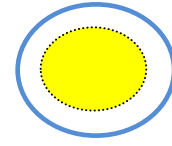




Correction of SW N°02 of Electricity

Gauss's theorem

Exercise 1:



The Gaussian surface is a sphere with center O and radius r.

For reasons of symmetry, the field is radial and constant at any point on the Gaussian surface.

$$\Phi = \oiint \vec{E} \cdot \vec{ds} = \frac{\Sigma Q_{int}}{\epsilon_0}$$

$$\vec{E} \parallel \vec{ds} \text{ Donc } : \oiint \vec{E} \cdot \vec{ds} = \iint E \cdot ds = E \iint ds = E \cdot S = E 4\pi r^2 \Rightarrow E 4\pi r^2 = \frac{\Sigma Q_{int}}{\epsilon_0}$$

$$\Rightarrow E = \frac{Q_{int}}{4\pi r^2 \epsilon_0} \quad (*)$$

1- The electrostatic field E(r) at any point in space.

we have 3 cases :

1st case $r < R_1$ ($r \in [0, R_1[$)

$$dq = \rho dv = \rho 4\pi r^2 dr \Rightarrow Q_{int} = \rho \iiint dv = \rho 4\pi \int_0^r r^2 dr$$

$$\Rightarrow Q_{int} = \rho \frac{4}{3} \pi r^3$$

$$\text{so } (*) \Rightarrow E_1 = \frac{\rho \frac{4}{3} \pi r^3}{4\pi r^2 \epsilon_0} \text{ then } E_1 = \frac{\rho}{3\epsilon_0} r_1$$

2nd case $R_1 \leq r < R_2$ ($r \in [R_1, R_2[$)

$$dq = \rho dv = \rho 4\pi r^2 dr \Rightarrow Q_{int} = \rho \iiint dv = \rho 4\pi \int_0^{R_1} r^2 dr$$

$$\text{so } Q_{int} = \rho \frac{4}{3} \pi R_1^3$$

$$(*) \Rightarrow E_2 = \frac{\rho \frac{4}{3} \pi R_1^3}{4\pi r^2 \epsilon_0} \text{ so } E_2 = \frac{\rho R_1^3}{3\epsilon_0 r^2}$$

3rd case $r \geq R_2$ ($r \in [R_2, +\infty[$)

$$Q_{int} = Q_1 + Q_2 \text{ with } Q_1 = \rho \frac{4}{3} \pi R_1^3 \text{ and } dq_2 = \sigma ds \Rightarrow Q_2 = \sigma 4\pi R_2^2$$

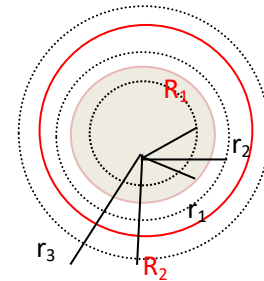
$$\text{So } Q_{int} = \rho \frac{4}{3} \pi R_1^3 + \sigma 4\pi R_2^2$$

$$\text{Then } (*) \Rightarrow E_3 = \frac{\rho \frac{4}{3} \pi R_1^3 + \sigma 4\pi R_2^2}{4\pi r^2 \epsilon_0} \text{ hence } E_3 = \frac{\rho R_1^3}{3\epsilon_0 r^2} + \frac{\sigma R_2^2}{\epsilon_0 r^2}$$

1- The electric potential v(r) at any point in space.

$$\vec{E} = -\overrightarrow{grad} v \Rightarrow E = -\frac{dv}{dr}$$

$$\text{so } v = -\int E dr$$





1st case : $r < R_1$ ($r \in [0, R_1[$)

$$E_1 = \frac{\rho}{3\epsilon_0} r \Rightarrow v_1 = -\frac{\rho}{3\epsilon_0} \int r dr \text{ so } v_1 = -\frac{\rho}{6\epsilon_0} r^2 + C_1$$

2st case $R_1 \leq r < R_2$ ($r \in [R_1, R_2[$)

$$E_2 = \frac{\rho R_1^3}{3\epsilon_0 r^2} \Rightarrow v_2 = -\frac{\rho R_1^3}{3\epsilon_0} \int \frac{1}{r^2} dr \text{ so } v_2 = \frac{\rho R_1^3}{3\epsilon_0} \frac{1}{r} + C_2$$

3rd case $r \geq R_2$ ($r \in [R_2, +\infty[$)

$$E_3 = \frac{\rho R_1^3}{3\epsilon_0 r^2} + \frac{\sigma R_2^2}{\epsilon_0 r^2} \Rightarrow v_3 = -\left(\frac{\rho R_1^3}{3\epsilon_0} + \frac{\sigma R_2^2}{\epsilon_0}\right) \int \frac{1}{r^2} dr$$

$$\text{So } v_3 = \left(\frac{\rho R_1^3}{3\epsilon_0} + \frac{\sigma R_2^2}{\epsilon_0}\right) \frac{1}{r} + C_3$$

$$\text{Infinite potential (} r \rightarrow \infty \text{) } v=0 \text{ so } C_3=0 \text{ and } v_3 = \left(\frac{\rho R_1^3}{3\epsilon_0} + \frac{\sigma R_2^2}{\epsilon_0}\right) \frac{1}{r}$$

- Potential is a continuous function:
- **at R_2 so $v_3(R_2) = v_2(R_2)$**

$$\frac{\rho R_1^3}{3\epsilon_0} \frac{1}{R_2} + \frac{\sigma R_2^2}{\epsilon_0} \frac{1}{R_2} = \frac{\rho R_1^3}{3\epsilon_0} \frac{1}{R_2} + C_2 \Rightarrow C_2 = \frac{\sigma R_2}{\epsilon_0}$$

$$\text{donc } v_2 = \frac{\rho R_1^3}{3\epsilon_0} \frac{1}{r} + \frac{\sigma R_2}{\epsilon_0}$$

- **at R_1 so $v_2(R_1) = v_1(R_1)$**

$$-\frac{\rho}{6\epsilon_0} R_1^2 + C_1 = \frac{\rho R_1^3}{3\epsilon_0} \frac{1}{R_1} + \frac{\sigma R_2}{\epsilon_0} \Rightarrow C_1 = \frac{\rho R_1^2}{3\epsilon_0} + \frac{\rho R_1^2}{6\epsilon_0} + \frac{\sigma R_2}{\epsilon_0} = \frac{3\rho R_1^2}{6\epsilon_0} + \frac{\sigma R_2}{\epsilon_0}$$

$$C_1 = \frac{\rho R_1^2}{2\epsilon_0} + \frac{\sigma R_2}{\epsilon_0}$$

$$\text{so } v_1 = -\frac{\rho}{6\epsilon_0} r^2 + \frac{\rho R_1^2}{2\epsilon_0} + \frac{\sigma R_2}{\epsilon_0}$$

Exercise 2

The Gaussian surface is a sphere with center O and radius r. For reasons of symmetry, the field is radial and constant at any point on the Gaussian surface.

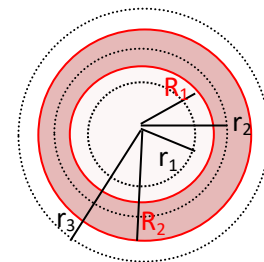
$$\Phi = \oiint \vec{E} \cdot \vec{ds} = \frac{\sum Q_{int}}{\epsilon_0}$$

$$\vec{E} \parallel \vec{ds} \text{ so : } \oiint \vec{E} \cdot \vec{ds} = \iint E \cdot ds = E \iint ds = E \cdot S = E 4\pi r^2 \Rightarrow E 4\pi r^2 = \frac{\sum Q_{int}}{\epsilon_0}$$

$$\Rightarrow E = \frac{Q_{int}}{4\pi r^2 \epsilon_0} \quad (*)$$

The electrostatic field E(r) at any point in space.

We have 3 cases:





1st case $r < R_1$

$$Q_{int} = 0 \quad \text{so } E_1 = 0$$

2nd case $R_1 \leq r < R_2$

$$dq = \rho dv = \rho 4\pi r^2 dr \Rightarrow Q_{int} = \rho \iiint dv = \rho 4\pi \int_{R_1}^r r^2 dr$$

$$\text{So } Q_{int} = \rho \frac{4}{3}\pi(r^3 - R_1^3)$$

$$(*) \Rightarrow E_2 = \frac{\rho \frac{4}{3}\pi(r^3 - R_1^3)}{4\pi r^2 \epsilon_0} \quad \text{so } E_2 = \frac{\rho (r^3 - R_1^3)}{3\epsilon_0 r^2} = \frac{\rho}{3\epsilon_0} \left(r - \frac{R_1^3}{r^2} \right)$$

3rd case $r \geq R_2$

$$dq = \rho dv = \rho 4\pi r^2 dr \Rightarrow Q_{int} = \rho \iiint dv = \rho 4\pi \int_{R_1}^{R_2} r^2 dr$$

$$\text{So } Q_{int} = \rho \frac{4}{3}\pi(R_2^3 - R_1^3)$$

$$(*) \Rightarrow E_3 = \frac{\rho \frac{4}{3}\pi(R_2^3 - R_1^3)}{4\pi r^2 \epsilon_0} \quad \text{donc } E_3 = \frac{\rho (R_2^3 - R_1^3)}{3\epsilon_0 r^2}$$

1- The electric potential $v(r)$ at any point in space.

$$\vec{E} = -\overrightarrow{grad}v \Rightarrow E = -\frac{dv}{dr} \quad \text{so } v = -\int E dr$$

1st case : $r < R_1$

$$E_1 = 0 \Rightarrow v_1 = C_1$$

2nd case $R_1 \leq r < R_2$

$$E_2 = \frac{\rho}{3\epsilon_0} \left(r - \frac{R_1^3}{r^2} \right) \Rightarrow v_2 = -\frac{\rho}{3\epsilon_0} \left(\int r dr - R_1^3 \int \frac{1}{r^2} dr \right)$$

$$\text{so } v_2 = -\frac{\rho}{3\epsilon_0} \left(\frac{r^2}{2} - R_1^3 \left(\frac{-1}{r} \right) \right) + C_2$$

$$v_2 = -\frac{\rho}{3\epsilon_0} \left(\frac{r^2}{2} + \frac{R_1^3}{r} \right) + C_2$$

3rd case $r \geq R_2$

$$E_3 = \frac{\rho (R_2^3 - R_1^3)}{3\epsilon_0 r^2} \Rightarrow v_3 = -\frac{\rho (R_2^3 - R_1^3)}{3\epsilon_0} \int \frac{1}{r^2} dr \quad \text{so } v_3 = \frac{\rho (R_2^3 - R_1^3)}{3\epsilon_0} \frac{1}{r} + C_3$$

Infinite potentiel at $(r \rightarrow \infty) v=0$ so $C_3=0$ and $v_3 = \frac{\rho (R_2^3 - R_1^3)}{3\epsilon_0} \frac{1}{r}$

- The potential is a continuous function:

- At R_2 so $v_3(R_2) = v_2(R_2)$

$$\frac{\rho (R_2^3 - R_1^3)}{3\epsilon_0} \frac{1}{R_2} = -\frac{\rho}{3\epsilon_0} \left(\frac{R_2^2}{2} + \frac{R_1^3}{R_2} \right) + C_2$$



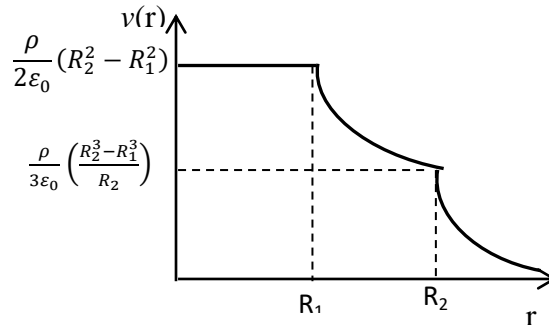
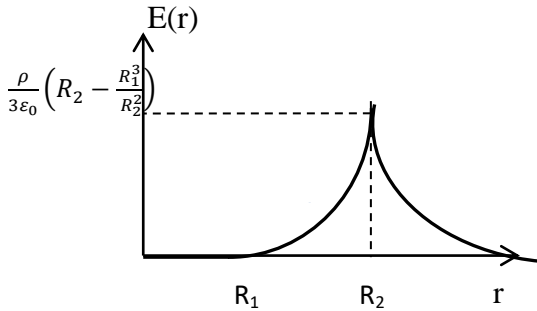
$$\Rightarrow C_2 = \frac{\rho R_2^2}{3\epsilon_0} + \frac{\rho R_2^2}{6\epsilon_0} - \frac{\rho (R_1^3)}{3R_2\epsilon_0} + \frac{\rho (R_1^3)}{3R_2\epsilon_0} = \frac{3\rho R_2^2}{6\epsilon_0} = \frac{\rho R_2^2}{2\epsilon_0}$$

$$\text{so } v_2 = -\frac{\rho}{3\epsilon_0} \left(\frac{r^2}{2} + \frac{R_1^3}{r} \right) + \frac{\rho R_2^2}{2\epsilon_0}$$

• at R_1 so $-\frac{\rho}{3\epsilon_0} \left(\frac{R_1^2}{2} + \frac{R_1^3}{R_1} \right) + \frac{\rho R_2^2}{2\epsilon_0} \Rightarrow C_1 = -\frac{\rho}{3\epsilon_0} \left(\frac{R_1^2}{2} + R_1^2 \right) + \frac{\rho R_2^2}{2\epsilon_0}$

$$\Rightarrow C_1 = -\frac{\rho}{3\epsilon_0} \left(\frac{3R_1^2}{2} \right) + \frac{\rho R_2^2}{2\epsilon_0}$$

then $v_1 = -\frac{\rho R_1^2}{2\epsilon_0} + \frac{\rho R_2^2}{2\epsilon_0}$



Exercise 3 :

Consider a cylinder of radius r and height h as a Gaussian surface.

Due to symmetry, the field is radial and constant at any point on the Gauss surface.

According to Gauss's Theorem:

$$\phi = \iint \vec{E} \cdot \vec{ds} = \frac{\sum Q_{int}}{\epsilon_0}$$

$$\phi = \iint \vec{E} \cdot \vec{ds} = 2 \iint \vec{E} \cdot \vec{ds}_{base} + \iint \vec{E} \cdot \vec{ds}_{lat}$$

$$\vec{E} \perp \vec{ds}_{base} \Rightarrow \iint \vec{E} \cdot \vec{ds}_{lat} = 0$$

$$\vec{E} \parallel \vec{ds}_{lat} \text{ so: } \phi = \iint \vec{E} \cdot \vec{ds}_{lat} = \iint E \cdot ds_{lat} = E \cdot \int ds_{lat} = E \cdot S_{lat}$$

$$\Rightarrow \phi = E 2\pi r h = \frac{\sum Q_{int}}{\epsilon_0} \text{ then } E = \frac{\sum Q_{int}}{2\pi r h \epsilon_0}$$

1- Electric field

1st case $r < R$ $dq = \lambda dl \Rightarrow Q_{int} = \lambda h$

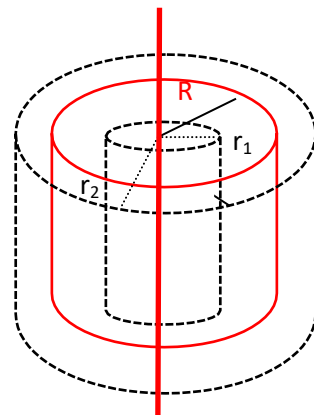
$$E_1 2\pi r h = \frac{\lambda h}{\epsilon_0} \Rightarrow E_1 = \frac{\lambda}{2\pi r \epsilon_0}$$

2nd case $r \geq R$ $Q_{int} = Q_1 + Q_2$

$$dq_2 = \sigma ds \Rightarrow Q_{int} = \sigma S = \sigma 2\pi R h$$

So $Q_{int} = \lambda h + \sigma 2\pi R h$

$$\text{Then } E_2 2\pi r h = \frac{\lambda h + \sigma 2\pi R h}{\epsilon_0} \Rightarrow E_2 = \frac{\lambda}{2\pi r \epsilon_0} + \frac{\sigma R}{\epsilon_0 r}$$





2- Let's find λ for which $E_2=0$

$$\frac{\lambda}{2\pi r \epsilon_0} + \frac{\sigma R}{\epsilon_0 r} = 0 \Rightarrow \frac{\lambda}{2\pi r \epsilon_0} = -\frac{\sigma R}{\epsilon_0 r} \text{ so } \lambda = -\frac{2\pi\sigma R}{h}$$

Exercise 4:

1- the field : Consider a cylinder of radius r and height h as a Gauss surface.

$$\text{Gauss's theorem : } \varphi = \iint \vec{E} \cdot \vec{ds} = \frac{\Sigma Q_{int}}{\epsilon_0}$$

$$\varphi = \iint \vec{E} \cdot \vec{ds} = 2 \iint \vec{E} \cdot \vec{ds}_{base} + \iint \vec{E} \cdot \vec{ds}_{lat}$$

$$\vec{E} \perp \vec{ds}_{lat} \text{ et } \vec{E} \perp \vec{ds}_{base} \Rightarrow \iint \vec{E} \cdot \vec{ds}_{lat} = 0$$

$$\text{so : } \varphi = \iint \vec{E} \cdot \vec{ds}_{lat} = \vec{E} \cdot \int \vec{ds}_{lat} = E \cdot S_{lat} = E 2\pi r h$$

For : $r < R_1$

$$Q_{int} = 0 \Rightarrow \vec{E}_1 = 0$$

For : $R_1 \leq r < R_2$

$$Q_{int} = \iint \sigma ds = \sigma \iint ds = \sigma s = \sigma 2\pi R_1 h$$

$$E_2 2\pi r h = \frac{\sigma 2\pi R_1 h}{\epsilon_0} \Rightarrow E_2 = \frac{\sigma R_1}{\epsilon_0} \frac{1}{r}$$

For : $r \geq R_2$

$$Q_{int} = Q_1 + Q_2$$

$$Q_1 = \sigma 2\pi R_1 h \quad \text{and} \quad Q_2 = -\sigma 2\pi R_2 h \quad \text{donc} \quad Q_{int} = \sigma 2\pi h (R_1 - R_2)$$

$$E_3 2\pi r h = \frac{\sigma 2\pi h (R_1 - R_2)}{\epsilon_0} \Rightarrow E_3 = \frac{\sigma (R_1 - R_2)}{\epsilon_0} \frac{1}{r}$$

2- The potentiel

$$\vec{E} = -\vec{grad} V$$

$$E = E(r) \Rightarrow E = -\frac{dV}{dr} \Rightarrow V = -\int E \cdot dr$$

- $V_1 = -\int E_1 \cdot dr \Rightarrow V_1 = C_1$
- $V_2 = -\int E_2 \cdot dr = -\frac{\sigma R_1}{\epsilon_0} \int \frac{1}{r} \cdot dr = -\frac{\sigma R_1}{\epsilon_0} \ln r + C_2$
- $V_3 = -\int E_3 \cdot dr = -\frac{\sigma (R_1 - R_2)}{\epsilon_0} \int \frac{1}{r} \cdot dr = -\frac{\sigma (R_1 - R_2)}{\epsilon_0} \ln r + C_3$

Supplementary exercises :

Exercise 1:

Consider a cylinder of radius r and height h as a Gauss surface.

Due to symmetry, the field is radial and constant at all points on the Gauss surface.

$$\text{According to Gauss's Theorem: } \varphi = \iint \vec{E} \cdot \vec{ds} = \frac{\Sigma Q_{int}}{\epsilon_0}$$



$$\begin{aligned}\phi &= \iint \vec{E} \cdot \vec{ds} = 2 \iint \vec{E} \cdot \vec{ds}_{base} + \iint \vec{E} \cdot \vec{ds}_{lat} \\ \vec{E} \perp \vec{ds}_{base} &\Rightarrow \iint \vec{E} \cdot \vec{ds}_{lat} = 0 \\ \vec{E} \parallel \vec{ds}_{lat} &\text{ so } : \phi = \iint \vec{E} \cdot \vec{ds}_{lat} = \iint E \cdot ds_{lat} = E \cdot \int ds_{lat} = E \cdot S_{lat}\end{aligned}$$

$$\Rightarrow \phi = E 2\pi r h = \frac{\Sigma Q_{int}}{\epsilon_0}$$

- Electric field

1st case $r < R_1$

$$Q_{int} = 0 \quad \text{so } E_1 = 0$$

2nd case $R_1 \leq r < R_2$

$$dq = \rho dv = \rho 2\pi r h dr \Rightarrow Q_{int} = \rho \iiint dv = \rho 2\pi h \int_{R_1}^r r dr$$

$$\text{so } Q_{int} = \rho 2 \pi h \left(\frac{r^2}{2} - \frac{R_1^2}{2} \right) = \rho \pi h (r^2 - R_1^2)$$

$$\text{or } Q_{int} = \rho (\pi r^2 h - \pi R_1^2 h)$$

$$\text{then } E_2 2\pi r h = \frac{\rho \pi h (r^2 - R_1^2)}{\epsilon_0} \quad \text{hence } E_2 = \frac{\rho (r^2 - R_1^2)}{2\epsilon_0 r} = \frac{\rho}{2\epsilon_0} \left(r - \frac{R_1^2}{r} \right)$$

3rd case $r \geq R_2$

$$dq = \rho dv = \rho 4\pi r^2 dr \Rightarrow Q_{int} = \rho \iiint dv = \rho 2\pi h \int_{R_1}^{R_2} r^2 dr \quad \text{so } Q_{int} = \rho \pi h (R_2^2 - R_1^2)$$

$$E_3 2\pi r h = \frac{\rho \pi h (R_2^2 - R_1^2)}{\epsilon_0} \quad \text{so } E_3 = \frac{\rho (R_2^2 - R_1^2)}{2\epsilon_0 r}$$

- The electric potential $v(r)$ at any point in space.

$$\vec{E} = -\text{grad}v \Rightarrow E = -\frac{dv}{dr} \quad \text{so } v = -\int E dr$$

1st case : $r < R_1$

$$E_1 = 0 \Rightarrow v_1 = C_1$$

2nd case $R_1 \leq r < R_2$

$$E_2 = \frac{\rho}{2\epsilon_0} \left(r - \frac{R_1^2}{r} \right) \Rightarrow v_2 = -\frac{\rho}{2\epsilon_0} \left(\int r dr - R_1^2 \int \frac{1}{r} dr \right) \quad \text{and } v_2 = -\frac{\rho}{2\epsilon_0} \left(\frac{r^2}{2} - R_1^2 \ln r \right) + C_2$$

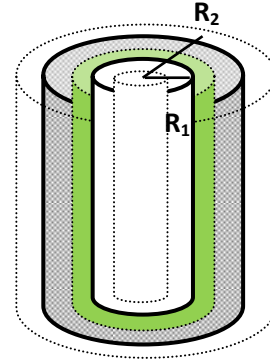
3rd cas $r \geq R_2$

$$E_3 = \frac{\rho (R_2^2 - R_1^2)}{2\epsilon_0 r} \Rightarrow v_3 = -\frac{\rho (R_2^2 - R_1^2)}{2\epsilon_0} \int \frac{1}{r} dr \quad \text{so } v_3 = -\frac{\rho (R_2^2 - R_1^2)}{2\epsilon_0} \ln r + C_3$$

Exercise 2:

The Gauss surface is a sphere with center O and radius r.

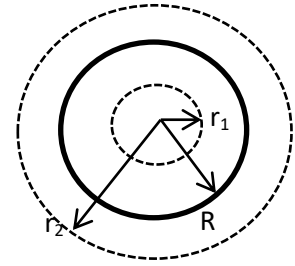
Due to symmetry, the field is radial and constant at any point on the Gauss surface.





The flux through the Gauss.

$$\phi = \iint \vec{E} \cdot \vec{ds} = \sum \frac{Q_{int}}{\epsilon_0}$$



1- Electric Field

Charge volume density $\rho = \frac{A}{r}$

$$\left\{ \begin{array}{l} \iint \vec{E} \cdot \vec{ds} = \sum \frac{Q_{int}}{\epsilon_0} \\ \vec{E} \parallel \vec{ds} \text{ and } E = cst \end{array} \right. \Rightarrow \iint E \cdot ds = E \cdot 4\pi r^2 = \sum \frac{Q_{int}}{\epsilon_0}$$

We have 2 cases :

1st case : $r < R$

$$\left\{ \begin{array}{l} dq = \rho dv \\ \rho = \frac{A}{r} \\ dv = 4\pi r^2 dr \end{array} \right. \Rightarrow \int dq = 4\pi \int_0^{r_1} \frac{A}{r} r^2 dr$$

So $Q_{int} = 2\pi A r_1^2$ where $E_1 4\pi r_1^2 = \frac{2\pi A}{\epsilon_0} r_1^2 \Rightarrow E_1 = \frac{A}{2\epsilon_0}$

2nd case : $r \geq R$

$$\int dq = 4\pi \int_0^R \frac{A}{r} r^2 dr \Rightarrow Q_{int} = 2\pi A R^2 \text{ where } E_2 4\pi r_2^2 = \frac{2\pi A}{\epsilon_0} R^2 \Rightarrow E_2 = \frac{AR^2}{2r_2^2 \epsilon_0}$$

2- The potentiel

$$\vec{E} = -\overrightarrow{grad}v \Rightarrow E = -\frac{dv}{dr} \text{ so } v = -\int E dr$$

1st case : $r < R$

$$E_1 = \frac{A}{2\epsilon_0} \Rightarrow v_1 = -\frac{A}{2\epsilon_0} \int dr \text{ so } v_1 = -\frac{A}{2\epsilon_0} r + c_1$$

2nd case : $r \geq R$

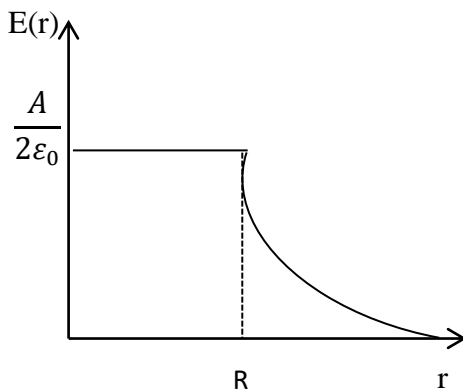
$$E_2 = \frac{AR^2}{2r^2 \epsilon_0} \Rightarrow v_2 = -\frac{AR^2}{2\epsilon_0} \int \frac{dr}{r^2} \text{ so } v_2 = \frac{AR^2}{2r\epsilon_0} + c_2$$

At infinity, the potential is zero: $\lim_{r \rightarrow \infty} v = 0 \Rightarrow c_2 = 0$ so $v_2 = \frac{AR^2}{2r\epsilon_0}$

The potential is a continuous function, so : $v_1(R) = v_2(R)$

$$\frac{AR}{2\epsilon_0} = -\frac{A}{2\epsilon_0} R + c_1 \Rightarrow c_1 = \frac{AR}{\epsilon_0} \text{ so } v_1 = -\frac{A}{2\epsilon_0} r + \frac{AR}{\epsilon_0}$$

3- The graph $E(r)=f(r)$



the graph $v(r)=f(r)$

