University of Tlemcen
Faculty of Sciences
Department of Mathematics

A. Y : 2023/2024

1ST YEAR LMD-MI

ELECTRICITY COURSE

Chapter II: Gauss's theorem

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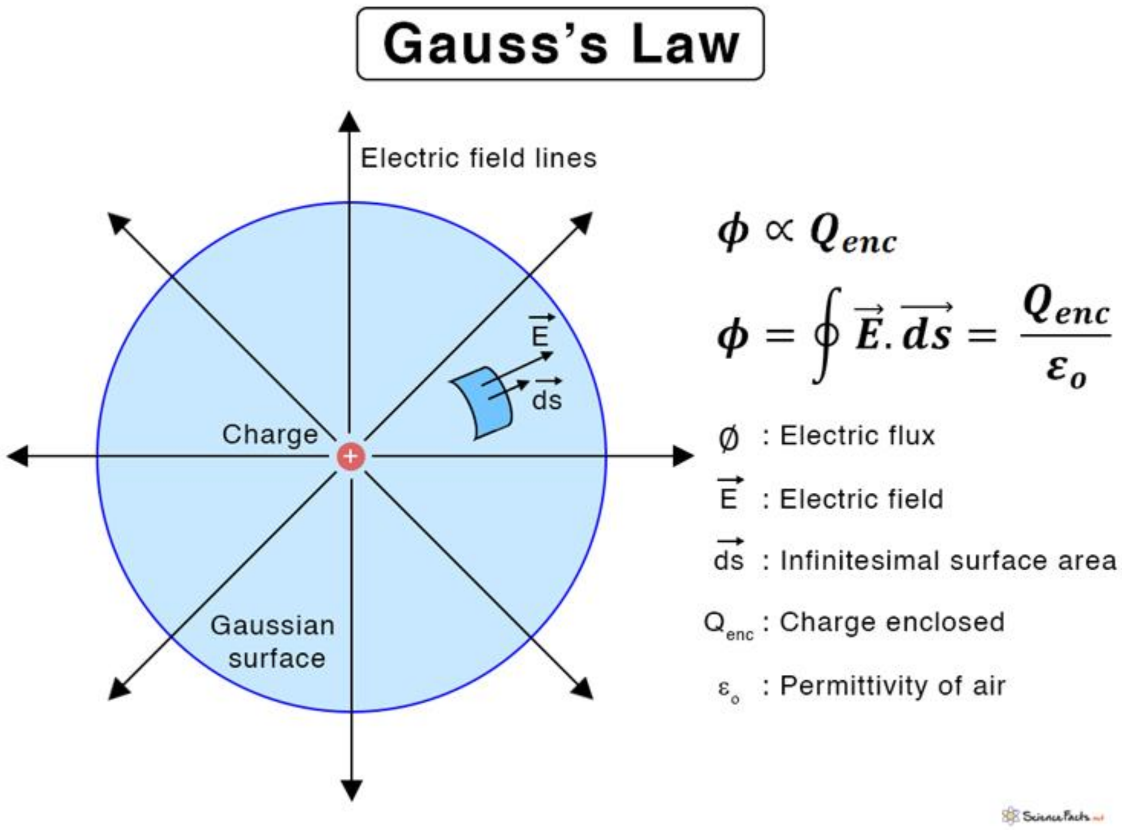


Sommaire

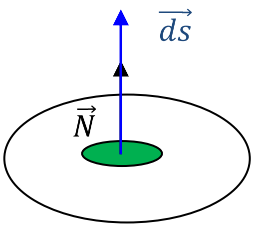
1. Introduction

Gauss's law is a mathematical model that can be used to obtain the electric fields of certain charge distributions with a high degree of symmetry, such as cylinders, spheres and infinite wires.

It is therefore a specialized method, but it is very useful for this class of problems to which it can be applied. At this stage, Gauss's law will help us to better understand the shapes of electric fields due to continuous charge distributions.

2. Definitions

A- Surface vector: The surface vector $\overrightarrow{ds}$ is a vector carried by the unit vector normal to the surface.



B- Flux of a vector field: The elementary flux dΦ is,

$$d\emptyset = \vec{E}.\overrightarrow{ds} \Rightarrow \emptyset = \iint \vec{E}.\overrightarrow{ds} = \iint \vec{E}.ds.\vec{N}$$

With $\overrightarrow{ds} = ds.\vec{N}$

The unit of flow is the Weber (Wb).

3. Electric field flow through a closed surface

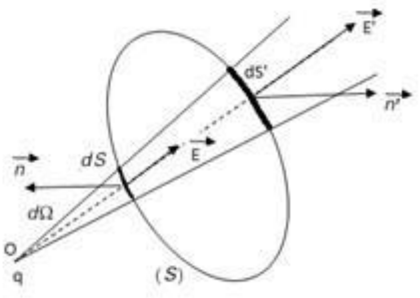
Let S be an arbitrary closed surface and q be the charge enclosed within the surface S. The elementary electric field flux created by the charge q across the closed surface S is given by:

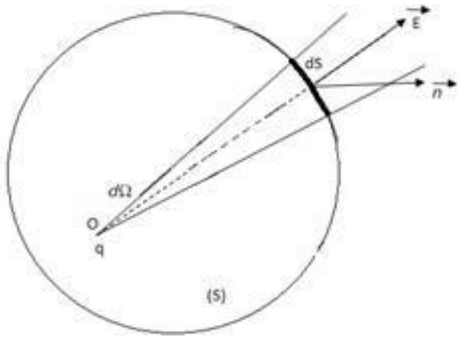
$$d\emptyset = \vec{E}.\overrightarrow{ds} = E.ds.\cos\alpha$$

α : the angle between $\vec{E}$ and $\vec{N}$ ($\overrightarrow{ds}$)

The electric field $\vec{E} = \frac{kq}{r^2}\vec{u}$ and $d\emptyset = \frac{kq}{r^2}\vec{u}\,.\overrightarrow{ds} = kq\frac{\vec{u}\,.\overrightarrow{ds}}{r^2}$

$$\vec{u}\,.\overrightarrow{ds} = ds\,|\vec{u}|\cos\alpha$$





The electric field will be :

$\vec{E} = \frac{kq}{r^2}\vec{u}$ and $\emptyset = \oiint \frac{kq}{r^2}\vec{u}\,.\overrightarrow{ds} = \oiint kq\frac{ds.cos\alpha}{r^2}$

$$with\ \frac{ds.cos\alpha}{r^2} = d\Omega = solid\ angle$$

Note: The unit of the solid angle is the steradian.

Since the area of a sphere of radius R is $S= 4\pi R^2$, we deduce that the largest measurable solid angle, which corresponds to an object covering the entire sphere, is 4π steradians.

$\Omega=4\pi=$ the solid angle to see all of space

$$\text{So}\quad \emptyset = \frac{q}{4\pi\varepsilon_0}.4\pi = \frac{q}{\varepsilon_0}$$

In the case of several point charges, the flux is written as:

$$\emptyset = \oiint \vec{E}.\overrightarrow{ds} = \frac{\sum Q_{int}}{\varepsilon_0}$$

4. Gauss Theorem

a- Statement of Gauss Theorem

« The flux inside a closed surface called a Gauss surface is equal to the sum of the net charges Q_{int} inside this surface divided by the dielectric permittivity in vacuum ε_0 »

$$\emptyset = \oiint \vec{E}.\overrightarrow{ds} = \frac{\sum Q_{int}}{\varepsilon_0}$$

b. The steps involved in applying Gauss's theorem

- Choosing a coordinate system
- Study the invariance of the system
- Study symmetry
- Choice of Gaussian surface; the table shows the different cases where a cylinder is chosen as the SG and the cases where a sphere is chosen as the SG:

	Possible cases	Possible cases	Possible cases	Possible cases
GS is a cylinder	An infinite wire	An infinite plane	A surface or volume charged cylinder	Two or more cylinders
GS is a sphere	A surface- or volume-charged sphere	Two or more spheres		

c. GT for different continuous charge distribution :

- **Linear distribution (dq= λdl)** $\emptyset = \oiint \vec{E}.\overrightarrow{ds} = \frac{\int \lambda dl}{\varepsilon_0}$
- **Surface distribution (dq= σds)** $\emptyset = \oiint \vec{E}.\overrightarrow{ds} = \frac{\int \sigma ds}{\varepsilon_0}$
- **Volume distribution (dq= ρdv)** $\emptyset = \oiint \vec{E}.\overrightarrow{ds} = \frac{\int \rho dv}{\varepsilon_0}$

5. Application examples

5.1. Case of an infinite wire

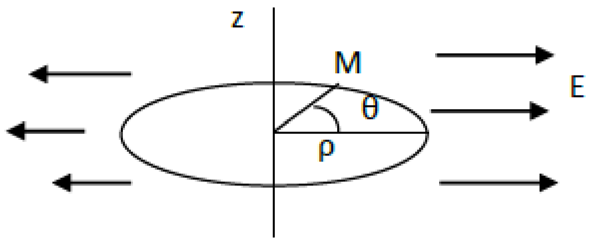
- Choice of coordinate system:

If we zoom in on the wire, we'll have a cylinder with an infinitely small radius, so we use cylindrical coordinates.

- **Study of invariance**

We study invariance with respect to ρ, θ and z (cylindrical coordinates).

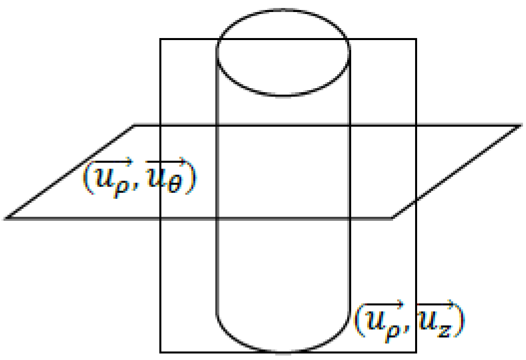
- If we change the angle θ, M rotates around the wire, but the electric field does not change.
- If we change z, M translates along (Oz) and since the wire is infinite, we still have the same wire, so $\vec{E}$ remains invariant remains invariant.
- If we change ρ, M can move away from or towards the wire, so $\vec{E}$ does not remain the same. So $\vec{E}$ depends only on ρ.



- **Study of symmetry**

In this case, we have two planes of symmetry:

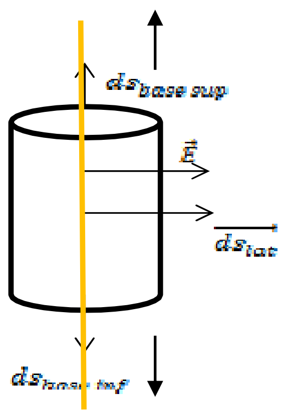
1. The plane intersecting the infinite wire horizontally $(\overrightarrow{u_\rho}, \overrightarrow{u_\theta})$
2. The plane intersecting the infinite wire vertically $(\overrightarrow{u_\rho}, \overrightarrow{u_z})$



So the axis of symmetry is the intersection of the two planes, this is the axis following $\overrightarrow{u_\rho}$ so the electric field is following $\overrightarrow{u_\rho}$.

- **Choice of Gauss surface**

The Gaussian surface is a cylinder of radius r and and height h. Because of symmetry, the field follows the radius ρ, **so we say the field** is said to be **"radial"** and constant in the Gaussian surface ($\vec{E}$ depends only on ρ).



According to Gauss's Theorem: $\emptyset = \iint \vec{E}.\overrightarrow{ds} = \frac{\sum Q_{int}}{\varepsilon_0}$

$$\emptyset = \iint \overrightarrow{E}.\overrightarrow{ds} = \iint \overrightarrow{E}.\overrightarrow{ds_{\ base\ 1}} + \iint \overrightarrow{E}.\overrightarrow{ds_{lat}} + \iint \overrightarrow{E}.\overrightarrow{ds_{\ base\ 2}}$$

$$\overrightarrow{E} \perp \overrightarrow{ds_{\ base}} \Rightarrow \iint \overrightarrow{E}.\overrightarrow{ds_{lat}} = 0$$

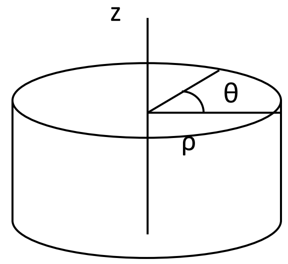
$\vec{E} \parallel \overrightarrow{ds_{lat}}$ so : $\emptyset = \iint \overrightarrow{E}.\overrightarrow{ds_{lat}} = \iint E.ds_{lat} = E.\int ds_{lat} = E.S_{lat}$

so $\emptyset = E\ 2\pi r$h

Let's find Q_{int}, the elementary charge is : $dq = \lambda dl \ \Rightarrow Q = \lambda \int_0^h dl = \lambda h$

$E 2\pi r h = \frac{\lambda h}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi r \varepsilon_0}$

5.1. Case of an infinite cylinder



- Choice of coordinate system

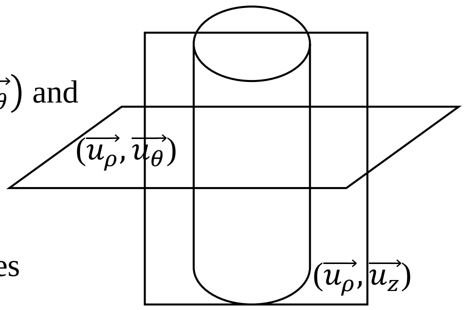
Since we're studying a cylinder, we use cylindrical coordinates.

- Study of invariance

It's the same as the wire. The electric field does not change by varying θ and z, however, $\vec{E}$ depends on ρ.

- Study of symmetry

In this case, too, we have two planes of symmetry:



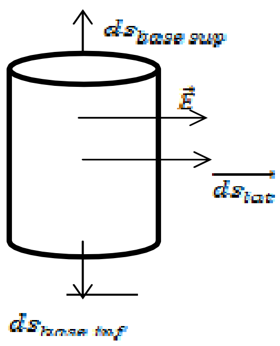
The plane intersecting the infinite wire horizontally $(\vec{u_\rho}, \vec{u_\theta})$ and

The plane intersecting the infinite wire vertically $(\vec{u_\rho}, \vec{u_z})$

So the axis of symmetry is the intersection of the two planes

It's the axis along $\vec{u_\rho}$ so the electric field is along $\vec{u_\rho}$.

- Choice of Gauss surface



The Gaussian surface is a cylinder of radius r and height h.

Because of symmetry, the radial field is constant in the Gaussian surface.

According to Gauss's Theorem: $\emptyset = \iint \vec{E}.\overrightarrow{ds} = \frac{\sum Q_{int}}{\varepsilon_0}$

$\emptyset = \iint \vec{E}.\overrightarrow{ds} = \iint \vec{E}.\overrightarrow{ds_{base\ 1}} + \iint \vec{E}.\overrightarrow{ds_{lat}} + \iint \vec{E}.\overrightarrow{ds_{base\ 2}}$

$\overrightarrow{E} \perp \overrightarrow{ds_{\ base}} \Rightarrow \iint \vec{E}.\overrightarrow{ds_{base}} = 0$

$\vec{E} \parallel \overrightarrow{ds_{lat}}$ so : $\emptyset = \iint \vec{E}.\overrightarrow{ds_{lat}} = \iint E.ds_{lat} = E.\int ds_{lat} = E.S_{lat}$

Then $\emptyset = \boldsymbol{E\ 2\pi r}$**h**=$\mathbf{Q_{int}/\varepsilon_0}$

The cylinder can be either surface or volume charged.

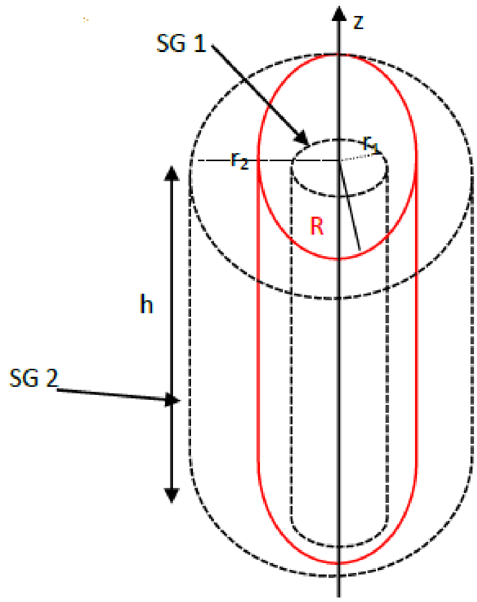
Important note:

The choice of Gaussian surface for a cylinder charged either on the surface or in volume, or two cylinders (one charged in volume and the other charged on the surface, or both charged on the surface...) is always a cylinder of radius r and height h. The flux calculation will be the same, only the Q_{int} charge will vary according to the given distribution.

a- For a surface-charded cylinder

- **The electric field**

We have tos cases ;



1st case $r < R$ we take the Gauss surface inside the charged cylinder to calculate the internal field. Then, in a surface distribution, we have :

$$Q_{int} = 0 \Longrightarrow \boldsymbol{E_1} = \boldsymbol{E_{ins}} = \boldsymbol{0}$$

2nd case r≥ R we take the Gauss surface outside the charged cylinder to calculate the field outside.

$$dq = \sigma ds \Rightarrow Q_{int} = \sigma S = \sigma 2\pi R h$$

So $E_2 2\pi r h = \frac{\sigma 2\pi R h}{\varepsilon_0} \Rightarrow \boldsymbol{E_2} = \boldsymbol{E_{out}} = \frac{\boldsymbol{\sigma R}}{\boldsymbol{\varepsilon_0 r}}$

- The potentiel

$$\vec{E} = -\overrightarrow{grad}\, V \; with \; E = E(r) \Longrightarrow E = -\frac{dV}{dr} so \; V = -\int E.dr$$

1st case $r < R$ $we \; have \; E_1 = 0 \Longrightarrow V_1 = \boldsymbol{C_1}$

2nd case r≥ R

$$\boldsymbol{E_2} = \frac{\boldsymbol{\rho R}}{\boldsymbol{\varepsilon_0}}\frac{\boldsymbol{1}}{\boldsymbol{r}} \Longrightarrow V_2 = -\frac{\sigma R}{\varepsilon_0}\int \frac{1}{r}.dr = -\frac{\boldsymbol{\sigma R}}{\boldsymbol{\varepsilon_0}}\boldsymbol{lnr} + \boldsymbol{C_2}$$

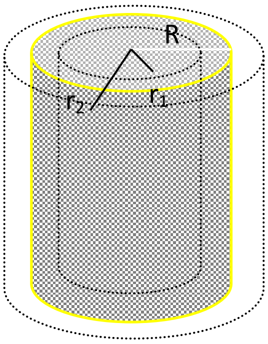
Note: In the case of a cylinder, the constants C_1 and C_2 cannot be calculated because the potential at infinity is non-zero.

b. Volume-charged cylinder

We have tow cases.

- **The electric field**

<u>1st case $r < R$</u> we take the Gauss surface inside the charged cylinder to calculate the internal field. Then, in a volume distribution dq= ρdV, we have :



$$Q_{int} = \iiint \rho dv = \rho \int_0^r 2\pi hrdr = \rho\pi hr^2 \Rightarrow E2\pi rh = \frac{\rho\pi hr^2}{\varepsilon_0}$$

Because $V = \pi hr^2 \Rightarrow dV = 2\pi hrdr$

$\Longrightarrow \boldsymbol{E_1} = \boldsymbol{E_{ins}} = \frac{\boldsymbol{\rho}}{\boldsymbol{2\varepsilon_0}}\boldsymbol{r}$

<u>2nd case r≥ R</u>

we take the Gauss surface outside the loaded cylinder to calculate E_2 =E_{ins}, so we integrate between 0 and R (because Q_{int} lies on the cylinder of radius R).

$$Q_{int} = \iiint \rho dv = \rho \int_0^R 2\pi hrdr = \rho\pi hR^2 \Rightarrow E2\pi rh = \frac{\rho\pi hR^2}{\varepsilon_0}$$

$\Longrightarrow \boldsymbol{E_2} = \frac{\boldsymbol{\rho R^2}}{\boldsymbol{2\varepsilon_0}}\frac{\boldsymbol{1}}{\boldsymbol{r}}$

- **The potentiel**

$$\vec{E} = -\overrightarrow{grad}\,V \; with \; E = E(r)$$

$\Longrightarrow E = -\frac{dV}{dr} so\ V = -\int E.dr$ (this calculation is valid for any cylinder).

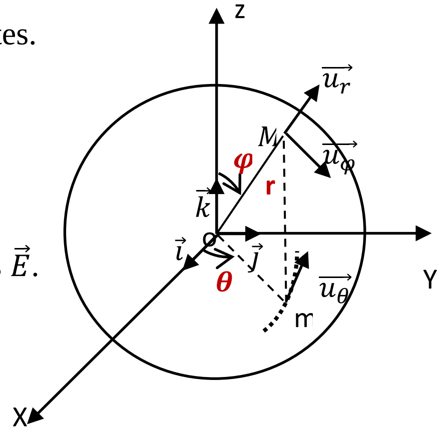
$V_1 = -\int E_1.dr \Longrightarrow V_1 = \frac{\rho}{2\varepsilon_0}\int rdr = -\frac{\boldsymbol{\rho}}{\boldsymbol{4\varepsilon_0}}\boldsymbol{r^2} + \boldsymbol{C_1}$

$V_2 = -\int E_2.dr = -\frac{\rho R^2}{2\varepsilon_0}\int \frac{1}{r}.dr = -\frac{\boldsymbol{\rho R^2}}{\boldsymbol{2\varepsilon_0}}\boldsymbol{lnr} + \boldsymbol{C_2}$

5.3. Case of a sphere

- Choice of coordinate system

Since we're studying a cylinder, we use spherical coordinates.



- Study of invariance

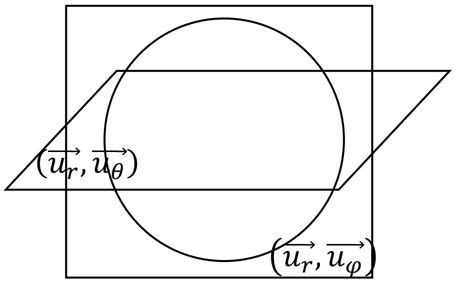
If we change the angle θ or the angle φ, the electric field $\vec{E}$

does not change, but by varying r the electric field $\vec{E}$ varies $\vec{E}$.

- Study of symmetry

We have two planes of symmetry:

The plane intersecting the infinite wire horizontally $(\overrightarrow{u_r}, \overrightarrow{u_\theta})$ and the plane intersecting the infinite wire vertically $(\overrightarrow{u_r}, \overrightarrow{u_\varphi})$.



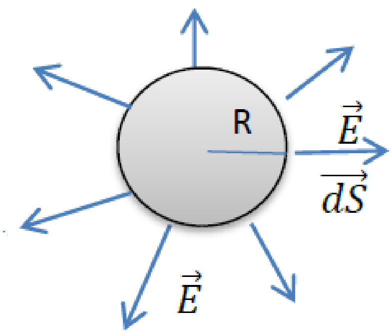
So the axis of symmetry is the intersection of the two planes,

This is the axis along $\overrightarrow{\boldsymbol{u_r}}$ so the electric field is along $\overrightarrow{\boldsymbol{u_r}}$

The field is then said to be radial

- Choosing the Gaussian surface

The Gaussian surface is a sphere with center O and radius r. Due to symmetry, the field is radial and constant in the Gaussian surface**.**



$$\emptyset = \oiint \vec{E}.\overrightarrow{ds} = \frac{\sum Q_{int}}{\varepsilon_0}$$

$\overrightarrow{E}$ // $\overrightarrow{ds}$:

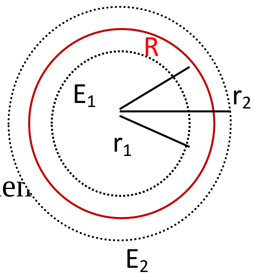
So : $\oiint \vec{E}.\overrightarrow{ds} = \iint E.ds = E \iint ds = E.S = E4\pi r^2 \Rightarrow \boldsymbol{E4\pi r^2} = \frac{\sum \boldsymbol{Q_{int}}}{\boldsymbol{\varepsilon_0}}$

Important note:

The choice of Gaussian surface for a sphere charged either on the surface or in volume, or two spheres (one charged in volume and the other charged on the surface, or both charged on the surface...) is always a sphere of radius r and center O. And the flux calculation will be the same, only the Q_{int} charge will vary according to the distribution.

a- Surface-charged sphere

- The electrostatic field E(r) at any point in space.



We have 2 cases :

1st case r<R

The Gaussian surface is inside the sphere to calculate E_1==E_{int} then

$$Q_{int} = 0 \Rightarrow \boldsymbol{E_1 = 0}$$

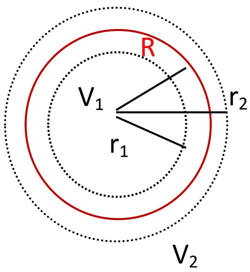
2nd case r≥ R

The Gauss surface is outside the sphere to calculate E_2=E_{outs}

$$dq = \sigma ds \Rightarrow Q_{int} = \sigma 4\pi R^2$$

So $E_2 4\pi r^2 = \frac{\sigma 4\pi R^2}{\varepsilon_0} \Rightarrow \boldsymbol{E_2 = \frac{\sigma R^2}{\varepsilon_0 r^2}}$

- The electrostatic potential V(r) at any point in space.



$$\vec{E} = -\overrightarrow{grad}v \Rightarrow E = -\frac{dv}{dr} \quad \text{so} \quad v = -\int E dr$$

1st case r<R $E_1 = 0 \Rightarrow v_1 = C_1$

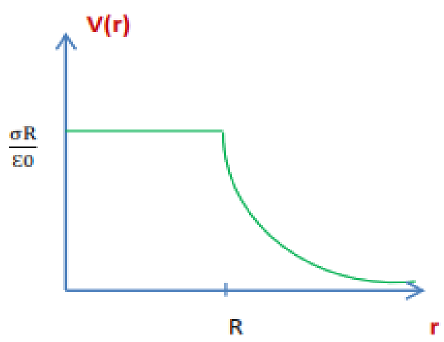
2nd case r≥ R $E_2 = \frac{\sigma R^2}{\varepsilon_0 r^2} \Rightarrow v_2 = -\sigma \frac{\sigma R^2}{\varepsilon_0} \int \frac{dr}{r^2} = \frac{\sigma R^2}{\varepsilon_0 r} + C_2$

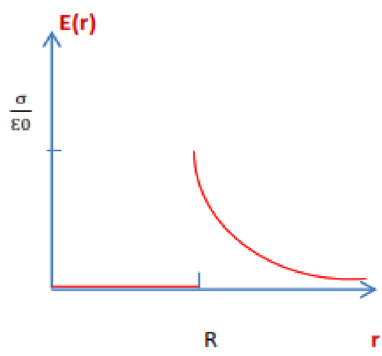
Calculating constants :

- The potential at infinity is zero (v_∞=0) so $\lim_{r \to \infty} v = 0$ so C_2=0 then $\boldsymbol{v_2 = \frac{\sigma R^2}{\varepsilon_0 r}}$
- The potential is a continuous function in R so $v_1(R) = v_2(R)$

then $v_1 = C_1 = \frac{\sigma R^2}{\varepsilon_0 R}$ so $\boldsymbol{v_1 = \frac{\sigma R}{\varepsilon_0}}$

- **Plot the graphs E(r) and V(r) as a function of r :**



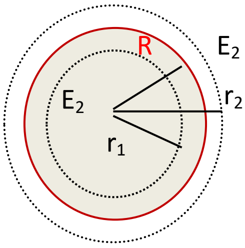


b- Sphere charged in volume

1- The electrostatic field E(r) at any point in space.

We have 2 cases:

1st case r<R



$dq = \rho dv = \rho 4\pi r^2 dr \Rightarrow Q_{int} = \rho \iiint dv = \rho 4\pi \int_0^r r^2 dr$

We integrate between 0 and r because the Q_{int} charge is located in the volume of the Gauss sphere of radius r.

$\Rightarrow Q_{int} = \rho\ \frac{4}{3}\pi r^3$ so $(*) \Rightarrow E_1 = \frac{\rho\ \frac{4}{3}\pi r^3}{4\pi r^2 \varepsilon_0}$ then $\boldsymbol{E_1 = \frac{\rho}{3\varepsilon_0} r = E_{ins}}$

2nd case r≥ R

$dq = \rho dv = \rho 4\pi r^2 dr \Rightarrow Q_{int} = \rho \iiint dv = \rho 4\pi \int_0^R r^2 dr$

We integrate between 0 and R because the Q_{int} charge is located in the volume of the sphere of radius R.

So $Q_{int} = \rho\ \frac{4}{3}\pi R^3$

$$(*) \Rightarrow E_2 = \frac{\rho\ \frac{4}{3}\pi R^3}{4\pi r^2 \varepsilon_0} \quad so \quad \boldsymbol{E_2 = \frac{\rho\ R^3}{3\varepsilon_0 r^2} = E_{out}}$$

2- The electric potential v(r) at any point in space.

$\vec{E} = -\overrightarrow{grad}v \Rightarrow E = -\frac{dv}{dr}$ So $v = -\int E dr$

1st case r<R:

$E_1 = \frac{\rho}{3\varepsilon_0} r \Rightarrow v_1 = -\frac{\rho}{3\varepsilon_0}\int r dr$ so $\boldsymbol{v_1 = -\frac{\rho}{6\varepsilon_0} r^2 + C_1}$

2nd case r≥ R :

$E_2 = \frac{\rho\ R^3}{3\varepsilon_0 r^2} \Rightarrow v_2 = -\frac{\rho\ R^3}{3\varepsilon_0}\int \frac{1}{r^2} dr$ so $\boldsymbol{v_2 = \frac{\rho\ R^3}{3\varepsilon_0}\frac{1}{r} + C_2}$

Calculating constants :

The potential at infinity is zero (v_∞=0) so $\lim_{r\to\infty} v = 0$ then $\boldsymbol{v_2} = \frac{\boldsymbol{\rho\ R^3}}{\boldsymbol{3\varepsilon_0}}\frac{\boldsymbol{1}}{\boldsymbol{r}}$

The potential is a continuous function in R, so $\boldsymbol{v_1(R)} = \boldsymbol{v_2(R)}$

$\frac{\rho\ R^3}{3\varepsilon_0}\frac{1}{R} = -\frac{\boldsymbol{\rho}}{\boldsymbol{6\varepsilon_0}}R^2 + C_1 \Rightarrow C_1 = \frac{\rho\ R^2}{\varepsilon_0}\left(\frac{1}{3}+\frac{1}{6}\right) = \frac{\rho\ R^2}{2\varepsilon_0}$

so $\boldsymbol{v_1} = -\frac{\boldsymbol{\rho\ r^2}}{\boldsymbol{6\varepsilon_0}} + \frac{\boldsymbol{\rho\ R^2}}{\boldsymbol{2\varepsilon_0}}$

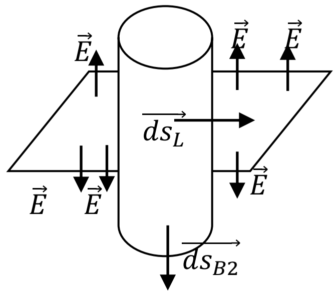
5.4. Case of an infinite plan

To find the electric field in an infinite plane,

we use Gauss's theorem.

The Gaussian surface is a cylinder intersecting the plane.

The cylinder has radius r and height h.



For reasons of symmetry, the field is radial and constant at any point on the Gaussian surface.

In the Gaussian surface.

$$\emptyset = \oiint \vec{E}.\overrightarrow{dS} = \frac{\sum Q_{int}}{\varepsilon_0}$$

$$\emptyset = \emptyset_{sbase1} + \emptyset_{Slat} + \emptyset_{sbase2} = \iint \vec{E}.\overrightarrow{dS_{B1}} + \iint \vec{E}.\overrightarrow{dS_{B2}} + \iint \vec{E}.\overrightarrow{dS_L}$$

$= 2\iint E.dS_{base} = 2E.S_{base}$

$\Rightarrow \emptyset = 2E.S_{base} = \frac{\sum Q_{int}}{\varepsilon_0}$ (1)

$dq = \sigma ds \ \Rightarrow Q_{int} = \sigma \iint ds = \sigma S_{base}$

(1)$\Rightarrow 2E.S_{base} = \frac{\sigma S_{base}}{\varepsilon_0}$ so $\quad \boldsymbol{E} = \frac{\boldsymbol{\sigma}}{\boldsymbol{2\varepsilon_0}}$

Then $\lim_{R\to\infty}|E| = \frac{\sigma}{2\varepsilon_0}$ and the field for an infinite plane is identical.