house. You are considering a 7-years or a 25-years loan.
A bank offers different interest rates, reflecting the differences in risks of intermediate-term and long-term lending.
* For the 7-years loan, the annual interest rate is 5.25% compounded monthly.

* For the 25-years loan, the annual interest rate is 6.75% compounded monthly.

→If you borrow $150000, what would be your monthly payments for each type of loan?

# Real life example: Loan Amortization

**Solution**

**First Scenario: 7-years loan**

**Second Scenario: 25-years loan**

# Real life example: Sinking Fund

**Example 1**

Suppose you decide to use a sinking fund to save \$150 000 for a house. If you plan to make 300 monthly payments (25 years × 12=300) and you receive 6.75% interest per annum, what is the required payment for an ordinary annuity?

**Solution**

# Real life example: Sinking Fund

**Example 2**

Suppose you use a sinking fund to save $50 000 for a car. If you plan on 60 monthly payments (5 years $\times$12= 60) and you receive 5% interest per annum, what is the required payment for an ordinary annuity?

**Solution**

# Time to Review !

- ✓Long term annuities is an extension to the ordinary and annuity due.

- ✓Making periodic payments to repay a debt, including the principal and interest, is called amortization.

- ✓A fund into which periodic payments necessary to realise a given sum of money in the future are made is called sinking fund.

# we will see in the next unit

- ✓ What's a bond
- ✓ Different types of bonds
- ✓ Bond valuation

| | |
|---|---|
| **Course** | **Financial Mathematics** |
| **Unit course** | **FIN 118** |
| **Number Unit** | **10** |
| **Unit Subject** | **Bond Valuation** |

# we will see in this unit

- ✓ What's a bond
- ✓ Different types of bonds
- ✓ Bond valuation

# LEARNING OUTCOMES

At the end of this unit, you should be able to:

1. Understand what is meant by "Bond".
2. Know the different types of bonds.
3. Calculate the value of a bond.

# What is a Bond

**Definition:**

- A debt instrument: When one purchases a bond, one essentially lends an organization such as the government or a corporation a specified amount of money which the borrower agrees to repay at a designated time.

- A promise to pay interest over a specific term at stated future dates and then pay lump sum at maturity (the end of the term).

- Issued by corporations and governments as a way to provide money to the company.

# Components of a bond

- Principal or Face value of the bond: The amount of money that is paid to the bondholders at maturity. For most bonds this amount is $1,000 (and its doubles). It also generally represents the amount of money borrowed by the bond issuer.

- Coupon rate or rate of interest : It is expressed as a percentage of the bond's face value. It also represents the interest cost of the bond to the issuer.

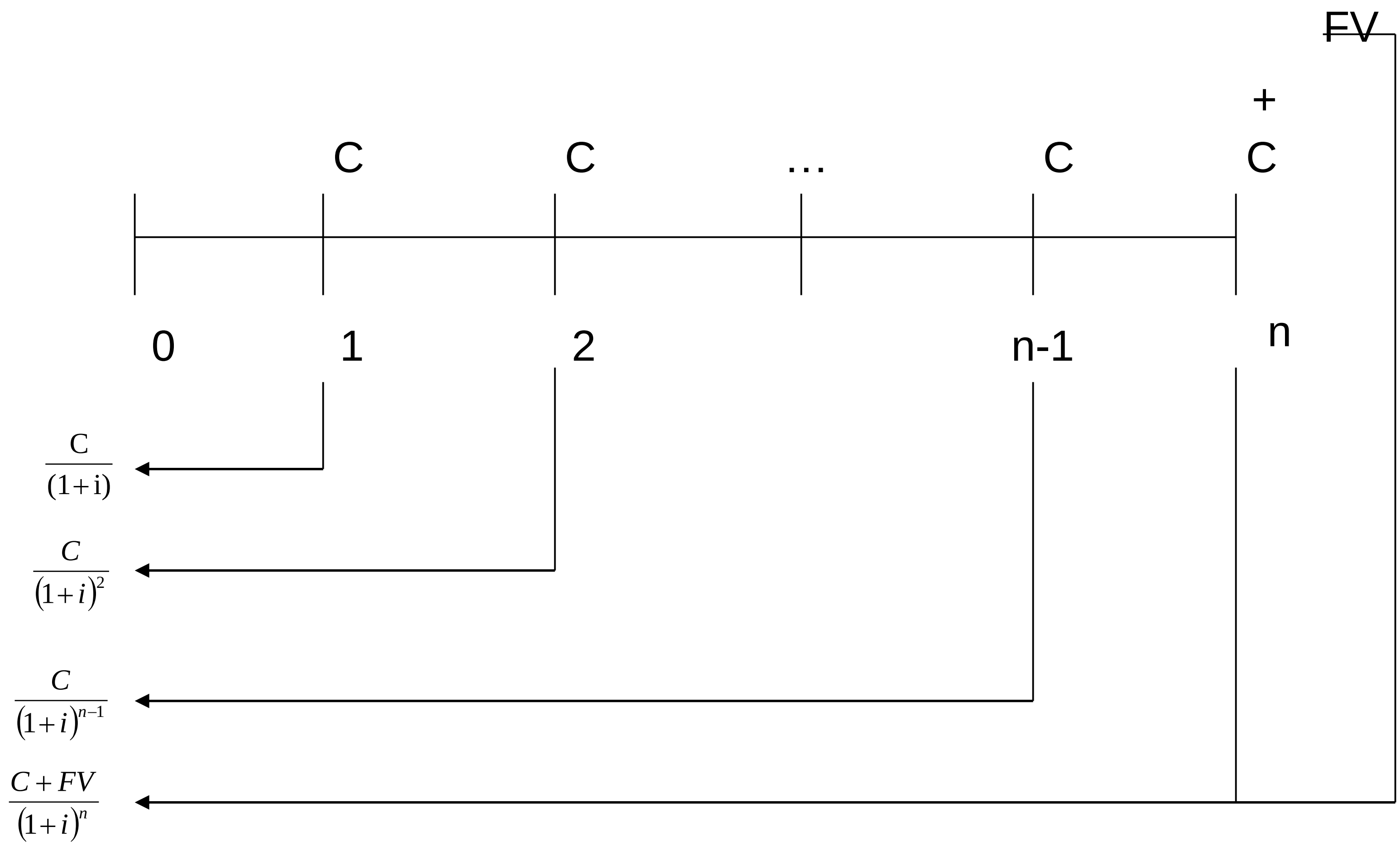
# Components of a bond

- Coupon Payments: It represent the periodic interest payments from the bond issuer to the bondholder. The annual coupon payment is calculated by multiplying the annual coupon rate by the bond's face value. Since most bonds pay interest semiannually, generally one half of the annual coupon is paid to the bondholders every six months.

- Maturity date: It represents the date on which the bond matures, i.e., the date on which the face value is repaid. The last coupon payment is also paid on the maturity date.

# Bond Valuation

Time line of payments



# Types of Bonds

- Government Bonds: or Treasury Bonds, a debt security issued by a government to support government spending.
- Corporate Bonds: a debt security issued by companies and sold to investors in order to finance expansion or raise funds for other expenses. Interest rates accorded to this type of bonds is higher than government bonds.
- Municipal Bonds: a debt security issued by a state, a municipality or a county in order to finance its projects or expenditures. Municipal bonds may be general obligations of the issuer or secured by specified revenues.

# Bond Valuation

- Bonds are valued using time value of money concepts.
- Their coupon, or interest, payments are treated like an equal cash flow stream (annuity).
- Their face value is treated like a lump sum.

$$PV(Bond) = \sum_{t=1}^{n} \frac{Coupon}{(1+i)^t} + \frac{Face\,Value\,of\;\,Bond}{(1+i)^n}$$

$$PV(bond) = Coupon \times \left[ \frac{1-(1+i)^{-n}}{i} \right] + \frac{Face\,Value\,of\;\,bond}{(1+i)^n}$$

$$Coupon = Face\,Value \times Coupon\;\,rate$$

C = Coupon; r = annual coupon rate; i = annual interest rate;
FV = face value; n = number of years.

# Bond Valuation

## Example 1: Annual Coupon Payments

Your father bought a 10-year bond from Al-Nasr Corporation Europe Ltd. The bond has a face value of $1000 and pays an annual coupon. The annual coupon rate is equal to 10%. The current market rate of return is 12% (or the discount rate).

1- Find the value of the coupon.

2- Find the Price of the Bond (Market value of the bond to day).

# Bond Valuation

Solution:

1/ The value of the coupon

2/ The bond price is the Present value of the coupon stream plus the Present Value of the Face Value.

# Bond Valuation
# Non-annual Coupon Payments

- The rule for valuing annual bonds is easily extended to valuing bonds paying interest even more frequently (semi-annually, quarterly, monthly).

- For example, to determine the value of a bond paying interest semi- annually, we can remark that we have to pay the coupon two times a year. Then, to calculate the price of the bond, we must double the number of annual periods and the annual coupon payment and divide the coupon rate and the discount rate by two.

# Bond Valuation
# Non-annual Coupon Payments

- In general, if we let t be equal to the number of payments per year, n be equal to the maturity in years and i be the annual discount rate, then the general formula for valuing a bond can be expressed as follows:

$$PV(bond) = Coupon \times \left[ \frac{1 - \left(1 + \frac{i}{t}\right)^{-n \times t}}{\frac{i}{t}} \right] + \frac{Face\,Value\ of\ bond}{\left(1 + \frac{i}{t}\right)^{n \times t}}$$

$$Coupon = Face\,Value \times \frac{Coupon\ \ rate}{t}$$

C = Coupon; r = Annual coupon rate; i = interest rate; FV = face value; n = number of years; t = number of times in 1 year.

# Bond Valuation

## Example 2: Semi-Annual Coupon Payments

Your father bought a 10-year bond from Al-Nasr Corporation Europe Ltd. The bond has a face value of $1000 and pays a semi-annual coupon. The annual coupon rate is 10%. The current market rate of return is 12% (or the discount rate).

1- Find the value of the coupon.
2- Find the Bond price.

# Bond Valuation

Solution:

1/ The value of the coupon

2/ The bond price is the Present value of the coupon stream plus the Present Value of the Face Value.

# Relation between coupon rate and discount rate

First Relation:

three cases are possible:

- ✓ Coupon rate = discount rate

  The price of the bond equal to the Face value of the bond $\Rightarrow$ par bond

- ✓ Coupon rate $>$ discount rate

  The price of the bond is greater than the Face value of the bond $\Rightarrow$ premium bond

- ✓ Coupon rate $<$ discount rate

  The price of the bond is smaller than the Face value of the bond $\Rightarrow$ discount bond

# Relation between bond price and discount rate

Second Relation:

Two cases are possible:

- ✓ Rate of return increase $\Rightarrow$ The price of the bond decreases
- ✓ Rate of return decrease $\Rightarrow$ The price of the bond increases
- ✓ Inverse relation between bond price and discount rate

$$\frac{\Delta BP}{\Delta i} < 0$$

## Example 3:

Your father bought a 15-year bond from Al-Nasr Corporation Europe Ltd. The bond has a face value of $2000 and pays an annual coupon. The annual coupon rate is fixed at 10%.

1- Find the price of the bond if the current market rate of return is 8%. What we can conclude?

2- Find the price of the bond if the current market rate of return is 10%. What we can conclude?

3- Find the price of the bond if the current market rate of return is 12%. What we can conclude?

# Bond Valuation

**Solution:**

$$Coupon =$$

1- If the market interest rate is 8% (the discount rate), the market value of the bond is:

2- If the market interest rate is 10% (the discount rate), the market value of the bond is:

# Bond Valuation

**Solution:**

3- If the market interest rate is 12% (the discount rate), the market value of the bond is:

# Time to Review !

- ✓Bonds are debt instruments with maturity date.

- ✓ If the coupon rate is greater than the market rate, the market value of the bond is greater than the Face value of the bond.

- ✓ If the coupon rate is smaller than the market rate, the market value of the bond is smaller than the Face value of the bond.

- ✓ If the two rates are equal, market value is equal to the Face value of the bond.

# we will see in the next unit

- ✓ Meant of Equal short term payments and settlement of short-term debt
- ✓ How to Calculate the amount of total payments.
- ✓ How to Calculate the amount of a new settlement

| | |
|---|---|
| **Course** | **Financial Mathematics** |
| **Unit course** | **FIN 118** |
| **Number Unit** | **11** |
| **Unit Subject** | **Equal short term payments**<br>**Settlement of short-term debt** |

# we will see in this unit

- ✓ Meant of Equal short term payments and settlement of short-term debt
- ✓How to Calculate the amount of total payments.
- ✓How to Calculate the amount of a new settlement

# LEARNING OUTCOMES

At the end of this unit, you should be able to:

1. Apply the rule of an ordinary annuity and an annuity due to Equal short term payments and settlement of short-term debt

2. Calculate the amount of total payments and the amount of total settlement of short-term debt.

# Introduction

In unit 10, we have seen that a multiple stream of cash flow that is made in an equal size and at a regular interval is known as simple annuity. Therefore we have seen that exists four types of Simple annuity: Ordinary Annuity, Annuity Due (unit10), Deferred Annuity, and Perpetuity. So we have applied the compound interest to this type of annuity to calculate future or present value of the amount of all stream.

In this unit we attempt to apply the simple interest rather than compound interest for multiple stream of cash flow known as Equal short term payments. We also explain how to apply the settlement of short-term debt via some practical examples.

# 1/ Equal short term payments

**Definition:** Equal short term payments is a series of equal cash payments made at the end or at beginning of each simple period.

**Examples :**

i/ When you buy some house wares in monthly installment

ii/ When financing operational activities of Firm.

# Ordinary Payments / Payments due

Definition: Ordinary payments, called the reimbursement installments, which payments are to be paid at the end of each period of time, it has paid at the end of every month or every two months, or every 3 months or etc.

Definition: Payments Due, called the reimbursement installments, which payments are to be paid at the beginning of each period of time, it has paid at the beginning of every month or every two months, or every 3 months or etc.

# How to calculate Total of installments?

Generally, the total of installments is equal to the principal plus interest. It is determined as follow:

$$Total\ Installments = Principal + Interest$$

$$Total\ Installments = PMT \times n + PMT \times i \times T$$

$$Total\ Installments = PMT\left(n + i \times T\right)$$

- PMT: the installment to be paid at each period,
- i : interest rate per period,
- n : number of payments,
- T: the number of time periods.

# How to calculate Total of installments?

The number of time periods (T) is generally, function of the number of payments.

$$T = \frac{n}{2}\begin{pmatrix} Number\ of\ \ periods \text{ of the first payment until maturity} \\ + \\ Number\ of\ \ periods \text{ of the last payment until maturity} \end{pmatrix}$$

Example1: Suppose you plan to deposit $1000 at the end of each month into an account for one year. If the account pays a simple interest equal to 15% annually, what is the value of your deposits at the end of the year?

Solution:

# How to calculate Total of installments?

Example2: Suppose you plan to deposit $1000 at the beginning of each month into an account for one year. If the account pays a simple interest equal to 15% annually, what is the value of your deposits at the end of the year?

Solution:

# How to calculate Total of installments?

Example3: Suppose you plan to deposit $1000 in the middle of each month into an account for one year. If the account pays a simple interest equal to 15% annually, what is the value of your deposits at the end of the year?

Solution:

# How to calculate Total of installments?

Example4: Suppose you plan to deposit $1000 at the end of each month into an account for six months. If the account pays a simple interest equal to 15% annually, what is the value of your deposits at the end of the year?

Solution:

# How to calculate Total of installments?

Example5: Suppose you plan to deposit $1000 at the end of each two months into an account for one year and half. If the account pays a simple interest equal to 15% annually, what is the value of your deposits at the end of the period?

Solution:

# 2/ the settlement of short-term debt

**Definition:** The settlement of short-term debt is intended to an agreement between the debtor and the creditor to replace old debt by new debt. Therefore the agreement includes a method of replacement and general rule used is the equality between the value of old debt and the value of new debt at a specific date which is called the settlement date.

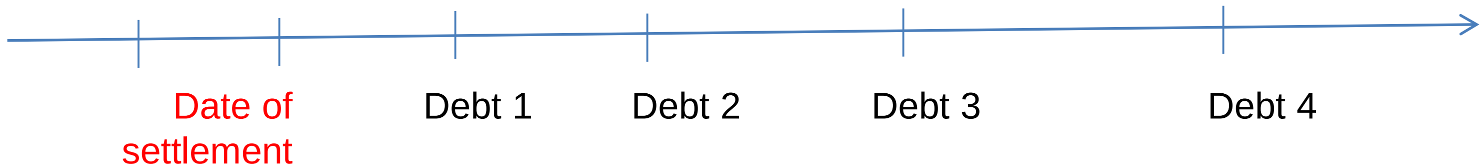
Rule :

| |
|---|
| The value of the old debt at the date of settlement |
| = |
| The value of the new debt at the date of settlement |

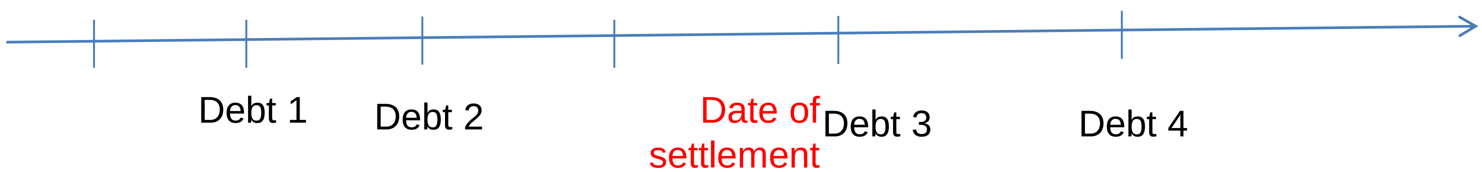
# How to calculate the value of New Debt ?

Generally, we have three cases:

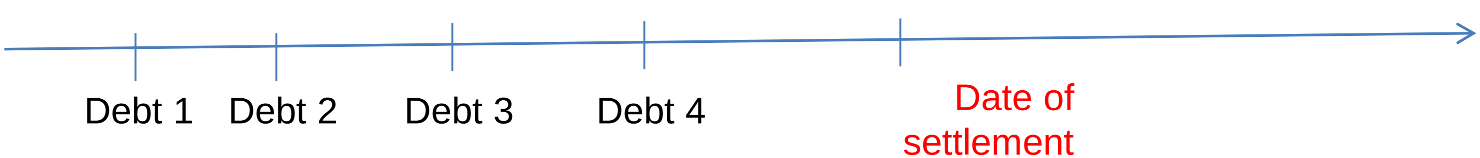
**Case 1 :** the settlement date is before all dates of maturity. So we must calculate the present value of each batch at the date of settlement.




**Case 2 :** the settlement date is before one date of maturity. So we must calculate the PV of each batch after the date of settlement and FV of each batch before the date of settlement .




**Case 3:** the settlement date is after all dates of maturity, we must calculate the future value of each batch at the date of settlement



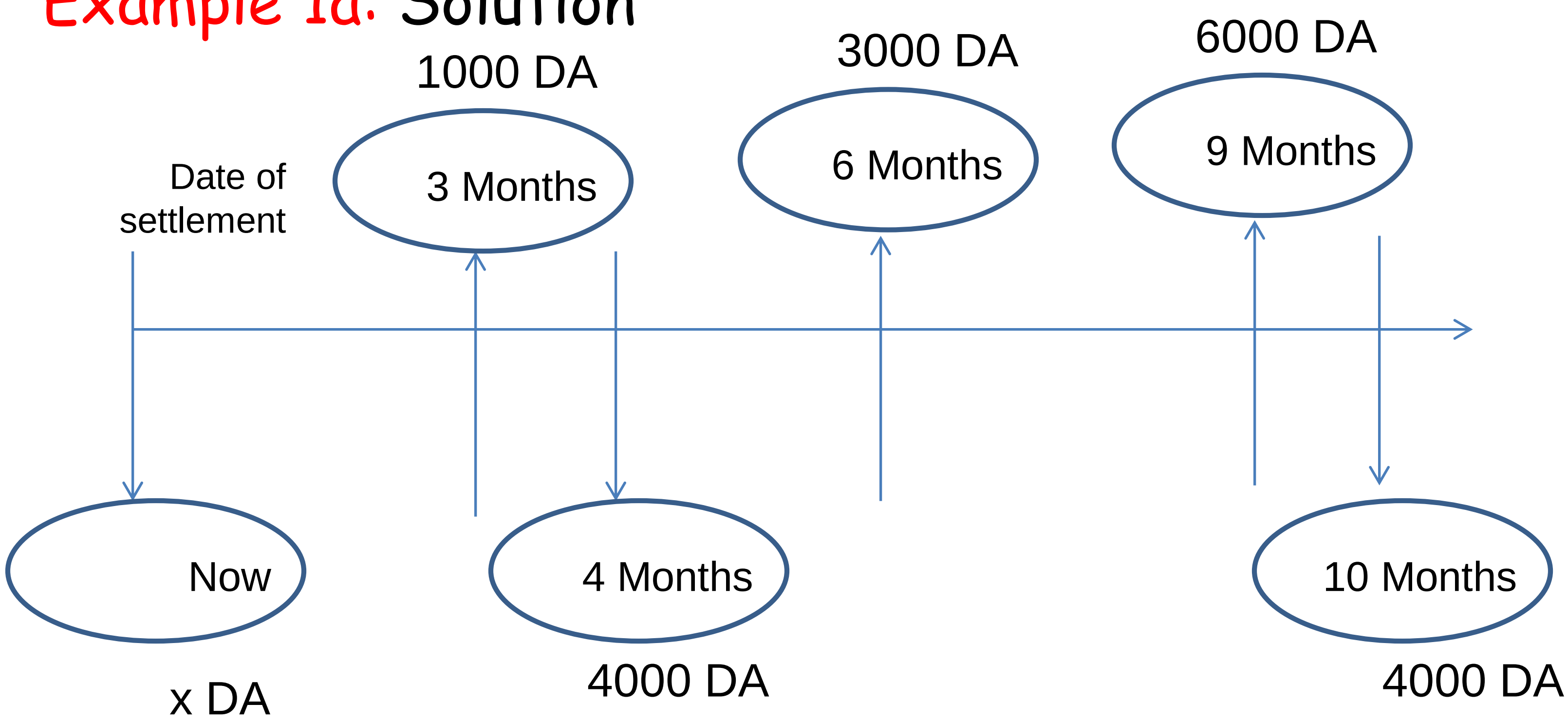
# How to calculate the value of New Debt and installments ?

Example 1a: Someone owes the following amounts:
1000 DA payable after 3 months
3000 DA payable after 6 months
6000 DA payable after 9 months
It was agreed with the creditor to sign two new promissory notes with the same amount fixed at 4000 DA. The first worth after 4 months, the second after 10 months and pay the rest owed cash immediately.

Calculate the amount of cash paid by the debtor if we set the discount rate at 5% per annum.

# How to calculate the value of New Debt and installments ?

Example 1a: Solution




$$PV_1 = 1000 \times \left[1 - 0.05 \times \frac{3}{12}\right] = 987.5$$

$$PV_2 = 3000 \times \left[1 - 0.05 \times \frac{6}{12}\right] = 2925$$

$$PV_3 = 6000 \times \left[1 - 0.05 \times \frac{9}{12}\right] = 5775$$

$$PV(olddebt) = 9687.5$$

# How to calculate the value of New Debt and installments ?

Example 1a: Solution (continued)

$$PV_1 = x$$

$$PV_2 = 4000\left[1 - 0.05 \times \frac{4}{12}\right] = 3933.33$$

$$PV_3 = 4000\left[1 - 0.05 \times \frac{10}{12}\right]$$
$$= 3833.33$$

$$PV(Newdebt) = x + 3933.33 + 3833.33$$

$$PV(Newdebt) = 7766.66 + x$$

$$PV(olddebt) = PV(Newdebt)$$

$$7766.66 + x = 9687.5 \Rightarrow x = 9687.5 - 7766.66 = 1920.84$$

The amount of cash paid by the debtor = 1920.84 DA

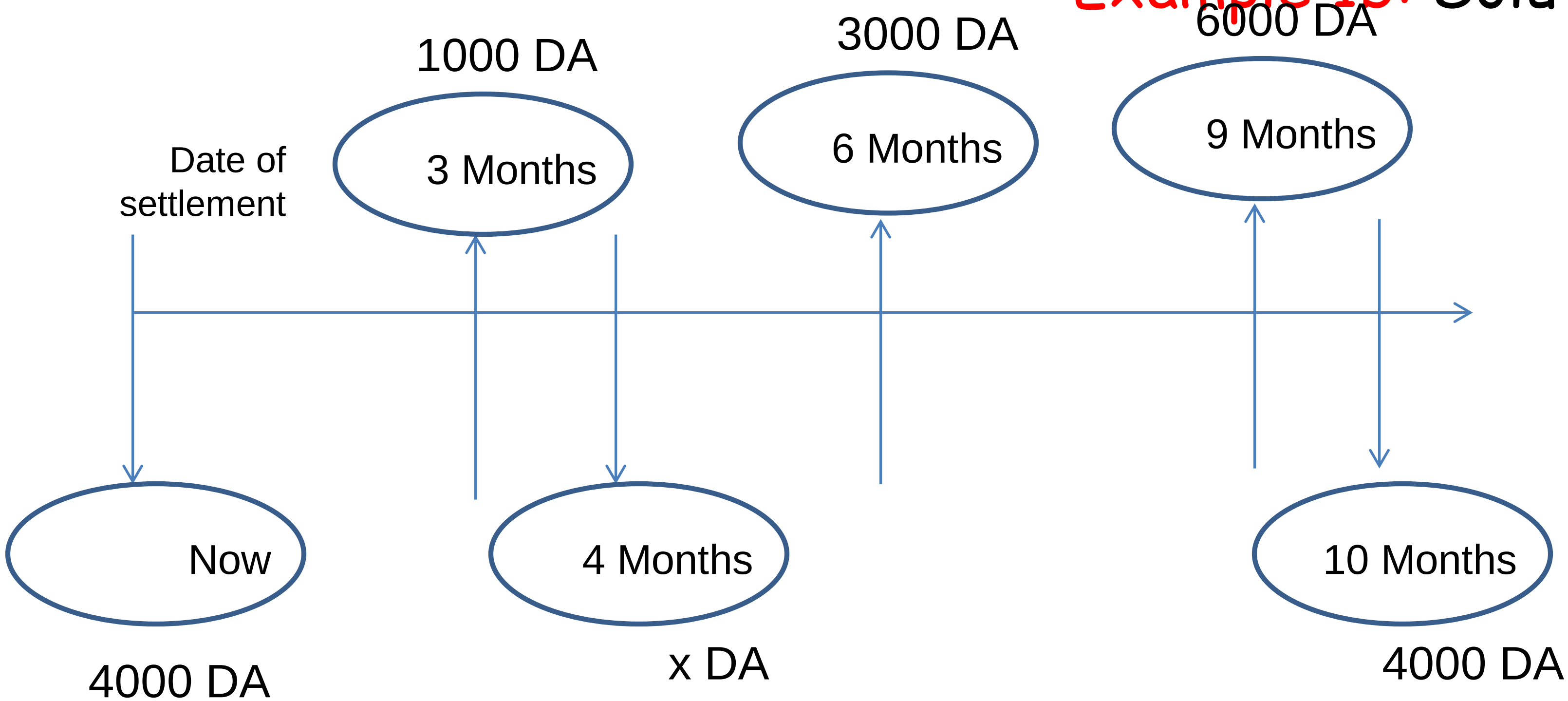
# How to calculate the value of New Debt and installments ?

Example 1b: Someone owes the following amounts:
1000 DA payable after 3 months
3000 DA payable after 6 months
6000 DA payable after 9 months

It was agreed with the creditor to pay immediately 4000 DA and sign two new promissory notes. The first worth after 4 months, the second after 10 months. If we set the discount rate at 5% per annum and the value of the second promissory note equal to 4000 DA, calculate the amount of the first promissory note.

# How to calculate the value of New Debt and installments ?

Example 1b: Solution




$$PV_1 = 1000 \times \left[1 - 0.05 \times \frac{3}{12}\right] = 987.5$$

$$PV_2 = 3000 \times \left[1 - 0.05 \times \frac{6}{12}\right] = 2925$$

$$PV_3 = 6000 \times \left[1 - 0.05 \times \frac{9}{12}\right] = 5775$$

$$PV(olddebt) = 9687.5$$

# How to calculate the value of New Debt and installments ?

Example 1b: Solution (continued)

$$PV_1 = 4000$$

$$PV_2 = x\left[1 - 0.05 \times \frac{4}{12}\right] = 0.983x$$

$$PV_3 = 4000\left[1 - 0.05 \times \frac{10}{12}\right] = 3833.33$$

$$PV(Newdebt) = 4000 + 0.983x + 3833.33$$

$$PV(Newdebt) = 7833.33 + 0.983x$$

$$PV(olddebt) = PV(Newdebt)$$

$$7833.33 + 0.983x = 9687.5 \Rightarrow x = \frac{9687.5 - 7833.33}{0.983} = 1886.24$$

The value of the first promissory note = 1886.24 DA

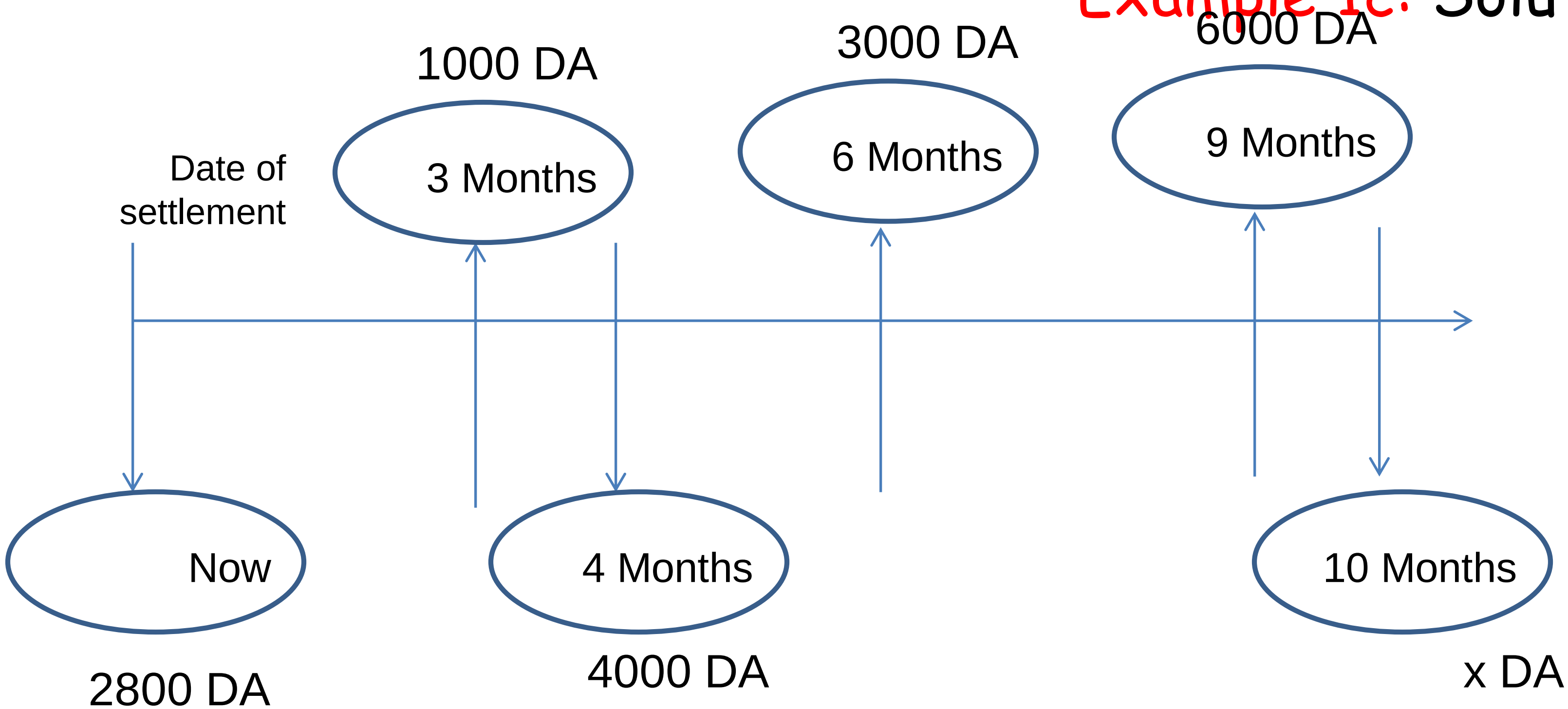
# How to calculate the value of New Debt and installments ?

Example 1c: Someone owes the following amounts:
1000 DA payable after 3 months
3000 DA payable after 6 months
6000 DA payable after 9 months

It was agreed with the creditor to pay immediately the amount of 2800 DA, and the rest of new debt is divided on two promissory notes. The value of the first promissory note is equal to 4000 DA payable after 4 months and the second payable after 10 months. If we set the discount rate at 5% per annum, calculate the nominal value of the second new promissory note.

# How to calculate the value of New Debt and installments ?

Example 1c: Solution




$$PV_1 = 1000 \times \left[1 - 0.05 \times \frac{3}{12}\right] = 987.5$$

$$PV_2 = 3000 \times \left[1 - 0.05 \times \frac{6}{12}\right] = 2925$$

$$PV_3 = 6000 \times \left[1 - 0.05 \times \frac{9}{12}\right] = 5775$$

$$PV(olddebt) = 9687.5$$

# How to calculate the value of New Debt and installments ?

Example 1c: Solution (continued)

$$PV_1 = 2800$$

$$PV_2 = 4000\left[1-0.05\times\frac{4}{12}\right] = 3933.33$$

$$PV_3 = x\left[1-0.05\times\frac{10}{12}\right]$$
$$= 0.958x$$

$$PV(Newdebt) = 2800 + 3933.33 + 0.958x$$

$$PV(Newdebt) = 6733.33 + 0.958x$$

$$PV(olddebt) = PV(Newdebt)$$

$$6733.33 + 0.958x = 9687.5 \Rightarrow x = \frac{9687.5 - 6733.33}{0.958} = 3083.69$$

The value of the second promissory note = 3083.69 DA

# Time to Review!

**Equal short term payments/ Settlement of short-term debt**

**Ordinary Payments / Payment Due**

$$Total\ Installments = Principal + Interest$$

$$Total\ Installments = PMT\left(n + i \times T\right)$$

$$T = \frac{n}{2}\left(Duration\ of\ the\ first\ payment + Duration\ of\ the\ last\ payment\right)$$

**the settlement of short-term debt**

The value of the old debt at the date of settlement

=

The value of the new debt at the date of settlement