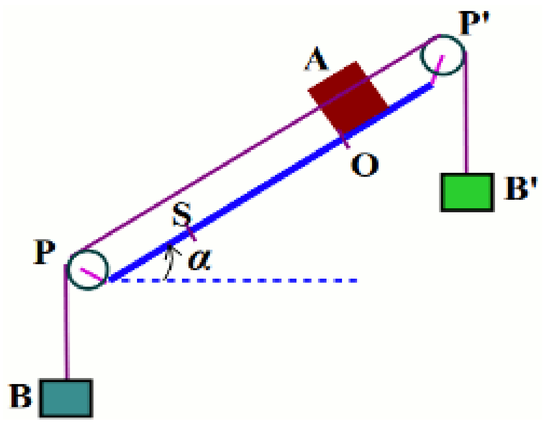
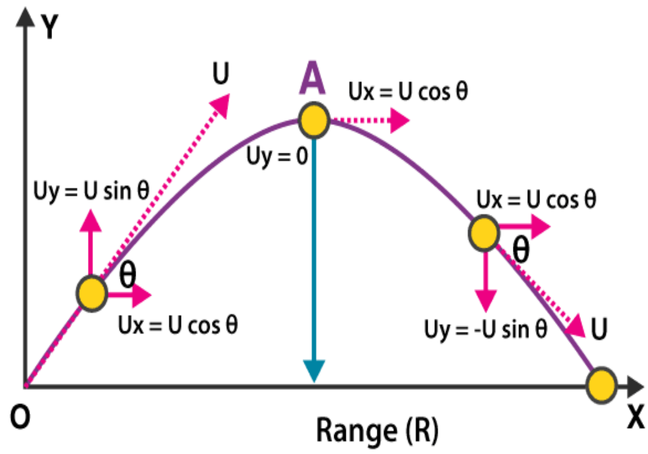
People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
University of Tlemcen, Abou Bakr Belkaïd
Faculty of Technology

Problems and Solved Exercises in Mechanics of material point

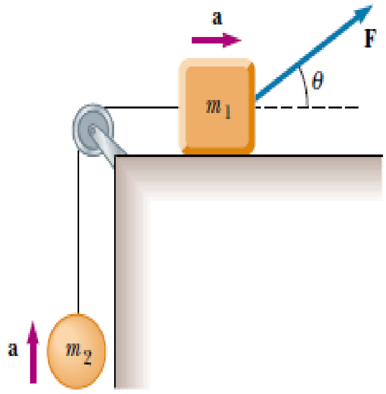
Intended for 1st year students (L.M.D), Industrial Engineering

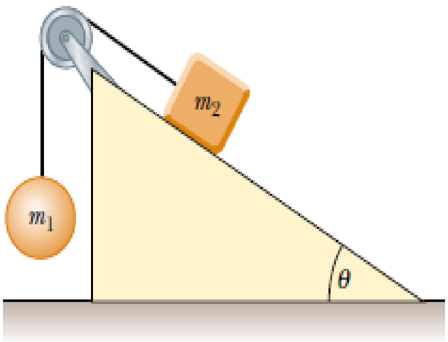
Prepared by: Dr Hicham Kadraoui





Academic year: 2023/2024


a
F
θ
m_1
a
m_2


m_2
m_1
θ

FOREWORD

The classical point mechanics module is the fundamental module taught in physics, this comes down to the fact that it serves as a basis for the other modules. His field of interest is the study of the movements of different objects as well as the causes which provoke them.

This manuscript is an educational support intended for students in the first year of the degree in industrial engineering. It includes exercises solved on the different chapters of the Physics 1 (Mechanics of points). These exercises cover the four chapters of the program course in the mechanics of the material point:

- ✓ Mathematical Reminder
- ✓ Kinematic of the material point.
- ✓ Dynamics of the material point.
- ✓ Work and energy.

All exercises and problems, followed by detailed answers to allow students to understand the concepts introduced in the section containing course reminders.

I must emphasize that this document does not in any way replace the face-to-face tutorial. As with all self-correcting exercises, the solutions are most beneficial to students who make the necessary effort to think and try to solve the proposed exercises.

I hope that this collection of solved exercise in mechanics of the material point can effectively help the majority of students. All constructive comments and suggestions from colleagues and students are welcome.

H.KADRAOUI

CONTENT

Unit	Chapter	Page	

SYLLABUS

Unit-I: Mathematical Reminder — 02 week

Chapter-1: Dimensional analysis

Physical Quantity, Fundamental and Derived Units, Dimensions of a Physical Quantity, uncertainty and precision of measuring instruments; measurement errors

Chapter-2: Vector calculation

Scalar and vector quantities ,Dot product, cross product, general vectors and their notations, equality of vectors; multiplication of vectors by a real number, addition and subtraction of vectors;

Unit-II: Kinematics — 05 week

Chapter-3: Kinematics

Position vector in coordinate systems (Cartesian, cylindrical, spherical, curvilinear) - law of motion – Trajectory. Velocity and acceleration in coordinate systems. Applications: Movement of the material point in the different coordinate systems.

Chapter-4: Relative Motion

Relative motion

Unit-III: Dynamics — 04 week

Chapter-5: Dynamics

Forces and newton's laws, Applications of the fundamental law of dynamics, conservation of momentum

Unit-IV: Work, Energy — 04 week

Chapter-6: Work, Energy

Work and energy, kinetic energy, potential energy, total energy conservation

---

Subject: Physics, Code: GI112

Credits: 6, Coefficient: 3

Evaluation method: CM: 40%; Exam: 60%.

Chapter1

DIMENSIONAL ANALYSIS

Learning Goals: After going through this chapter, students will be able to

- Understand physical quantities, fundamental and derived dimensions.
- Associate physical quantities with their International System of Units (SI) and perform conversions among SI units.
- Understand the principle of dimensional homogeneity, and show how it is applied to any equations involving physical quantities in order to identify the dimension of any physical quantities and check the correctness of equations.
- Write dimensional equations and apply these to verify various formulations.
- Define what an error is; explain the effect errors can have on precision and accuracy.
- Calculate the uncertainties in measurements.

CHAPTER 1 EXERCISES

Exercise n°1:

Using the dimensions of the basic quantities, complete the following table:

Physical Quantity	Symbol	Formula used	Dimension	Unit (IS)
Speed or velocity	v			
Angular velocity	ω			
Acceleration	a			
Force	F			
Pressure , Stress	p			
Work	W			
Energy (all form)	E			
Electrical charge	q			
Electric field	$\vec{E}$			
Electric Potential	u			
Electrical resistance	R			
Electric power	P_e			

Exercise n°2:

1- Find the dimensions of the following quantities: (G) the gravitational constant, $(\varepsilon_0)$the vacuum permittivity and(μ_0) the vacuum magnetic permeability, knowing that these three constants appear in the following equations:

$$F = -G.\frac{m_1.m_2}{r^2} \quad .... (1), \qquad F = \frac{1}{4\pi.\varepsilon_0}.\frac{q_1.q_2}{r^2} \quad .... (2) \qquad \text{and} \qquad F = \frac{\mu_0}{2\pi}.\frac{L.I^2}{r} \quad .... (3)$$

Where: (F) is a force, (m_1) and (m_2) are masses, (q_1) and (q_2) electric charges, (L) and (r) distances and (I) Electric Current Intensity.

2- The speed of light (c) is a function of these three constants:$\left(c = k.G^{\alpha}.\varepsilon_0{}^{\beta}.\mu_0{}^{\gamma}\right)$.

Find the expression for c.

Exercise n°3:

1- The vacuum permittivity and the vacuum magnetic permeability are related by the relation, $\varepsilon_0 \mu_0 c^2 = 1$ where c is the speed of propagation of light in a vacuum. By dimensional analysis, check the homogeneity of this relationship using the results of the previous exercise.

2- Check the correctness of the following formulae by dimensional analysis.

- The expression of instantaneous velocity as a function of time, $v = v_0 + at^2$.
- The centripetal force $F = m\,\omega^2\, r$
- The expression of kinetic energy of a particle $\frac{1}{2}mv^2 = \sqrt{mgh}$.

3- Using the principle of homogeneity of dimensions, determine the dimension and unit of the physical quantities, F, g and h in the fallowing homogeneous equation,

$$\vec{F} = 2m\frac{d\vec{v}}{dt} - g\cos(\omega t)\vec{i} - \lambda h \exp(-\lambda r)\vec{j}$$

Where $m, t, r, v, \omega, \vec{i}, \vec{j}$ are physical quantities representing, mass, time, distance, linear velocity, angular velocity and unit vectors respectively

Exercise n°4:

The oscillation period of a simple pendulum depends on its length, the gravity field g and $\theta_{\max}$ the angular amplitude of the oscillations. We offer several formulas; specify which are the homogeneous formulas:

1) $T = 2\pi.\sqrt{\frac{l+\theta_{\max}}{g-\theta_{\max}}}$

2) $T = 2\pi.\sqrt{\frac{l}{g\theta_{\max}}}$

3) $T = 2\pi.\sqrt{\frac{l}{g}}\left(1+\frac{\theta_{\max}^2}{16}\right)$

4) $T = 2\pi.\sqrt{\frac{l}{g}}\left(1+\frac{\theta_{\max}}{l}\right)$

Exercise n°5:

One of the four relations below gives the volume of a conical trunk of height h, radius r and R. From a dimensional analysis, find it

1) $V = 2\pi(R-r)\frac{h}{3}$

2) $V = \frac{\pi h}{3}\left(r^2 - R + R^2\right)$

3) $V = \frac{\pi h}{3}\left(r^2 + rR + R^2\right)$

4) $V = \frac{\pi h}{3}\left(r - \frac{r}{R}\right)^2$

Exercise n°6:

Specify which of these formulas are homogeneous: P the momentum (mass multiplied by speed), m the mass, c the speed of light and E is energy.

1) $E = \sqrt{pc^2 + m^2c^4}$
2) $E^2 - \frac{P^2C^4}{m} = m^2$
3) $E = \sqrt{p^2c^2 + m^2c^4}$
4) $E = \sqrt{p^2c^2 + m^2c^2}$

Exercise n°7:

In an optics laboratory we use a halogen type lamp which allows very intense lighting. The law which governs the light emission of this lamp was given by Wilhelm Wien in 1896;

it is written: $\phi_\lambda = \frac{A}{\lambda^5}\frac{1}{e^{\frac{B}{\lambda T}} - 1}$

Where ϕ_λ is called spectral exitance, A and B are constants, λ the wavelength in a vacuum and T the temperature.

1. What is the dimension of the constant B? Give your unity in the international system of units.

2. In the International System of Units the constant A is expressed in $m^4.kg.s^{-3}$. What is the unit SI of spectral emitting? We will give the result using the watt and the meter.

Exercise n°8:

The frequency (ν) of a musical note emitted by a guitar depends on the tension (T) of the wire, its density (ρ) as well as its length (l) according to the law: $\nu = KT^{\alpha}\rho^{\beta}l^{\gamma}$

Where k is a dimensionless constant.

- Determine α, β and γ using the dimensional equations.

Exercise n°9:

A cylindrical container of radius R contains a height H of an incompressible liquid. This container is made to rotate around its axis of symmetry at an angular velocityω. The minimum pressure experienced by the bottom of the container is givenby the relationship:

$$P = Kg\left(H - \frac{\omega^2R^2}{4K'}\right)$$

Where g is the acceleration of gravity.

Determine the dimensions of the quantities $K$and K'.

Exercise n°10:

We suppose that $F^x \rho^y v^z = (n + tg\theta)m^3$ is the formula that links the force F, density ρ, and velocity v to the mass m. If x, y, and z are real numbers, find:

1. The values of x, y and z?
2. The dimension of n?

Exercise n°11:

A ball of radius r moving with speed v in a fluid, is subjected to a friction force given by $F = -6\pi\eta r v\eta$, where η is the viscosity.

1) What is the dimension of η ?

2) When the ball is released without initial speed at instant $\mathrm{t} = 0$, its speed is written for $\mathrm{t} > 0$: $v = a(1 - \exp\left(-\frac{\mathrm{t}}{b}\right)$ where a and b are two quantities which depend on the characteristics of the fluid. What are the dimensions of a and b

3) If ρ denotes the density of the fluid, find a simple combination $Re = \rho^\alpha v^\beta r^\gamma \eta^\delta$ which is dimensionless (among the different possible choices we will take $= 1$). Thus, we determine the Reynolds number, which allows us to describe the fluid's flow regime (laminar or turbulent).

Exercise n°12:

A. Using the fundamental quantities of electrical conductivity at a conductor's point, ascertain the dimension by confirming "the local form of Ohm's law" $\vec{J} = \gamma\vec{E}$; knowing that the current density vector can also be expressed as a function of the volume density of mobile charges and the speed of a carrier, by the relation: $\vec{J} = \rho\vec{v}$.

B. Check the homogeneity of the following equations:

- Time constant of the dipole $\tau = RC$
- Time constant of the dipole $\tau = \frac{L}{R}$
- Period of an electric oscillator $\tau = \frac{2\pi}{\sqrt{LC}}$.

C. Using a quick dimensional analysis, for each of the following literal expressions, circle the result likely to be correct.

1. Maximum height attained by a projectile of mass m launched vertically at speed v, g is the acceleration of gravity:

$h = \frac{mv^2}{g}$	$h = \frac{1}{2}gt^2 + vt$	$h = \frac{v^2}{mg}$

2. Horizontal range x of the firing of mass projectile m whose initial speed makes an angle with the horizontal:

$x=\frac{mv^2\sin 2\alpha}{g}$	$x=\frac{g}{m{v_0}^2\cos\alpha^2}+\tan(\alpha)x$	$x=\frac{v\tan(2\alpha)}{2g}$

3. Altitude h of a satellite in circular orbit around the Earth of radius R, knowing the period T and acceleration of gravity g at ground:

$h=\sqrt[3]{\frac{T^2R^2g}{4\pi}}-R$	$h=\sqrt[3]{\frac{T^2R^2g}{4\pi^2}}-R$	$h=\sqrt[3]{\frac{T^4Rg^2}{4\pi^2}}-R$

4. The Voltage U within an electrical circuit (with E is a voltage, R_1, R_2, R_3 are electrical resistances:

$U=\frac{R_1R_2E}{R_1R_2+R_3(1+R_2)}$	$U=\frac{R_1E}{R_1R_2+R_3(1+R_2)}$	$U=\frac{R_1R_2E}{R_1R_2+R_3(R_1+R_2)}$

Exercise n°13:

A student noticed, in a physics laboratory, that the position **x** of an electron as a function of its acceleration (a) and time (t) is written in the form $x=K\cdot a^{\alpha}\cdot t^{\beta}$, K is a dimensionless constant.

1. Determine the exact expression of the position by calculating α and β**.**
2. Using the logarithmic method, find the relative uncertainty about the position x based on Δa and Δt .

Exercise n°14:

Find the dimension of (G) with $G=\frac{(t^2.g.x)}{4\pi}-x^2$ where **t** denotes time, **x** is a length and g is an acceleration of gravity

1- Find the absolute uncertainty of the physical quantity (G).

2- Find the relative uncertainty of the centripetal force $F=m\frac{v^2}{R}$

Exercise n°15:

The vibration frequency of a drop of water can be written in the form: $f=KR^{\alpha}\rho^{\beta}\tau^{\gamma}$ where K is a dimensionless constant. R is the radius of the drop, ρ its density. τ is the surface tension defined as force per unit length.

- Determine by dimensional analysis the values of the parametersα,β and γ.
- Determine the relative error on f as a function of ΔR, $\Delta\rho$ and $\Delta\tau$.

Exercise n°16:

The energy of a photon is given by the expression,$E = h\nu$ Or h is Planck's constant and ν the frequency of the photon.

1- Give the dimension of h
2- Find the expression for the wavelength by assuming the form $\lambda = Kh^x m^y v^z$ Or K is a constant, m and v represent respectively the mass and the speed of the photon.
3- Determine the relative error on λ as a function of Δm, Δh and Δv.

Exercise n°17:

1-The ideal gas law states that$(pV = nRT)$, where (p) is the absolute pressure of gas, (V) is the volume it occupies, (n) is the number of atoms and molecules in the gas, and (T) its absolute temperature.

a) What is the dimension of the universal gas constant R?
b) What is its unity in the international system?

2-The equation of state of a real gas molecule is given by:

$$p = \left(\frac{R.T}{V - b}\right).\exp\left(\frac{-a}{R.T.V}\right)$$

In this expression (a) and (b) are coefficients such that a=3.5 .10^{-3} SI and b=2.5 .10^{-5} SI.

- Give the equations for the dimensions of (a) and(b).

Exercise n°18:

Kepler's third law, which relates the period (T) and the semi-major axis (a) of the orbit of a planet around the sun, is written in the form: $\frac{T^2}{a^2} = \frac{4\pi^2}{G \cdot M_S}$

Where (G) is the universal gravitational constant and (Ms) the mass of the sun. For planet earth, T=(365,25636567±0,00000001) days and a=(1.4960±0.0003).10^{+11} m.

We also give G=(6.668±0.005).10^{+11} SI.

1- Determine the dimension and unit of (G).
2- Calculate the mass of the sun and the absolute error of this mass (ΔM_s) using the total differential.

Exercise n°19:

A sphere of radius a and density ρ, falling into a viscous medium with coefficient of viscosity η and density ρ' has a limiting speed v, which can be found using the following formula:

$$v = \frac{1}{9}\frac{a^2 g(\rho - \rho')}{\eta}$$

Where g is the acceleration of gravity.

1- Check the consistency of this formula.
2- Using the dimensional equations, determine the dimension of η .
3- Determine the relative error on η as a function of Δa, $\Delta\,\rho$, $\Delta\,\rho'$ and $\Delta\,v$.

Exercise n°20:

1) A liquid's flow speed depends on its density ρ and pressure difference (ΔP) in accordance with the following relationship:

$$v = K\sqrt{\frac{2\Delta P}{\rho}}$$

Where K is a dimensionless constant.

Check the homogeneity of the relationship.

2) Let X represents a gas's compressibility. We consider that is homogeneous at $\frac{1}{P}$, Where P is the pressure.
   - Find the dimension of X .
3) Based on experiment, the speed of sound in a gas is determined by:

$$v = K\rho^{\alpha}X^{\beta}$$

- Determine the expression of v by using the dimensional equations.

4) Determine the relative error on v as a function of $\Delta\rho$ and ΔX.

SOLUTIONS TO EXERCISES

Exercise n°01:

Physical Quantity	Symbol	Formula used	Dimension	Unit (SI)
Speed or velocity	v	$v = x/t$	$[v] = L.T^{-1}$	$m.s^{-1}$
Angular velocity	ω	$v = \omega.r$	$[\omega] = T^{-1}$	$rad.s^{-1}$
Acceleration	a	$a = v/t$	$[a] = L.T^{-2}$	$m.s^{-2}$
Force	F	$F = m.a$	$[F] = M.L.T^{-2}$	kg m s^{-2} Newton (N)
Pressure	P	$p = F/s$	$[p] = M.L^{-1}.T^{-2}$	kg $m^{-1}s^{-2}$ Pascal (Pa)
Work	W	$W = F.l$	$[W] = M.L^2.T^{-2}$	kg m^2 s^{-2} Joule (J)
Energy	E	$E = 1/2 .m.v^2$	$[E] = M.L^2.T^{-2}$	kg m^2 s^{-2} Joule (J)
Electrical charge	q	dq = i.dt	$[q] = I.T$	Coulomb
Electric field	E	F = q E	$[E] = M.L.T^{-3}.I^{-1}$	Volt/m
Electric Potential	u	u= r E	$[u]=M^r. L^2.T^{-3} I^{-1}$	Volt
Electrical resistance	R	R=u/i	$[R] = M.L^2.T^{-3}.I^{-2}$	Ohm
Electric power	P	$P = R.i^2$	$[P] = M.L^2.T^{-3}$	Watt

Exercise n°02:

1)

a) [G]= ?

$$G = \frac{-F.r^2}{m_1.m_2} \Rightarrow [G] = \frac{[-1].[F].[r^2]}{[m_1].[m_2]} \Rightarrow [G] = M^{-1}.L^3.T^{-2}$$

b) $[\varepsilon_0] = ?$

$$F = \frac{1}{4\pi\varepsilon_0}.\frac{q_1.q_2}{r^2} \Rightarrow \varepsilon_0 = \frac{q_1.q_2}{4\pi.F.r^2} \Rightarrow [\varepsilon_0] = \frac{[q_1].[q_2]}{[4\pi].[F].[r]^2} \Rightarrow [\varepsilon_0] = M^{-1}.L^{-3}.T^4.I^2$$

c) $[\mu_0] = ?$

$$F = \frac{\mu_0}{2\pi}.\frac{L.I^2}{r} \Rightarrow \mu_0 = \frac{2\pi.F.r}{L.I^2} \Rightarrow [\mu_0] = \frac{[F].[2\pi].[r]}{[L].[I]^2} \Rightarrow [\mu_0] = M.L.T^{-2}.I^{-2}$$

2) Expression of (c):

$c = k.G^{\alpha}.\varepsilon_0^{\beta}.\mu_0^{\gamma} \quad \Rightarrow \quad [c]=[k].[G]^{\alpha}.[\varepsilon_0]^{\beta}.[\mu_0]^{\gamma}$

With, c: speed$\Rightarrow \quad [c]=L.T^{-1}$

By identification: α=0 and β = γ =$-\frac{1}{2}$

Finally $c = k.G^{\alpha}.\varepsilon_0^{\beta}.\mu_0^{\gamma} = k.G^{0}.\varepsilon_0^{-1/2}.\mu_0^{-1/2} = k.\frac{1}{\varepsilon_0^{+1/2}.\mu_0^{+1/2}} \quad \Rightarrow \quad c = k.\frac{1}{\sqrt{\varepsilon_0.\mu_0}}$

Exercise n°03

1- $\varepsilon_0\mu_0 c^2 = 1$

$[\varepsilon_0][\mu_0][c^2] = M^{-1}L^{-3}I^{2}T^{4}.MLI^{-2}T^{-2}.L^{2}T^{-2} = 1 \Rightarrow \varepsilon_0\mu_0 c^2 = 1$ is homogeneous.

2-

1) $\boldsymbol{v = v_0 + at^2}$

Dimensions of various quantities in the equation are:

Velocity, $[v]$ =[v_0] = LT^{-1} Time, $[t]^2 = T^2$, acceleration, [a] = LT^{-2}

Thus, the dimension of the term on L.H.S is $[v]$ = LT^{-1} and dimensions of terms on R.H.S is$[at^2] = LT^{-2}\,T^2 = L$

Here, the dimensions of all the terms on both sides of the equation are not the same. Therefore, the equation is non-homogeneous.

2) $F = m\,\omega^2\, r$

Dimensions of the term on L.H.S. Force, [F] = MLT^{-2}

Dimensions of the term on R.H.S. $[m][\omega]^2[r] = MT^{-2}L = [F]$

The dimensions of the term on the L.H.S are equal to the dimensions of the term on R.H.S. Therefore, the relation is correct.

3) $\frac{1}{2}mv^2 = \sqrt{mgh}$

Dimensions of the term on L.H.S. Force, $[\frac{1}{2}mv^2] = ML^2T^{-2}$

Dimensions of the term on R.H.S.

$[\sqrt{m.g.h}] = [m.g.h]^{1/2} = M^{1/2}(LT^{-2})^{1/2}L^{1/2} = M^{1/2}LT^{-1} \neq \left[\frac{1}{2}mv^2\right]$

The dimensions of all the terms on both sides of the equation are not the same. Therefore, the equation is non-homogeneous.

4) The equation is homogeneous, so we can write, on the one hand,

$[F] = [m]\frac{[v]}{[t]} = [g] = [\lambda][h]$ and on the other hand the argument of the function exp(-λr) is dimensionless, which allows us to write:

$$[\lambda][r]=1 \Rightarrow [\lambda]=L^{-1}$$

Therefore as a result, $[F]=[g]=MLT^{-2}$

$[h]=ML^2T^{-2}$ Its SI unit is kgm^2s^{-2}, Joule (J)

Exercise n°04:

1. $T=2\pi.\sqrt{\dfrac{l}{g\theta_{\max}}}$ is dimensionally correct if:

$$[T]=[2\pi].\left[\sqrt{\frac{l}{g\theta_{\max}}}\right]$$

The dimension of the first term of the expression is: T : Period → $[T] = T$

The dimension of the second term of the expression is:

2π : Constant → $[2\pi] = \theta_{max} = 1$, l: length → [l]=L , g: gravity → $[g]=LT^{-2}$

Then we find:

$$[2\pi].\left[\sqrt{\frac{l}{g\theta_{\max}}}\right]=[2\pi][l]^{\frac{1}{2}}[g]^{\frac{-1}{2}}[\theta_{\max}]^{\frac{-1}{2}}=L^{(1/2)}L^{(-1/2)}T^{1}=T=L^{\frac{1}{2}}L^{\frac{-1}{2}}T=T$$

$$\Rightarrow \left[2\pi\sqrt{\frac{l}{g\theta_{\max}}}\right]=T$$

So the two terms have the same dimension, that of a time. We conclude that the expression is homogeneous.

2. In the same way the expression $T=2\pi.\sqrt{\dfrac{l}{g}}\left(1+\dfrac{\theta_{\max}^2}{16}\right)$ is homogeneous.

In $T=2\pi.\sqrt{\dfrac{l+\theta_{\max}}{g-\theta_{\max}}}$ and $T=2\pi.\sqrt{\dfrac{l}{g}}\left(1+\dfrac{\theta_{\max}}{l}\right)$, the dimensions of all the terms on both sides of the equations are not the same. Therefore, the expressions are non-homogeneous.

Exercise n°05:

$V=\dfrac{\pi h}{3}\left(r^2+rR+R^2\right)$ is dimensionally correct if:

$$[V]=\left[\frac{\pi}{3}\right][h][R]^2=1.L.L^2=L^3$$

The two terms have the same dimension. We conclude that the expression is homogeneous. In other expressions, the dimensions of all the terms on both sides of the equations are not the same. Therefore, the expressions are non-homogeneous.

Exercise n°06:

$E = \sqrt{pc^2 + m^2c^4}$

$$\begin{cases} E^2 = pc^2 + m^2c^4 \\ \Rightarrow [pc^2 + m^2c^4] = [p][c]^2 + [m]^2[c]^4 \\ [E^2] = (ML^2T^{-2})^2 \end{cases}$$

$= [mv][c]^2 + [m]^2[c]^4$

$= (MLT^{-1})(LT^{-1})^2 + M^2L^2T^{-2}$

$(ML^2T^{-2})^2 \neq (ML^3T^{-3}) + M^2L^2T^{-2}$

So the expression is not homogeneous.

$E = \sqrt{p^2c^2 + m^2c^4}$

$E^2 = p^2c^2 + m^2c^4$

With $\begin{cases} [E^2] = (ML^2T^{-2})^2 \\ [p^2c^2 + m^2c^4] = [p]^2[c]^2 + [m]^2[c]^4 \end{cases}$

Knowing that

$$\begin{cases} [m]^2[c]^4 = (M)^2(LT^{-1})^4 = M^2L^4T^{-4} \\ [mv]2\ [c]^2 = (MLT^{-1})^2(LT^{-1})^2 = M^2L^4T^{-4} \end{cases} \Rightarrow$$

$$\begin{cases} [E^2] = M^2L^4T^{-4} \\ [p^2c^2 + m^2c^4] = M^2L^4T^{-4} + M^2L^4T^{-4} \end{cases}$$

It is homogeneous.

Exercise n°07:

1. The argument of the function exp $(\frac{B}{\lambda T})$ is dimensionless, which allows us to write:

$$\left[\frac{B}{\lambda T}\right] = 1 \Rightarrow [B] = [\lambda][T] = L\theta$$, its SI unit is mK

2. $\phi_\lambda = \frac{A}{\lambda^5}\frac{1}{e^{\frac{B}{\lambda T}} - 1}$ is dimensionally correct if:

$$[\phi_\lambda] = \frac{[A]}{[\lambda]^5}\left[\frac{1}{e^{\frac{B}{\lambda T}} - 1}\right] \Rightarrow [\phi_\lambda] = \frac{[A]}{[\lambda]^5} = \frac{L^4MT^{-3}}{L^5} = \frac{MT^{-3}L^2}{L^3}$$, its SI unit is W m^{-3}

Exercise n°08:

$\nu = KT^{\alpha}\rho^{\beta}l^{\gamma}$

$\nu = \frac{1}{T} \Rightarrow [\nu] = \frac{1}{[T]} = \frac{1}{T} = T^{-1} \Rightarrow [\nu] = T^{-1}$

$$\nu = KT^{\alpha}\rho^{\beta}l^{\gamma} \Rightarrow [\nu]=[K][T]^{\alpha}[\rho]^{\beta}[l]^{\gamma}$$

$\rho = \frac{m}{v} \Rightarrow [\rho] = \frac{[m]}{[v]} = ML^{-3}$, T : Tension (force) $\Rightarrow$[T] = MLT^{-2}, l : length [l] = L

$$[\nu]=\left(MLT^{-2}\right)^{\alpha}\left(ML^{-3}\right)^{\beta}(L)^{\gamma} \Rightarrow T^{-1} = M^{x}L^{x}T^{-2\alpha}M^{y}L^{-3\beta}L^{\gamma}$$

$$\Rightarrow T^{-1} = M^{\alpha+\beta}L^{\alpha-3\beta+\gamma}T^{-2\alpha}$$

By identification:

$$\begin{cases} \alpha+\beta=0 \\ \alpha-3\beta+\gamma=0 \\ -2\alpha=-1 \end{cases} \Rightarrow \begin{cases} \beta=-\alpha=-\frac{1}{2} \\ \gamma=-2 \\ \alpha=\frac{1}{2} \end{cases}$$

$$\nu = KT^{1/2}\rho^{-1/2}l^{-2} \Rightarrow \nu=\frac{K}{l^{2}}\sqrt{\frac{T}{\rho}}$$

Exercise n°09:

$$P = Kg\left(H-\frac{\omega^{2}R^{2}}{4K'}\right)$$

-$[K]$ =? and $[K']$ =?

- We know that two physical quantities only add up if they have the same dimension

$$[H]=\frac{[\omega]^{2}[R]^{2}}{[4][K]'}$$

$$\Rightarrow [K'] = \frac{[\omega]^{2}[R]^{2}}{[4][H]}$$

$$\begin{cases} \omega : \text{angular velocity} \Rightarrow \omega = T^{-1} \\ R : \text{radius} \Rightarrow [R]=L \\ H : \text{height} \Rightarrow [H]=L \\ 4 : \text{number} \Rightarrow [4]=1 \end{cases}$$

$$\Rightarrow [K'] = \frac{(T^{-1})^{2}(L)^{2}}{1.L}$$

$$\Rightarrow [K'] = LT^{-2}$$

We also know that:

$$[P]=[K][g]\,[H] \Rightarrow [K]=\frac{[P]}{[g]\,[H]}$$

$$\begin{cases} p : \text{pressure} \Rightarrow [p]=M.L^{-1}.T^{-2} \\ g : \text{acceleration} \Rightarrow [g]=LT^{-2} \\ H : \text{height} \Rightarrow [H]=L \end{cases}$$

$$\Rightarrow [K] = ML^{-3}$$

Exercise n°10:

$$F^x \rho^y v^z = (n + tg\theta) m^3$$

$$\Rightarrow [\,F]^x\ [\,\rho]^y\ [\,v]^z = [\,n.m^3] = [tg\theta.m^3]$$

$$\Rightarrow \begin{cases} [F]^x[p]^y[v]^z = [tg\theta.m^3]\ .\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1) \\ et \\ [F]^x[p]^y[v]^z = [n.m^3]\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2) \end{cases}$$

$$(1) \Rightarrow (MLT^{-2})^x (ML^{-3})^y (LT^{-1})^z = [tg\theta][m]^3 = 1.M^3$$

$$M^{x+y} L^{x-3y+z} T^{-2x-z} = M^3 L^0 T^0$$

$$\Rightarrow \{x - 3y + z = 0 - 2x - z = 0x + y = 3 \Rightarrow \begin{cases} x = \frac{9}{2} \\ y = \frac{-3}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3) \\ = -9 \end{cases}$$

$$(2) \Rightarrow [F]^x[p]^y[v]^z = [n.m^3]$$

$$\Rightarrow (MLT^{-2})^x (ML^{-3})^y (LT^{-1})^z = [n][m]^3 = [n]M^3$$

$$\Rightarrow [n] = (MLT^{-2})^x (ML^{-3})^y (LT^{-1})^z M^{-3}$$

$$\Rightarrow [n] = M^{x+y-3} L^{x-3y+z} T^{-2x-z}\ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ .(4)$$

Replacing values of x,y and z ((3) in (4)) :

$[\,n] = M^0 L^0 T^0 = 1$, so n is dimensionless.

Exercise n°11:

1) $F = -6\pi\eta r v \Rightarrow \eta = \frac{F}{6\pi r v} \Rightarrow [\eta] = \frac{[F]}{[r][v]} == \frac{M.\cdot(L.\cdot T^{-2})}{L(LT^{-1})}$

$$= M.L^{-1}.T^{-1}$$

2) The argument of the exponential is dimensionless so [b] $= T$. The right hand side has the dimension of a speed so [a] $= L.T^{-1}$.

3) $Re = p^\alpha v^\beta r^\gamma \eta^\delta \Rightarrow [Re] = [p]^\alpha [v]^\beta [r]^\gamma [\eta]^\delta = (M.L^{-3})^\alpha\ (L.T^{-1})^\beta L^\gamma\ (M.L^{-1}T^{-1})^\delta$

$$\Rightarrow [Re] = M^{\alpha+\delta}.L^{-3\alpha+\beta+\gamma-\delta}.T^{-\beta-\delta}$$

Re is dimensionless (according to the statement),taking $\alpha = 1$, we can deduce:

$$\{-3 + \beta + \gamma - \delta - \beta - \delta = 01 + \delta = 0 = 0 \Rightarrow \begin{cases} \delta = -1 \\ \beta = 1 \\ \gamma = 1 \end{cases}$$

Finally: $Re = \frac{pvr}{\eta}$

Exercise n°12:

A.

$\vec{J} = \gamma \vec{E}$

$\vec{J}$: Surface current density

γ: Electrical conductivity

$\vec{E}$: Electric field

$\vec{J} = \rho\vec{v}$

$[\rho] = \frac{[q]}{[Volume]}$

$\vec{J}$: Surface current density

ρ: The volume charge density

$\vec{v}$: Drift velocity

$q = I.t \Rightarrow [q] = [I].[t] \Rightarrow [q] = IT$

$\Rightarrow [\rho] = L^{-3}IT$

$[\vec{J}] = [\rho][\vec{v}] \Rightarrow [\vec{J}] = L^{-3}IT.LT^{-1}$

$\Rightarrow [\vec{J}] = L^{-2}I$

$E = \frac{F}{q} \Rightarrow [E] = \frac{[F]}{[q]} \Rightarrow [E] = \frac{MLT^{-2}}{IT} \Rightarrow [E] = MLT^{-3}I^{-1}$

$\vec{J} = \gamma\vec{E} \Rightarrow \gamma = \frac{\vec{J}}{\vec{E}} \Rightarrow [\gamma] = \frac{[\vec{J}]}{[\vec{E}]} = \frac{L^{-2}I}{MLT^{-3}I^{-1}}$

$\Rightarrow [\gamma] = M^{-1}L^{-3}T^{3}I^{2}$

B.

$U = \vec{E}.l \Rightarrow [U] = [\vec{E}].[l] = MLT^{-3}I^{-1}.L$

$\Rightarrow [U] = ML^{2}T^{-3}I^{-1}$

- $q = CU \Rightarrow [C] = [q].[U]^{-1} = IT.M^{-1}L^{-2}T^{3}I^{1} \Rightarrow [C] = M^{-1}L^{-2}T^{4}I^{2}$
- $U = RI \Rightarrow [R] = [U].[I]^{-1} = ML^{2}T^{-3}I^{-1}.I^{-1} \Rightarrow [R] = ML^{2}T^{-3}I^{-2}$
- $U = -L\frac{dI}{dt} \Rightarrow [L] = [U].[t].[I]^{-1} = ML^{2}T^{-3}I^{-1}T.I^{-1} \Rightarrow [L] = ML^{2}T^{-2}I^{-2}$

Check the homogeneity of :

- Time constant of the dipole

  $\tau = RC \Rightarrow [\tau] = [R].[C] = ML^{2}T^{-3}I^{-2}.M^{-1}L^{-2}T^{4}I^{2} \Rightarrow [\tau] = T$

  ⇒(Homogeneous equation)

- Time constant of the dipole $\tau = \frac{L}{R}$

  $\tau = \frac{L}{R} \Rightarrow [\tau] = [\tau] = \left[\frac{[L]}{R}\right] = ML^{2}T^{-2}I^{-2}.M^{-1}L^{-2}T^{3}I^{2} \Rightarrow [\tau] = T$

  ⇒(Homogeneous equation)

- Period of an electric oscillator $\tau = \frac{2\pi}{\sqrt{LC}}$.

  $\tau = \frac{2\pi}{\sqrt{LC}}. \Rightarrow [\tau] = [\tau] = [L]^{-1/2}[C]^{-1/2} = MLT^{-1}I^{-1}.M^{-1}L^{-1}T^{2}I \Rightarrow [\tau] = T$

  ⇒(Homogeneous equation)

C.

Using a quick dimensional analysis, for each of the following literal expressions, circle the result likely to be correct.

$h=\frac{mv^2}{g}$	$h=\frac{1}{2}gt^2+vt$	$h=\frac{v^2}{mg}$

$x=\frac{mv^2\sin 2\alpha}{g}$	$x=\frac{v\tan(2\alpha)}{2g}$	$x=\frac{g}{m{v_0}^2\cos\alpha^2}+\tan(\alpha)x$

$h=\sqrt[3]{\frac{T^2R^2g}{4\pi^2}}-R$	$h=\sqrt[3]{\frac{T^2R^2g}{4\pi}}-R$	$h=\sqrt[3]{\frac{T^4Rg^2}{4\pi^2}}-R$

$U=\frac{R_1R_2E}{R_1R_2+R_3(1+R_2)}$	$U=\frac{R_1E}{R_1R_2+R_3(1+R_2)}$	$U=\frac{R_1R_2E}{R_1R_2+R_3(R_1+R_2)}$

Exercise n°13:

Expression of (x):

We have: $x=K.a^\alpha.t^\beta \Rightarrow [x]=[K].[a]^\alpha.[t]^\beta$

With, x: length;$\Rightarrow [x]=L$

(a): acceleration;$\Rightarrow [a]=LT^{-2}$, t: time$\Rightarrow [t]=T,\ \ k: constant\ \Rightarrow [K]=1$

$\Rightarrow\ [x]=[K].[a]^\alpha.[t]^\beta$

$\Rightarrow\ L=1.(LT^{-2})^\alpha.(T)^\beta$

$\Rightarrow\ L=L^\alpha T^{-2\alpha+\beta}$

By identification: α=1 and β=-2

Finally: $x=K.a^1.t^2 \Rightarrow x=K.a.t^2$

$x=K.a.t^2$

$x=K.a.t^2 \Rightarrow \log(x)=\log(K.a.t^2)$

$\log(x)=\log(k)+\log(a)+2\log(t)$

Let's derive this expression:

$[\log(x)]'=[\log(K)]'+[\log(a)]'+2[\log(t)]'$

$\Rightarrow\ \frac{dx}{x}=\frac{dK}{K}+\frac{da}{a}+2\frac{dt}{t}$

$\Rightarrow \quad \frac{dg}{g} = \frac{da}{a} + 2\frac{dt}{t} \quad because\ dK = 0\ , K: constant$

Let's move on to absolute notation:

$\Rightarrow \quad \frac{\Delta \boldsymbol{x}}{\boldsymbol{x}} = \frac{\Delta \boldsymbol{a}}{\boldsymbol{a}} + 2\frac{\Delta \boldsymbol{t}}{\boldsymbol{t}}$

Exercise n°14:

1. $G = \frac{t^2.g.x}{4\pi} - x^2$

$$[G] = \frac{[t]^2.[g].[x]}{[4][\pi]} = [x]^2 = L^2$$

2. $\Delta G = ?$

$$dG(\pi, t, g, 4, x) = \left(\frac{\partial G}{\partial \pi}\right) d\pi + \left(\frac{\partial G}{\partial t}\right) dt + \left(\frac{\partial G}{\partial g}\right) dg + \left(\frac{\partial G}{\partial 4}\right) d4 + \left(\frac{\partial G}{\partial x}\right) dx$$

π, 4 and g are constants; $d\pi = d4 = 0$

$$dG(t, g, x) = \left(\frac{\partial G}{\partial t}\right) dt + \left(\frac{\partial G}{\partial x}\right) dx$$

$\left(\frac{\partial G}{\partial t}\right) = ? \quad \Rightarrow \quad \frac{\partial G}{\partial t} = \frac{2.t.g.x}{4\pi} - 0$

$\left(\frac{\partial G}{\partial x}\right) = ? \quad \Rightarrow \quad \frac{\partial G}{\partial x} = \frac{t^2.g}{4\pi} - 2x$

$$dG = \frac{2.t.g.x}{4\pi} dt + (\frac{t^2.g}{4\pi} - 2x) dx$$

$$\Delta G = \left|\frac{t.g.x}{2\pi}\right| \Delta t + \left|\frac{t^2.g}{4\pi} - 2x\right| \Delta x$$

3. $F = m\frac{V^2}{R}$

$F = m\frac{V^2}{R} \quad \Rightarrow \quad \log(F)\log(m\frac{V^2}{R})$

$\log(F) = \log(m) + \log(V)^2 - \log(R)$

Let's derive this expression:

$\Rightarrow \quad [\log(F)]' = [\log(m)]' + [2.\log(V)]' - [\log(R)]'$

$\Rightarrow \quad \frac{dF}{F} = + \frac{d(m)}{m} + 2.\frac{d(V)}{V} - \frac{d(R)}{R}$

Let's move on to absolute notation:

$\Rightarrow \quad \frac{\Delta F}{F} = \frac{\Delta m}{m} + 2.\frac{\Delta V}{V} + |-1|.\frac{\Delta R}{R}$

$$\boxed{\frac{\Delta F}{F} = \frac{\Delta m}{m} + 2.\frac{\Delta V}{V} + \frac{\Delta R}{R}}$$

Exercise n°15:

1- $f = KR^{\alpha}\rho^{\beta}\tau^{\gamma}$

$f = KR^{\alpha}\rho^{\beta}\tau^{\gamma} \Rightarrow [f] = [K][R]^{\alpha}[\rho]^{\beta}[\tau]^{\gamma}$

$[R] = L\,,\ [\rho] = ML^{-3} \quad \Rightarrow [\tau] = \dfrac{[F]}{[l]} = M.T^{-2}$

$T^{-1} = L^{\alpha}M^{\beta}L^{-3\beta}.\,M^{\gamma}T^{-2\gamma}$

$\Rightarrow T^{-1} = M^{\beta+\gamma}L^{\alpha-3\beta}T^{-2\gamma}$

$$\begin{cases} \beta+\gamma = 0 \\ \alpha - 3\beta = 0 \\ -2\gamma = -1 \end{cases} \Rightarrow \begin{cases} \alpha = -3/2 \\ \beta = -1/2 \\ \gamma = 1/2 \end{cases}$$

$\Rightarrow \ f = KR^{-3/2}\rho^{-1/2}\tau^{1/2} \ \Rightarrow \ f = K\dfrac{\tau}{\sqrt{\rho R^3}}$

2-Δf =?

$\Rightarrow \log(f) = \log(K) + \frac{1}{2}\log(\tau) - \frac{1}{2}\log(\rho) - \frac{3}{2}\log(R)$

Let's derive this expression:

$\Rightarrow [\log(f)]' = [\log(K)]' + \dfrac{1}{2}[\log(\tau)]' - \dfrac{1}{2}[\log(\rho)]' - \dfrac{3}{2}[\log(R)]'$

$$\Rightarrow \frac{df}{f} = \frac{dK}{K} + \frac{1}{2}\frac{d\tau}{\tau} - \frac{1}{2}\frac{d\rho}{\rho} - \frac{1}{2}\frac{dR}{R}$$

$$\Rightarrow \frac{df}{f} = \frac{1}{2}\frac{d\tau}{\tau} - \frac{1}{2}\frac{d\rho}{\rho} - \frac{3}{2}\frac{dR}{R} \quad , car\ dK = 0\,, K\!: constant$$

Let's move on to absolute notation: $\Rightarrow \dfrac{\Delta f}{f} = \dfrac{1}{2}\dfrac{\Delta\tau}{\tau} + \dfrac{1}{2}\dfrac{\Delta\rho}{\rho} + \dfrac{3}{2}\dfrac{\Delta R}{R}$

Exercise n°16:

1- $E = h\nu \Rightarrow h = \frac{E}{\nu} \Rightarrow [h] = \frac{[E]}{[\nu]}$

$[E] = M.L^2.T^{-2}\,,\ [\nu] = T^{-1} \quad \Rightarrow [h] = M.L^2.T^{-1}$

2- $\lambda = Kh^x m^y v^z \Rightarrow [\lambda] = [K][h]^x[m]^y[v]^z$

$[\lambda] = L\ ,\ [K] = 1, [m] = M, [v] = LT^{-1}$

$L = M^x L^{2x}T^{-x}.\,M^y.L^zT^{-z}$

$\Rightarrow L = M^{x+y}L^{2x+z}T^{-x-z}$

$$\begin{cases} 2x + z = 1 \\ x + y = 0 \\ -x - z = 0 \end{cases} \Rightarrow \begin{cases} x = -y \\ x = -z \\ x = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -1 \\ z = -1 \end{cases}$$

$\Rightarrow \ \lambda = \dfrac{Kh}{mv}$

3-$\Delta\lambda$ =?

$$d\lambda(h,m,v)=\left(\frac{\partial\lambda}{\partial h}\right)dh+\left(\frac{\partial\lambda}{\partial m}\right)dm+\left(\frac{\partial\lambda}{\partial v}\right)dv$$

$$\left(\frac{\partial\lambda}{\partial h}\right)=\frac{K}{mv},\left(\frac{\partial\lambda}{\partial m}\right)=-\frac{Kh}{m^2v},\left(\frac{\partial\lambda}{\partial v}\right)=-\frac{Kh}{mv^2}$$

$$\frac{d\lambda}{\lambda}=\frac{mv}{Kh}\left(\frac{K}{mv}dh-\frac{Kh}{m^2v}dm-\frac{Kh}{mv^2}dv\right)$$

$$\frac{d\lambda}{\lambda}=\frac{dh}{h}-\frac{dm}{m}-\frac{dv}{v}$$

$$\Rightarrow\frac{\Delta\lambda}{\lambda}=\frac{\Delta h}{h}-\frac{\Delta m}{m}-\frac{\Delta v}{v}$$

Exercise n°17:

$[R]=?$

$$p.V=n.R.T \quad\Rightarrow\quad \mathrm{R}=\frac{\mathrm{p.V}}{\mathrm{n.T}} \ \ldots\ldots (1)$$

Equation (1) allows us to write $[\mathrm{R}]=\dfrac{[\mathrm{p}].[\mathrm{V}]}{[\mathrm{n}].[\mathrm{T}]}$ …..(2)

Knowing that:

$$\begin{cases}\mathrm{p}: \text{ pressure} & \Rightarrow \quad [\mathrm{p}]=\mathrm{M.L^{-1}.T^{-2}}\\ \mathrm{V}: \text{ volume} & \Rightarrow \quad [\mathrm{V}]=\mathrm{L^3}\\ \mathrm{n}: \text{ number of moles} & \Rightarrow \quad [\mathrm{n}]=\mathrm{N}\\ \mathrm{T}: \text{ temperature} & \Rightarrow \quad [\mathrm{T}]=\theta\end{cases}$$

$$(2)\Rightarrow\ [\mathrm{R}]=\frac{(\mathrm{M.L^{-1}.T^{-2}}).(\mathrm{L^3})}{(\mathrm{N}).(\theta)}=\mathrm{M.L^2.T^{-2}.N^{-1}.\theta^{-1}}$$

$$\Rightarrow\ \boxed{[\mathrm{R}]=\mathrm{M.L^2.T^{-2}.N^{-1}.\theta^{-1}}}$$

- The unit of (R) in (SI) is therefore: $\mathrm{kg.m^2.K^{-1}.s^{-2}mol^{-1}}$ Or $\mathrm{J.K^{-1}.mol^{-1}}$.

b- $[a]=?$ and $[b]=?$

$$p=\left(\frac{RT}{V-b}\right).\exp\left(\frac{-a}{R.T.V}\right)\ \ldots\ (3)$$

- We know that two physical quantities only add up if they have the same dimension

$$\Rightarrow[\mathrm{V}]=[\mathrm{b}]=\mathrm{L^3}$$

$$\Rightarrow\ \boxed{[\mathrm{b}]=\mathrm{L^3}}$$

- We also know that the argument $f = \frac{-a}{R.T.V}$ of the exponential function is dimensionless:

$$\Rightarrow [f] = \left[\frac{-a}{R.T.V}\right] = 1$$

$$\Rightarrow [-a] = [R.T.V]$$

$$\Rightarrow [a] = [R].[T].[V] = (M.L^2.T^{-2}.N^{-1}.\theta^{-1}).(\theta).(L^3)$$

$$\Rightarrow \boxed{[a] = M.L^5.T^{-2}.N^{-1}}$$

Exercise n°18:

2. Dimension of (G)? $\frac{T^2}{a^3} = \frac{4\pi^2}{G.M_S}$

$$G = \frac{a^3.4\pi^2}{T^2.M_S} \Rightarrow [G] = \frac{[a]^3.[4][\pi]^2}{[T]^2[M_S]}$$

We have: a: length$\Rightarrow [a] = L$, Period, $T \Rightarrow [T] = T$, Mass of sun $\boldsymbol{M_s} \Rightarrow [\boldsymbol{M_s}] = M$

4 *et* π : *constantes* $\Rightarrow [4] = 1$;$[\pi] = 1$

Thus$[G] = \frac{L^3.1.1}{M.T^2} = L^3.M^{-1}.T^{-2}$ Its unit in (SI) is.$\boldsymbol{m^3 kg^{-1}.S^{-2}}$

$$M_S = \frac{\boldsymbol{a}^3.4\pi^2}{T^2.\boldsymbol{G}}$$

$$M_S = \frac{a^3.4\pi^2}{T^2.G} = \frac{.(1{,}4960.10^{11})^3.4\pi^2}{(365{,}25636567.24.\ 3600)^2..6{,}668.10^{11}} = 1.99.10^{+30}\text{kg}.$$

3. Calculation of absolute uncertainty using the Total Differential Method M_S

$$dM_S(a,4,\pi,T,G) = \left(\frac{\partial M_S}{\partial a}\right)da + \left(\frac{\partial M_S}{\partial 4}\right)\underbrace{d4}_{0} + \left(\frac{\partial M_S}{\partial \pi}\right)\underbrace{d\pi}_{0} + \left(\frac{\partial M_S}{\partial T}\right)dT + \left(\frac{\partial M_S}{\partial G}\right)dG$$

Thus $$dM_S = \left(\frac{\partial M_S}{\partial a}\right)da + \left(\frac{\partial M_S}{\partial T}\right)dT + \left(\frac{\partial M_S}{\partial G}\right)dG$$

$$\frac{\partial M_S}{\partial a} = \frac{\boldsymbol{3.a^2}.4\pi^2}{T^2.\boldsymbol{G}} \quad , \quad \frac{\partial M_S}{\partial T} = \frac{\boldsymbol{-2a^3}.4\pi^2}{T^3.\boldsymbol{G}} \quad , \quad \frac{\partial M_S}{\partial G} = \frac{\boldsymbol{-a^3}.4\pi^2}{T^2.\boldsymbol{G}}$$

$$dM_S = \left(\frac{\boldsymbol{3.a^2.4}\pi^2}{T^2.\boldsymbol{G}}\right)da + \left(\frac{-2\boldsymbol{a^3}.4\pi^2}{T^3.\boldsymbol{G}}\right)dT + \left(\frac{\boldsymbol{-a^3}.4\pi^2}{T^2.\boldsymbol{G^2}}\right)dG$$

$$\Delta M_S = \left|\frac{\boldsymbol{3.a^2.4}\pi^2}{T^2.\boldsymbol{G}}\right|\Delta a + \left|\frac{-2\boldsymbol{a^3}.4\pi^2}{T^3.\boldsymbol{G}}\right|\Delta T + \left|\frac{\boldsymbol{-a^3}.4\pi^2}{T^2.\boldsymbol{G^2}}\right|\Delta G$$

$$\Delta M_S = \frac{\boldsymbol{3.a^2.4}\pi^2}{T^2.\boldsymbol{G}}\Delta a + \frac{2\boldsymbol{a^3}.4\pi^2}{T^3.\boldsymbol{G}}\Delta T + \frac{\boldsymbol{a^3}.4\pi^2}{T^2.\boldsymbol{G^2}}\Delta G = M_S(3\frac{\Delta a}{a} + 2\frac{\Delta T}{T} + \frac{\Delta G}{G})$$

$$\Rightarrow \Delta M_S = 1.99.10^{+30}(3\frac{0{,}0003.10^{+11}}{1{,}4960.10^{+11}} + 2\frac{0{,}00000001}{365{,}25636567} + \frac{0{,}005}{6{,}668}) = 0{,}003.10^{+30}kg$$

Exercise n°19:

$$v = \frac{1}{9}\frac{a^2 g(\rho - \rho')}{\eta}$$

1- The left member has the dimension:

$$[v] = L.T^{-1}$$

The right-hand side has the dimension:

$$[a]^2[(\rho - \rho')][\eta^{-1}][g] = L^2\ (M.L^{-3})(M.L^{-1}.T^{-1})^{-1}(L.T^{-2})$$
$$= L.T^{-1}$$

The formula is homogeneous.

2- $[\eta]$=?

$$v = \frac{1}{9}\frac{a^2 g(\rho - \rho')}{\eta} \Rightarrow \eta = \frac{1a^2 g(\rho - \rho')}{9v}$$

$$\Rightarrow [\eta] = \left[\frac{1}{9}\right][a^2]\frac{[\rho - \rho']}{[v]}[g]$$

We recall that

v: represents a speed, so $[v]$ = LT^{-1}, a : is a radius, [a]=L, g : is gravity, thus $[g]=LT^{-2}$

ρ, and ρ' represent densities, so $[\rho - \rho'] = [\rho] = [\rho'] = ML^{-3}$.

$\frac{1}{9}$: is a number and $[1/9] = 1$

Finally, we obtain:

$$[\eta] = \frac{L^2 M L^{-3} L T^{-2}}{L T^{-1}} = M L^{-1} T^{-1}.$$

3-$\Delta\eta$ =?

$$\eta = \frac{1a^2 g(\rho - \rho')}{9v} \Rightarrow \log(\eta) = \log\left(\frac{1}{9}\right) + \log(a)^2 + \log(\rho - \rho') + \log(g) - \log(v)$$

$$\Rightarrow \log(\eta) = \log\left(\frac{1}{9}\right) + 2\log(a) + \log(\rho - \rho') + \log(g) - \log(v)$$

Let's derive this expression:

$$\Rightarrow \left[\log(\eta)\right]' = \left[\log\left(\frac{1}{9}\right)\right]' + \left[2\,.\log(a)\right]' + \left[\log(\rho-\rho')\right]' + \left[\log(g)\right]' - \left[\log(v)\right]'$$

$$\Rightarrow \frac{d\eta}{\eta} = 2.\frac{da}{a} + \frac{d\rho}{\rho-\rho'} - \frac{d\rho'}{\rho-\rho'} + \frac{dg}{g} - \frac{dv}{v}$$

Let's move on to absolute notation:

$$\Rightarrow \frac{\Delta\eta}{\eta} = 2.\frac{\Delta a}{a} + \frac{\Delta\rho}{\rho-\rho'} - \frac{\Delta\rho'}{\rho-\rho'} + \frac{\Delta g}{g} - \frac{\Delta v}{v}$$

($\frac{\Delta g}{g} \rightarrow 0$)

$$\Rightarrow \frac{\Delta\eta}{\eta} = 2.\frac{\Delta a}{a} + \frac{\Delta\rho}{\rho-\rho'} - \frac{\Delta\rho'}{\rho-\rho'} - \frac{\Delta v}{v}$$

Exercise n°20:

1) $v = K\sqrt{\frac{2\Delta P}{\rho}} = K\left(\frac{2\Delta P}{\rho}\right)^{\frac{1}{2}} = K\,\frac{(2\Delta P)^{\frac{1}{2}}}{(\rho)^{\frac{1}{2}}}$

The left member has the dimension: $v = LT^{-1}$

The right-hand side has the dimension:

$$\left[\frac{(2\Delta P)^{\frac{1}{2}}}{(\rho)^{\frac{1}{2}}}\right] = \left[\frac{P^{\frac{1}{2}}}{\rho^{\frac{1}{2}}}\right] = \frac{(ML^{-1}T^{-2})^{\frac{1}{2}}}{(ML^{-3})^{\frac{1}{2}}} = \frac{M^{\frac{1}{2}}L^{\frac{-1}{2}}T^{-1}}{M^{\frac{1}{2}}\rho^{\frac{-3}{2}}} = L^{\frac{+3-1}{2}}.T^{-1} = LT^{-1} = v$$

The relationship is homogeneous.

2) $P = \frac{F}{S} \Rightarrow [P] = \frac{[F]}{[S]} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$

$[X] = \frac{1}{[P]} = [P]^{-1} = M^{-1}L\,T^{+2}$

3) $v = K\rho^{\alpha}X^{\beta}$

$[v] = [K][\rho]^{\alpha}[X]^{\beta}$

On a : $[\rho] = ML^{-3}$; $[CK] = 1$; $[X] = M^{-1}L\,T^{+2}$

Donc : $LT^{-1} = 1.\left(ML^{-3}\right)^{\alpha}.\left(M^{-1}L\,T^{+2}\right)^{\beta} = M^{\alpha}L^{-3\alpha}.M^{\beta}.L^{\beta}\,T^{+2\beta}$

$\Rightarrow LT^{-1} = M^{\alpha-\beta}L^{\beta-3\alpha}\,T^{+2\beta}$

By identification we obtain:

$$\begin{cases} \alpha - \beta = 0 \\ -3\alpha + \beta = 1 \\ 2\beta = -1 \end{cases} \Rightarrow \alpha = \beta = -\frac{1}{2}$$

Therefore: $v = K\frac{1}{\sqrt{\rho X}}$

4) $v = K\frac{1}{\sqrt{\rho X}} \Rightarrow v = K\rho^{-1/2}X^{-1/2}$

$\Rightarrow \log(v) = \log(K) - \frac{1}{2}\log(\rho) - \frac{1}{2}\log(X)$

Let's derive this expression:

$\Rightarrow [\log(v)]' = [\log(K)]' - \frac{1}{2}[\log(\rho)]' - \frac{1}{2}[\log(X)]'$

$$\Rightarrow \frac{dv}{v} = \frac{dK}{K} - \frac{1}{2}\frac{d\rho}{\rho} - \frac{1}{2}\frac{dX}{X}$$

$$\Rightarrow \frac{dv}{v} = -\frac{1}{2}\frac{d\rho}{\rho} - \frac{1}{2}\frac{dX}{X} \quad , car\ dK = 0\,, K: constant$$

Let's move on to absolute notation: $\Rightarrow \frac{\Delta v}{v} = +\frac{1}{2}\frac{\Delta\rho}{\rho} + \frac{1}{2}\frac{\Delta X}{X}$

Chapter2

VECTOR CALCULATION

Learning Goals: After going through this chapter, students will be able to

- Describe the difference between vector and scalar quantities.
- Identify the magnitude and direction of a vector.
- Explain the effect of multiplying a vector quantity by a scalar.
- Describe how one-dimensional vector quantities are added or subtracted.
- Explain the geometric construction for the addition or subtraction of vectors in a plane.
- Distinguish between a vector equation and a scalar equation.

CHAPTER 2 EXERCISES

Exercise n°01:

Let $\vec{A}$ and $\vec{B}$ be two vectors identified in the triad (Oxyz), defined by:

$$\vec{A} = 3\vec{i} + 4\vec{j} - 5\vec{k} \quad \text{and} \quad \vec{B} = -\vec{i} + \vec{j} + 2\vec{k}$$

1- Calculate their magnitudes.

2- Calculate, then represent the two vectors: $\vec{A} + \vec{B}$ and $\vec{A} - \vec{B}$.

3- a- Calculate the dot products $\left(\vec{A}.\vec{B}\right)$ and $\left(\vec{B}.\vec{A}\right)$. What do you notice?

b- Determine the angle $\theta = \left(\vec{A}, \vec{B}\right)$.

4- Calculate the vector products $\left(\vec{A} \wedge \vec{B}\right)$ and $\left(\vec{B} \wedge \vec{A}\right)$. What do you notice, and what does $\left|\vec{A} \wedge \vec{B}\right|$?

5- Consider a vector $\vec{C} = x.\vec{i} + y.\vec{j} + z.\vec{k}$, find the variables x, y and z so that $\vec{A} + \vec{B} + \vec{C} = \vec{0}$

Exercise n°02:

In a three dimensional coordinate system (O, x, y, z); we have three vectors defined by:

$$\vec{A} = -2\vec{i} - 3\vec{j} + 2\vec{k} \quad , \quad \vec{B} = 5\vec{i} + 2\vec{j} - \vec{k} \ , \ \vec{C} = \vec{i} + 2\vec{j} + 3\vec{k}$$

1. Calculate their magnitudes $\left|\vec{A}\right|$, $\left|\vec{B}\right|$ and $\left|\vec{C}\right|$.
2. Calculate: $\vec{A} + \vec{B} + \vec{C}$ and $\vec{A} + \vec{B} - \vec{C}$.
3. Calculate the unit vector $\vec{u}$ carried by the vector $\vec{A} + \vec{B} - \vec{C}$
4. Calculate the dot products $\left(\vec{A}.\vec{B}\right)$ and $\left(\vec{B}.\vec{A}\right)$.
5. Calculate the vector products $\left(\vec{A} \wedge \vec{B}\right)$ and $\left(\vec{B} \wedge \vec{A}\right)$.
6. Calculate the dot products $\left(\vec{B}.\vec{C}\right)$ and determine the angle $\theta = \left(\vec{B}, \vec{C}\right)$.

Exercise n°03:

In a three dimensional coordinate system (O, x, y, z); we have three vectors defined by:

$$\vec{r}_1 = 2\vec{i} - \vec{j} + 2\vec{k} \quad , \quad \vec{r}_2 = \vec{i} - 3\vec{j} - 2\vec{k} \ , \ \vec{r}_3 = 2\vec{i} + 2\vec{j} - \vec{k}$$

1. Calculate their magnitudes $\left|\vec{r}_1\right|$, $\left|\vec{r}_2\right|$ and $\left|\vec{r}_2\right|$.

2. Calculate the unit vector $\vec{u}$ carried by the vector $\vec{r}_2$
3. Calculate the angles α, β and γ that form the vector $\vec{r}_1$ with the three axes (ox, oy, oz)
4. What are they called $\cos\alpha, \cos\beta, \cos\gamma$, what do they represent?
5. Calculate the dot products $(\vec{r}_1.\vec{r}_2)$ and $(\vec{r}_2.\vec{r}_1)$.
6. Calculate the vector products $(\vec{r}_1 \wedge \vec{r}_2)$ and $(\vec{r}_2 \wedge \vec{r}_1)$.
7. Calculate the moment of $\vec{V_1}$ passing through point A (1, 2, 3) with respect to the origin O.

Exercise n°04:

1- In three-dimensional orthonormal basis space $(O, \vec{i}, \vec{j}, \vec{k})$, we consider two vectors given by: $\vec{A} = a_1.\vec{i} + a_2.\vec{j} + a_3.\vec{k}$ et $\vec{B} = b_1.\vec{i} + b_2.\vec{j} + b_3.\vec{k}$

Calculate $\cos(\vec{A}, \vec{B})$ depending on the parameters a_i and b_i with the index i varying from 1 to 3.

2- Consider the points M_1 (1, 1, 1), M_2 (3, 1, 1) and M_3 (3, 3, 1), calculate the angle $M_2\hat{M}_1M_3$

3- Determine the equation of the plane (P) passing through the point $M_0(x_0, y_0, z_0)$, and perpendicular to the vector $\vec{A}$. Application of the special case where M_0 is in M_2 and $\vec{A} = 3\vec{i} - 2\vec{j} + \vec{k}$

Exercise n°05:

In a rectangular coordinate system $(O, \vec{i}, \vec{j}, \vec{k})$, we consider four points A(2,-1,0), B(3,0,-2), C(-1, 1,2) et D(1,-3,-2) .

1. Find the coordinates of the following vectors: $\vec{U} = \vec{AB}, \vec{V} = \vec{AC}$ and $\vec{W} = \vec{AD}$
2. Calculate the cross product or vector product, $\vec{U} \wedge \vec{V}$
3. Find the area of the triangle $\widehat{ABC}$.
4. Calculate the angle between the vectors $\vec{U}$ and $\vec{V}$.
5. Calculate the mixed or scalar triple product, $(\vec{U} \wedge \vec{V}).\vec{W}$
6. Determine the volume of the parallelepiped formed by three vectors $\vec{U}, \vec{V}$ and $\vec{W}$.

Exercise n°06:

In a three-dimensional orthonormal basis space $(0, \vec{i}, \vec{j}, \vec{k})$, given three vectors

$\vec{A}\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\vec{B}\begin{pmatrix}1\\0\\-1\end{pmatrix}$ et $\vec{C}\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

1. Determine the magnitude, and direction of cross product $\vec{K} = \vec{A} \wedge \vec{B}$.

2. Calculate the area of the parallelogram formed by the vectors $\vec{A}$ et $\vec{B}$.

3. Deduce the angle between the vectors $\vec{A}$ et $\vec{B}$.

4. Calculate the volume of the parallelepiped formed by three vectors$\vec{A}$, $\vec{B}$et $\vec{C}$.

5. Deduce the angle between the vectors $\vec{K}$ et $\vec{C}$.

Exercise n°07:

- Given vector $\vec{r}$ defined by $\vec{r} = \cos(\omega t)\,\vec{i} + \sin(\omega t)\vec{j} + e^{-\omega t}\vec{k}$.

- Calculate derivative vectors $\frac{d\vec{r}}{dx}, \frac{d^2\vec{r}}{dx^2}$ and evaluate their magnitudes for t=0.

Exercise n°08:

I- Given vector $\vec{r}$ defined by $\vec{r} = \cos(2x).\vec{i} + \sin(5x).\vec{j} + e^{-\alpha x}.\vec{k}$. (α is a real constant).

- Calculate derivative vectors $\frac{d\vec{r}}{dx}, \frac{d^2\vec{r}}{dx^2}$ and evaluate their magnitudes for x=0, we take $\alpha = 1$

II- Consider two vectors $\vec{A} = 3x\vec{i} + x^2\vec{j} - x^3\vec{k}$ and $\vec{B} = -x\vec{i} + 4x\vec{j} + x\vec{k}$, calculate the derivatives $\frac{d}{dx}(\vec{A}.\vec{B})$ and $\frac{d}{dx}(\vec{A}\Lambda\vec{B})$ by two different methods:

1- In applying the rules of derivation vector.

2- In calculating the product scalar or the product vector, then drifting the result.

Exercise n°09:

Let a vector $\vec{r} = x.\vec{i} + y.\vec{j} + z.\vec{k}$ and a scalar function. $\phi(x, y, z) = x^2 + y^2 + z^2$

- Calculate: $\overrightarrow{grad}(\phi)$, $div(\vec{r})$, $\overrightarrow{grad}\left(\frac{1}{r}\right)$ et $\overrightarrow{rot}(\vec{r})$

Exercise n°10:

Consider a vector $\vec{A} = xz\vec{i} + (2x^2 - y)\vec{j} - yz^2\vec{k}$ and $\emptyset(x, y, z) = 3x^2y + 2y^2z^3$ a scalar function

1. Calculate: $\overrightarrow{grad}(\emptyset)$, $div(\vec{A})$ and $\overrightarrow{rot}(\vec{A})$
2. Calculate at point (1, 0,1):
   a. $\overrightarrow{grad}(\emptyset)$
   b. $div(\vec{A})$
   c. $\overrightarrow{rot}(\vec{A})$

Exercise n°11:

Consider a scalar function $\varphi\ (x, y, z) = 4x^3 \cos(3y) \exp(-2z)$ and two vector felds

$\vec{F} = 3x^{2\rightarrow}i + y^2x^3\vec{j} + y\sqrt{x^2z^2}\vec{k}$ and $\vec{G} = x^3 \cos(yz^2)\vec{\iota} + y^2 \ln(xyz)\vec{j} + \sqrt{xz} \exp(yz^3)\vec{k}$

1. Calculate the gradient vector field of the scalar function φ (denoted by $\vec{\nabla}\varphi(x, y, z)$).

2. Calculate the divergence of the vector field $\vec{F}$ (denoted by $\vec{\nabla} \wedge \vec{F}$).

3. Find the curl of the vector field ∂ (denoted by curl $\vec{G} = \vec{\nabla} \wedge \vec{G}$).

Exercise n°12:

Given the vector field $\vec{A} = 3x^2y\,\vec{\iota} + yz^2\,\vec{j} - xz\,\vec{k}$ and the scalar function
$\varphi\ (x, y, z) = x^2yz$

1. Calculate the gradient $\vec{\nabla}\varphi(x,y,z)$.

2. Calculate the divergence $\vec{\nabla}.\vec{A}$.

3. Find the curl of the gradient $(\vec{\nabla} \wedge \vec{\nabla}\varphi(x,y,z))$.

4. Find the divergence of the curl $(\vec{\nabla}.\vec{\nabla}\varphi(x,y,z))$.

5. Find $\frac{\partial^2(\varphi\vec{A})}{\partial y \partial z}$ at the point M(1,-2,-1)

Exercise n°13:

We consider the following vectors:

$$\vec{V_1} = 5t^3\,\vec{\iota} + 3t\,\vec{j} - 2t^4\,\vec{k} \quad \text{and} \quad \vec{V_2} = \sin t\,\vec{\iota} - \cos t\,\vec{j} - 3t\,\vec{k}$$

1. Find the expressions of the quantities: $\frac{d}{dt}(\vec{V_1}.\vec{V_2})$ and $\frac{d}{dt}(\vec{V_1}\Lambda\,\vec{V_2})$
2. Calculate the moment of $\vec{V_1}$ passing through point A (1, 2, 3) with respect to the origin O.
3. Calculate the moment of $\vec{V_1}$ with respect to an axis (Δ) of unit vector $\vec{u}(1,0,-1)$.

Exercise n°14:

In a direct orthonormal coordinate system (OXYZ), consider the vector $\vec{V} = \vec{i} + 2\vec{j} + 3\vec{k}$ passing through point A (3, 4, 2).

1- Calculate the moment of $\vec{V}$ with respect to the origin O.

2- Calculate the moment of $\vec{V}$ in relation to the three main axes.

3- Calculate the moment of $\vec{V}$ with respect to an axis $(\Delta)$of unit vector $\vec{u}$ passing through point O. Where, $\vec{u}(-\frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2})$

SOLUTIONS TO EXERCISES

Exercise n°01:

$\vec{A}=3\vec{i}+4\vec{j}-5\vec{k}$

$\vec{B}=-\vec{i}+\vec{j}+2\vec{k}$

1) $|\vec{A}|=\sqrt{3^2+4^2+(-5)^2}=\sqrt{50}=5\sqrt{2}$, $|\vec{B}|=\sqrt{(-1)^2+1^2+2^2}=\sqrt{6}$

2) $\vec{A}+\vec{B}=(3+(-1))\vec{i}+(4+1)\vec{j}+((-5)+2)\vec{k}=2\vec{i}+5\vec{j}-3\vec{k}$

$\vec{A}-\vec{B}=(3-(-1))\vec{i}+(4-1)\vec{j}+((-5)-2)\vec{k}=4\vec{i}+3\vec{j}-7\vec{k}$

3) $$\vec{A}.\vec{B}=\begin{pmatrix}3\\4\\-5\end{pmatrix}.\begin{pmatrix}-1\\1\\2\end{pmatrix}=3.(-1)+4.1+(-5).2=-9$$

$$\vec{B}.\vec{A}=\begin{pmatrix}-1\\1\\2\end{pmatrix}.\begin{pmatrix}3\\4\\-5\end{pmatrix}=(-1).3+1.4+2.(-5)=-9$$

$\Rightarrow\quad \vec{A}.\vec{B}=\vec{B}.\vec{A}$ The dot product is commutative.

We know that $\vec{A}.\vec{B}=|\vec{A}|.|\vec{B}|.\cos(\vec{A},\vec{B})$

$$\cos(\vec{A},\vec{B})=\frac{\vec{A}.\vec{B}}{|\vec{A}|.|\vec{B}|}\quad\Rightarrow\quad \theta=(\vec{A},\vec{B})=121.33°$$

$$\vec{A}\wedge\vec{B}=\begin{vmatrix}\overset{(+)}{\vec{i}} & \overset{(-)}{\vec{j}} & \overset{(+)}{\vec{k}}\\ 3 & 4 & -5\\ -1 & 1 & 2\end{vmatrix}=+\vec{i}(8+5)-\vec{j}(6-5)+\vec{k}(3+4)=13\vec{i}-\vec{j}+7\vec{k}$$

$$\vec{B}\wedge\vec{A}=\begin{vmatrix}\overset{(+)}{\vec{i}} & \overset{(-)}{\vec{j}} & \overset{(+)}{\vec{k}}\\ -1 & 1 & 2\\ 3 & 4 & -5\end{vmatrix}=+\vec{i}(-5-8)-\vec{j}(5-6)+\vec{k}(-4-3)=-13\vec{i}+\vec{j}-7\vec{k}$$

$\vec{B}\wedge\vec{A}=-13\vec{i}+\vec{j}-7\vec{k}=-\vec{A}\wedge\vec{B}\mapsto$ The cross product is not commutative.

4) $$\vec{A}+\vec{B}+\vec{C}=\begin{pmatrix}3\\4\\-5\end{pmatrix}+\begin{pmatrix}-1\\1\\2\end{pmatrix}+\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3-1+x\\4+1+y\\-5+2+z\end{pmatrix}=\begin{pmatrix}2+x\\5+y\\-4+z\end{pmatrix}=\vec{0}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\vec{C}=-2.\vec{i}-5.\vec{j}+3.\vec{k}$$

Exercise n°02:

$$\vec{A}=-2\vec{i}-3\vec{j}+2\vec{k} \quad , \quad \vec{B}=5\vec{i}+2\vec{j}-\vec{k} \ , \ \vec{C}=\vec{i}+2\vec{j}+3\vec{k}$$

1) $$\left|\vec{A}\right|=\sqrt{(-2)^2+(-3)^2+(2)^2}=\sqrt{17},\left|\vec{B}\right|=\sqrt{(5)^2+2^2+(-1)^2}=\sqrt{30}$$

$$\left|\vec{C}\right|=\sqrt{(1)^2+2^2+(3^2}=\sqrt{14}$$

2) $$\vec{A}+\vec{B}+\vec{C}=\begin{pmatrix}-2\\-3\\2\end{pmatrix}+\begin{pmatrix}5\\2\\-1\end{pmatrix}+\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}4\\1\\4\end{pmatrix}$$

$$\vec{A}-\vec{B}-\vec{C}=\begin{pmatrix}-2\\-3\\2\end{pmatrix}-\begin{pmatrix}5\\2\\-1\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-2-5-1\\-3-2-2\\2+1-3\end{pmatrix}=\begin{pmatrix}-8\\-7\\0\end{pmatrix}$$

3) $$\vec{A}.\vec{B}=\begin{pmatrix}-2\\-3\\2\end{pmatrix}.\begin{pmatrix}5\\2\\-1\end{pmatrix}=(-2).(5)+(-3).2+2.(-1)=-18$$

The dot product is commutative $\Rightarrow$ $\vec{A}.\vec{B}=\vec{B}.\vec{A}=-18$

4) $$\vec{u}=\frac{\vec{A}+\vec{B}-\vec{C}}{\left|\vec{A}+\vec{B}-\vec{C}\right|}$$

$$\vec{A}+\vec{B}-\vec{C}=\begin{pmatrix}-2+5-1\\-3+2-2\\2-1-3\end{pmatrix}=\begin{pmatrix}2\\-3\\-2\end{pmatrix}$$

$$\left|\vec{A}+\vec{B}-\vec{C}\right|=\sqrt{(2)^2+(-3)^2+(-2)^2}=\sqrt{17}$$

$$\vec{u}=\begin{pmatrix}2/\sqrt{17}\\-3/\sqrt{17}\\-2/\sqrt{17}\end{pmatrix}\Rightarrow\vec{u}=\frac{2}{\sqrt{17}}\vec{i}-\frac{3}{\sqrt{17}}\vec{j}-\frac{2}{\sqrt{17}}\vec{k}$$

5) $$\vec{A}\wedge\vec{B}=\begin{vmatrix}\overset{(+)}{\vec{i}} & \overset{(-)}{\vec{j}} & \overset{+}{\vec{k}}\\-2 & -3 & 2\\5 & 2 & -1\end{vmatrix}=+\vec{i}\,(3-4)-\vec{j}\,(2-10)+\vec{k}\,(-4+15)=-\vec{i}+8\vec{j}+11\vec{k}$$

The cross product is not commutative $\Rightarrow \quad \vec{B}\wedge\vec{A} = -\vec{A}\wedge\vec{B} = \vec{i} - 8\vec{j} - 11\vec{k}$

6) $\vec{B}.\vec{C} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (5).(1) + 2.2 + (-1).3 = 6$

We know that $\qquad \vec{B}.\vec{C} = |\vec{B}|.|\vec{C}|.\cos(\vec{B},\vec{C})$

$$\cos(\vec{B},\vec{C}) = \frac{\vec{B}.\vec{C}}{|\vec{B}|.|\vec{C}|} = \frac{6}{\sqrt{30}\sqrt{14}} \quad \Rightarrow \quad \theta = (\vec{B},\vec{C}) = 72.97°$$

Exercise n°03:

$\vec{r}_1 = 2\vec{i} - \vec{j} + 2\vec{k}$, $\vec{r}_2 = \vec{i} - 3\vec{j} - 2\vec{k}$, $\vec{r}_3 = 2\vec{i} + 2\vec{j} - \vec{k}$

1) $|\vec{r}_1| = \sqrt{(2)^2 + (-1)^2 + (2)^2} = \sqrt{9} = 3, |\vec{r}_2| = \sqrt{(1)^2 + (-3)^2 + (-2)^2} = \sqrt{14}$

$|\vec{r}_3| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

2) $\vec{u} = \dfrac{\vec{r}_2}{|\vec{r}_2|}$

$$\Rightarrow \vec{u} = \begin{pmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ -2/\sqrt{14} \end{pmatrix} \Rightarrow \vec{u} = \frac{1}{\sqrt{14}}\vec{i} - \frac{3}{\sqrt{14}}\vec{j} - \frac{2}{\sqrt{14}}\vec{k}$$

3) $\alpha(\vec{r}_1, ox) \Leftrightarrow \alpha(\vec{r}_1, \vec{i}), \vec{i} \in ox, \vec{i}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\vec{r}_1.\vec{i} = |\vec{r}_1|.|\vec{i}|.\cos\alpha \Rightarrow \cos\alpha = \frac{\vec{r}_1.\vec{i}}{|\vec{r}_1|.|\vec{i}|} \Rightarrow \cos\alpha = \frac{2}{3}$$

$\beta(\vec{r}_1, oy) \Leftrightarrow \beta(\vec{r}_1, \vec{j}), \vec{j} \in oy, \vec{j}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\vec{r}_1.\vec{j} = |\vec{r}_1|.|\vec{j}|.\cos\beta \Rightarrow \cos\beta = \frac{\vec{r}_1.\vec{j}}{|\vec{r}_1|.|\vec{j}|} \Rightarrow \cos\beta = \frac{-1}{3}$$

$\gamma(\vec{r}_1, oz) \Leftrightarrow \gamma(\vec{r}_1, \vec{k}), \vec{k} \in oz, \vec{k}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\vec{r}_1.\vec{k} = |\vec{r}_1|.|\vec{k}|.\cos\gamma \Rightarrow \cos\gamma = \frac{\vec{r}_1.\vec{k}}{|\vec{r}_1|.|\vec{k}|} \Rightarrow \cos\gamma = \frac{2}{3}$$

4) $(\cos\alpha, \cos\beta, \cos\gamma)$ are called the direction cosines of the vector $\vec{r}_1$,they represent the unit vector $\vec{u}$ carried by the vector $\vec{r}_1$

5) $\vec{r}_2.\vec{r}_3 = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}.\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = (2).(1)+(-3).2+(-2).1 = -2$

The dot product is commutative $\Rightarrow$ $\vec{r}_2.\vec{r}_3 = \vec{r}_3.\vec{r}_2 = -2$

6) $\vec{r}_1 \wedge \vec{r}_2 = \begin{vmatrix} \overset{(+)}{\vec{i}} & \overset{(-)}{\vec{j}} & \overset{(+)}{\vec{k}} \\ 2 & -1 & 2 \\ 1 & -3 & -2 \end{vmatrix} = \vec{i}\,(2+6) - \vec{j}\,(-4-2) + \vec{k}\,(-6+1) = 8\vec{i} + 6\vec{j} - 5\vec{k}$

The cross product is not commutative$\Rightarrow$ $\vec{r}_2 \wedge \vec{r}_1 = -\vec{r}_1 \wedge \vec{r}_2 = -8\vec{i} - 6\vec{j} + 5\vec{k}$

7) With respect to the origin O: $\vec{M}_{/O}(\vec{r}_1) = \overrightarrow{OM} \wedge \vec{r}_1 = \begin{vmatrix} \overset{(+)}{\vec{i}} & \overset{(-)}{\vec{j}} & \overset{(+)}{\vec{k}} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix}$

So that $\vec{M}_{/O}(\vec{r}_1) = \overrightarrow{OM} \wedge \vec{r}_1 = (4+3)\vec{i} - (2-6)\vec{j} + (-1-4)\vec{k} = 7\vec{i} + 4\vec{j} - 5\vec{k}$

Exercise n°04:

1- $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ And $\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

We know that $\vec{A}.\vec{B} = |\vec{A}|.|\vec{B}|.\cos(\vec{A}, \vec{B})$

- $\vec{A}.\vec{B} = a_1b_1 + a_2b_2 + a_3b_3$ with $\begin{cases} \vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1 \\ \vec{i}.\vec{j} = \vec{i}.\vec{k} = \vec{j}.\vec{k} = 0 \end{cases}$
- $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ And $|\vec{B}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

$$\cos(\vec{A}, \vec{B}) = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}.\sqrt{b_1^2 + b_2^2 + b_3^2}} \qquad \ldots(1)$$

2- The angle $M_2\hat{M}_1M_3$ between the two vectors $\overrightarrow{M_1M_2}$ And $\overrightarrow{M_1M_3}$.

$$\overrightarrow{M_1M_2} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \overrightarrow{2i} \Rightarrow a_1 = 2, a_2 = 0 \text{ And } a_3 = 0$$

$$\overrightarrow{M_1M_3} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 2\vec{i} - 2\vec{j} \Rightarrow a_1 = 2, a_2 = 2 \text{ And } a_3 = 0$$

Replacing a_i and b_i by their values in expression (1), we find:

$$(1) \Rightarrow \cos(\phi) = \cos\left(\overrightarrow{M_1M_2}, \overrightarrow{M_1M_3}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \boxed{\phi = \pm\frac{\pi}{4}}$$

3- The plane equation (P) which passes through the point $M_0(x_0, y_0, z_0) \in (P)$?

Assuming that another point $M(x, y, z) \in (P)$ check the condition $\overrightarrow{MM_0} \perp \vec{A}$

$$\Rightarrow \overrightarrow{MM_0}.\vec{A} = 0$$

$$\Rightarrow \begin{pmatrix} x_0 - x \\ y_0 - y \\ z_0 - z \end{pmatrix} . \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 0 \Rightarrow 3(x_0 - x) - 2(y_0 - y) + (z_0 - z) = 0 \qquad \ldots(2)$$

Particular case M_0 is M_2 $\Rightarrow \begin{cases} x_0 = 2 \\ y_0 = 2 \\ z_0 = 1 \end{cases}$

By replacing these values in equation (4) we find: $3(2-x) - 2(2-y) + (1-z) = 0$

Where again after the development we find:

$$\boxed{-3x + 2y - z + 3 = 0}$$

Exercise n°05:

1. The coordinates of vectors $\vec{U}$, $\vec{V}$ et $\vec{W}$ are given by:

$$\vec{U} = \overrightarrow{AB} = (x_B - x_A)\vec{\imath} \rightarrow +(y_B - y_A)\vec{\jmath} + (z_B - z_A)\vec{k} = 1\vec{\imath} \rightarrow +1\vec{\jmath} - 2\vec{k},$$

$$\vec{V} = \overrightarrow{AC} = (x_C - x_A)\vec{\imath} + (y_C - y_A)\vec{\jmath} + (z_C - z_A)\vec{k} = -3\vec{\imath} + 2\vec{\jmath} + 2\vec{k},$$

$$\vec{W} = \overrightarrow{AD} = (x_D - x_A)\vec{\imath} + (y_D - y_A)\vec{\jmath} + (z_D - z_A)\vec{k} = -\vec{\imath} - 2\vec{\jmath} - 2\vec{k}.$$

2. The cross product: $\vec{U} \wedge \vec{V} = 6\vec{\imath} + 4\vec{\jmath} + 5\vec{k}$.

3. The area of the triangle $\overline{ABC}$ is: $S = \frac{\|\vec{U}\wedge\vec{V}\|}{2} = 4.39m^2$.

4. The angle between the vectors $\vec{U}$ et $\vec{V}$ is : $\theta = \widehat{BAC} = \arcsin\left(\frac{\|\vec{U}\wedge\vec{V}\|}{\|\vec{U}\|\|\vec{V}\|}\right) = 60.32°$.

5. The mixed product $(\vec{U} \wedge \vec{V}) \cdot \vec{W} = -24$.

6. The volume of the parallelepiped formed by three vectors $\vec{U}$, $\vec{V}$ et $\vec{W}$ is given by:

$V = \|(\vec{U} \wedge \vec{V}) \cdot \vec{W}\| = 24m^3$.

Exercise n°06:

1.The cross product $\vec{K} = \vec{A} \wedge \vec{B}$ is given by

$$\vec{K} = \vec{A} \wedge \vec{B} \begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \vec{\imath}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$\Rightarrow \vec{K} = \vec{A} \wedge \vec{B} = -\vec{\imath} + \vec{\jmath} - \vec{k}$

Then deduces the characteristics of the vector $\vec{K}$:

− The magnitude is equal to $\|\vec{K}\| = \sqrt{3}$

− The direction is along the axis z′Oz

2. The area of the parallelogram formed by vectors $\vec{A}$ and $\vec{B}$ is equal to $\|\vec{A} \wedge \vec{B}\|$=1.73m^2

3. The angle θ between vectors $\vec{A}$ and B is given by,

$$\arcsin\left(\frac{\|\vec{K}\|}{\|\vec{A}\|\|\vec{B}\|}\right) \quad \arcsin\left(\frac{\sqrt{3}}{\sqrt{2}\sqrt{2}}\right) \Rightarrow \theta = \frac{\pi}{3}$$

4. The volume of the parallelepiped constructed on the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is given by,

$$(\vec{A} \wedge \vec{B}).\vec{C} = \vec{K}.\vec{C} = -3$$

5. The angle α between vectors $\vec{K}$ and $\vec{C}$ is then,

$$\arccos\left(\frac{\vec{K}.\vec{C}}{\|\vec{K}\|\|\vec{C}\|}\right) \quad \arccos\left(\frac{-3}{\sqrt{3}\sqrt{3}}\right) \Rightarrow \theta = \pi$$

Exercise n°07:

$$\vec{r} = \cos(\omega t)\,\vec{i} + \sin(\omega t)\,\vec{j} + e^{-\omega t}\vec{k}$$

$$\frac{d\vec{r}}{dt} = -\omega\sin(\omega t)\vec{i} + \omega\cos(\omega t)\vec{j} - \omega e^{-\omega t}\vec{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\omega^2\sin^2(\omega t) + \omega^2\cos^2(\omega t) + \omega^2\left(e^{-\omega t}\right)^2}$$

$$= \sqrt{\omega^2\left[\sin^2(\omega t) + \cos^2(\omega t)\right] + \omega^2\left(e^{-\omega t}\right)^2} \qquad \Rightarrow \left|\frac{d\vec{r}}{dt}\right| = \omega\left(1 + \omega e^{-2\omega t}\right)^{\frac{1}{2}}$$

$$\Rightarrow \left|\frac{d\vec{r}}{dt}\right|_{t=0} = \omega\sqrt{2}$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\cos(\omega t)\vec{i} - \omega^2\sin(\omega t)\vec{j} + \omega^2 e^{-\omega t}\vec{k} \Rightarrow \left|\frac{d^2\vec{r}}{dt^2}\right| = \omega^2\left(1 + e^{-2\omega t}\right)^{\frac{1}{2}}$$

$$\Rightarrow \left|\frac{d\vec{r}}{dt}\right|_{t=0} = \omega^2\sqrt{2}$$

Exercise n°08:

$$\vec{r} = \cos(2x)\vec{i} + \sin(5x)\vec{j} + e^{-\alpha x}\vec{k}$$

$$\frac{d\vec{r}}{dx} = -2\sin(2x)\vec{i} + 5\cos(5x)\vec{j} - \alpha e^{-\alpha x}\vec{k} \text{ if x=0} \left|\frac{d\vec{r}}{dx}\right| = \sqrt{25 + \alpha^2} = \sqrt{26}$$

$$\frac{d^2\vec{r}}{dx^2} = -4\cos(2x)\vec{i} - 25\sin(5x)\vec{j} + \alpha^2 e^{-\alpha x}\vec{k} \text{ if x=0} \left|\frac{d^2\vec{r}}{dx^2}\right| = \sqrt{16 + \alpha^4} = \sqrt{17}$$

II- $\vec{A} = 3x\vec{i} + x^2\vec{j} - x^3\vec{k}$ and $\vec{B} = -x\vec{i} + 4x\vec{j} + x\vec{k}$

1- By applying the rules of vector differentiation.

- $\dfrac{d(\vec{A}.\vec{B})}{dx} = \vec{A}.\dfrac{d\vec{B}}{dx} + \dfrac{d\vec{A}}{dx}.\vec{B}$ ...(1)

We have $\begin{cases} \dfrac{d\vec{A}}{dx} = 3.\vec{i} + 2x.\vec{j} - 3x^2.\vec{k} \\ \dfrac{d\vec{B}}{dx} = -\vec{i} + 4.\vec{j} + \vec{k} \end{cases}$

$$(1) \Rightarrow \frac{d(\vec{A}.\vec{B})}{dx} = \left(3x\vec{i} + x^2\vec{j} - x^3\vec{k}\right).\left(-\vec{i} + 4\vec{j} + \vec{k}\right) + \left(3\vec{i} + 2x\vec{j} - 3x^2\vec{k}\right).\left(-x\vec{i} + 4x\vec{j} + x\vec{k}\right)$$

$$\Rightarrow \boxed{\frac{d(\vec{A}.\vec{B})}{dx} = -6x + 12x^2 - 4x^3} \quad ...(2)$$

- $\dfrac{d(\vec{A}\wedge\vec{B})}{dx} = \dfrac{d\vec{A}}{dx}\wedge\vec{B} + \vec{A}\wedge\dfrac{d\vec{B}}{dx}$

$$= \begin{vmatrix} \overset{(+)}{\vec{i}} & \overset{(-)}{\vec{j}} & \overset{(+)}{\vec{k}} \\ 3 & 2x & -3x^2 \\ -x & 4x & x \end{vmatrix} + \begin{vmatrix} \overset{(+)}{\vec{i}} & \overset{(-)}{\vec{j}} & \overset{(+)}{\vec{k}} \\ 3x & x^2 & -x^3 \\ -1 & 4 & 1 \end{vmatrix}$$

$$= \left[\vec{i}.\left(2x^2 + 12x^3\right) - \vec{j}.\left(3x - 3x^3\right) + \vec{k}.\left(12x + 2x^2\right)\right] + \left[\vec{i}.\left(x^2 + 4x^3\right) - \vec{j}.\left(3x - x^3\right) + \vec{k}.\left(12x + x^2\right)\right]$$

$$\Rightarrow \boxed{\frac{d(\vec{A}\wedge\vec{B})}{dx} = \vec{i}.\left(3x^2 + 16x^3\right) - \vec{j}.\left(6x - 4x^3\right) + \vec{k}.\left(24x + 3x^2\right)} \quad ...(3)$$

2- By calculating the scalar product or the vector product, then deriving the result:

- $\vec{A}.\vec{B} = -3x^2 + 4x^3 - x^4$ ...(4)
- $\vec{A}\wedge\vec{B} = \begin{vmatrix} \overset{(+)}{\vec{i}} & \overset{(-)}{\vec{j}} & \overset{(+)}{\vec{k}} \\ 3x & x^2 & -x^3 \\ -x & 4x & x \end{vmatrix} = \vec{i}.\left(x^3 + 4x^4\right) - \vec{j}.\left(3x^2 - x^4\right) + \vec{k}.\left(12x^2 + x^3\right)$ ...(5)

The derivatives:

- $(4) \Rightarrow \dfrac{d(\vec{A}.\vec{B})}{dx} = -6x + 12x^2 - 4x^3$ = result (2)
- $(5) \Rightarrow \dfrac{d(\vec{A}\wedge\vec{B})}{dx} = \vec{i}\left(3x^2 + 16x^3\right) - \vec{j}\left(6x - 4x^3\right) + \vec{k}\left(24x + 3x^2\right)$ = result (3)

Exercise n°09:

$$\vec{r} = x.\vec{i} + y.\vec{j} + z.\vec{k} \text{ And } \varphi(x,y,z) = x^2 + y^2 + z^2$$

- $$\overrightarrow{grad}(\varphi) = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = 2\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = 2\vec{r} \quad \Rightarrow \quad \boxed{\overrightarrow{grad}(\varphi) = 2.\vec{r}}$$

- $$\operatorname{div}(\vec{r}) = \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} = 1+1+1 = 3 \quad \Rightarrow \quad \boxed{\operatorname{div}(\vec{r}) = 3}$$

- $$\overrightarrow{grad}\left(\frac{1}{r}\right) = \overrightarrow{grad}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)}{\partial x}\vec{i} + \frac{\partial\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)}{\partial y}\vec{j} + \frac{\partial\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)}{\partial z}\vec{k}$$

$$= \frac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}.\vec{i} + \frac{-y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}.\vec{j} + \frac{-z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}.\vec{k}$$

$$= \frac{-\left(x.\vec{i} + y.\vec{j} + z.\vec{k}\right)}{\left[\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\right]^3} = \frac{-\vec{r}}{r^3} \quad \Rightarrow \quad \boxed{\overrightarrow{grad}\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}}$$

- We found that

$$\overrightarrow{grad}(\varphi) = 2.\vec{r} \Rightarrow \vec{r} = \frac{1}{2}\overrightarrow{grad}(\varphi) \quad \Rightarrow \quad \overrightarrow{rot}(\vec{r}) = \overrightarrow{rot}\left(\frac{1}{2}\overrightarrow{grad}(\varphi)\right) = \frac{1}{2}.\overrightarrow{rot}\left(\overrightarrow{grad}(\varphi)\right)$$

$$\overrightarrow{rot}\left[\overrightarrow{grad}(\varphi)\right] = \vec{0} \text{ Or } \vec{\nabla}\Lambda\left[\vec{\nabla}.\varphi\right] = \vec{0} \quad \Rightarrow \quad \boxed{\overrightarrow{rot}(\vec{r}) = \vec{0}}$$

Exercise n°10:

1) $\overrightarrow{grad}(\emptyset) = \vec{\nabla}.\emptyset = \frac{\partial\emptyset}{\partial x}\vec{\imath} + \frac{\partial\emptyset}{\partial y}\vec{\jmath} + \frac{\partial\emptyset}{\partial z}\vec{k} = 6xy\vec{\imath} + (3x^2 + 4yz^3)\vec{\jmath} + 6y^2z^2)\vec{k}$

$$div(\vec{A}) = \vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = z - 1 - 2yz$$

$$\text{curl}(\vec{A}) = \vec{\nabla} \wedge \vec{A} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = z^2\vec{\imath} + x\vec{\jmath} + 4x\vec{k}$$

a) At the point (1,0,1) we have: $\overrightarrow{grad}(\emptyset) = 0$; $div(\vec{A}) = 3\vec{\jmath}$; $\overrightarrow{rot}(\vec{A}) = \vec{\imath} + \vec{\jmath} + 4\vec{k}$

Exercise n°11:

1. The partial derivatives with respect to x, y et z of the scalar function are given by:

$\emptyset(x,y,z) = 4x^3 \cos(3y) \exp(-2z)$

$\frac{\partial \emptyset(x,y,z)}{\partial x} = 12x^2 \cos(3y) \exp(-2z)$

$\frac{\partial \emptyset(x,y,z)}{\partial y} = -12x^3 \sin(3y) \exp(-2z)$.

$\frac{\partial \emptyset(x,y,z)}{\partial z} = -8x^3 \cos(3y) \exp(-2z)$.

3. The gradient of f is given by

$\vec{\nabla}\emptyset(x,y,z) = \frac{\partial \emptyset(x,y,z)}{\partial x}\vec{\imath} + \frac{\partial \emptyset(x,y,z)}{\partial y}\vec{\jmath} + \frac{\partial \emptyset(x,y,z)}{\partial z}\vec{k}$

$\vec{\nabla}\emptyset(x,y,z) =$

$12x^2 \cos(3y) \exp(-2z)\,\vec{\imath} - 12x^3 \sin(3y) \exp(-2z)\,\vec{\jmath} - 8x^3 \cos(3y) \exp(-2z)\vec{k}$

$\vec{F} = 3x^{2\vec{}}i + y^2x^3\vec{\jmath} + y\sqrt{x^2z^2}\vec{k}$

$div\ \vec{F} = \vec{\nabla}.\vec{F}$

Thus, the divergence of a vector $\vec{F} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}$ is given by:

$$div\ \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$div\ \vec{F} = 6x + 2yx^3 + xy$

curl $\vec{G} = \vec{\nabla} \wedge \vec{G}$

$$\vec{\nabla} \wedge \vec{G} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_x & G_y & G_z \end{vmatrix}$$

$$\text{curl}(\vec{G}) = \left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right).\vec{i} + \left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}\right).\vec{j} + \left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right).\vec{k}$$

$$\text{curl}(\vec{G}) = \left(\sqrt{xz}\ \exp(yz^3) - \frac{1}{z}\right).\vec{i} + \left(-2x^3 \sin(yz^2) - \frac{1}{2\sqrt{xz}} \exp(yz^3)\right).\vec{j}$$

$$+\left(\frac{1}{x} + x^3 \sin(yz^2)\right).\vec{k}$$

Exercise n°12:

$\vec{A} = 3x^2y\,\vec{\imath} + yz^2\,\vec{\jmath} - xz\,\vec{k}$, $\varphi(x,y,z) = x^2yz$

1- $\overrightarrow{grad}(\emptyset) = \vec{\nabla}.\emptyset = \frac{\partial\emptyset}{\partial x}\vec{\imath} + \frac{\partial\emptyset}{\partial y}\vec{\jmath} + \frac{\partial\emptyset}{\partial z}\vec{k} = 2xyz\vec{\imath} + x^2z\,\vec{\jmath} + x^2y\vec{k} = \vec{B}$

2- $div(\vec{A}) = \vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 6xy + z^2 - x$

3- $\text{curl}\left(\overrightarrow{grad}(\emptyset)\right) = \vec{\nabla} \wedge \vec{B} = \vec{\nabla} \wedge \vec{\nabla}\varphi(x,y,z) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} = 0$

4- $div(\text{curl}) = \vec{\nabla}.(\vec{\nabla} \wedge \vec{A})$

$$(\vec{\nabla} \wedge \vec{A}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = -2yz\vec{\imath} + z\,\vec{\jmath} - 3x^2\vec{k} = \vec{C}$$

$$div(\text{curl}) = \vec{\nabla}.(\vec{\nabla} \wedge \vec{A}) = \vec{\nabla}.\vec{C} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z} = 0$$

5-$(\varphi\vec{A}) = (x^2yz)(3x^2y\,\vec{\imath} + yz^2\,\vec{\jmath} - xz\,\vec{k}) = 3x^4y^2z\,\vec{\imath} + x^2y^2z^3\,\vec{\jmath} - x^3yz^2\,\vec{k}$

$$\frac{\partial(\varphi\vec{A})}{\partial z} = \frac{\partial(3x^4y^2z\,\vec{\imath} + x^2y^2z^3\,\vec{\jmath} - x^3yz^2\,\vec{k})}{\partial z} = 3x^4y^2\,\vec{\imath} + 3x^2y^2z^2\,\vec{\jmath} - 2x^3yz\,\vec{k}$$

$$\frac{\partial^2(\varphi\vec{A})}{\partial y\partial z} = \frac{\partial(3x^4y^2\,\vec{\imath} + 3x^2y^2z^2\,\vec{\jmath} - 2x^3yz\,\vec{k})}{\partial y} = 6x^4y\,\vec{\imath} + 6x^2yz^2\,\vec{\jmath} - 2x^3z\,\vec{k}$$

At the point M (1,-2,-1) we have: $\frac{\partial^2(\varphi\vec{A})}{\partial y\partial z} = -12\,\vec{\imath} - 12\,\vec{\jmath} + 2\,\vec{k}$

Exercise n°13:

a. Calculate $\frac{d}{dt}(\vec{V_1}.\vec{V_2})$ and $\frac{d}{dt}(\vec{V_1}\wedge\vec{V_2})$

$$\vec{V_1} = 5t^3\,\vec{\imath} + 3t\vec{\jmath} - 2t^4\,\vec{k}\ , \qquad \vec{V_2} = \sin t\ \vec{\imath} - \cos t\,\vec{\jmath} - 3t\,\vec{k}$$

8)
1) $\vec{V_1}.\vec{V_2} = 5t^3\sin t - 3t\cos t + 6t^5$

$\frac{d}{dt}(\vec{V_1}.\vec{V_2}) = 15t^2\sin t + 5t^3\cos t - 3\cos t + 3t\sin t + 30\,t^4$

$$\vec{V_1}\wedge\vec{V_2} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 5t^3 & 3t & -2t^4 \\ \sin t & -\cos t & -3t \end{vmatrix}$$

$\vec{V_1}\wedge\vec{V_2} = (-9t^2 - 2t^4\cos t)\vec{\imath} - (-15t^4 + 2t^4\sin t)\vec{\jmath} + (-5t^3\cos t - 3t\sin t)\vec{k}$

$$\frac{d}{dt}(\vec{V_1}\wedge\vec{V_2}) = (-18t - 8t^3\cos t + 2t^4\sin t)\vec{\imath} - (-60t^3 + 8t^3\sin t + 2t^4\cos t)\vec{\jmath} + (-15t^2\cos t + 5t^3\sin t - 3\sin t - 3t\cos t)\vec{k}$$

2) With respect to the origin O: $\vec{M}_{/O}(\vec{V_1}) = \vec{OA}\wedge\vec{V_1} = \begin{vmatrix} \overset{(+)}{\vec{i}} & \overset{(-)}{\vec{j}} & \overset{(+)}{\vec{k}} \\ 1 & 2 & 3 \\ 5t^3 & 3t & -2t^4 \end{vmatrix}$

So that $\vec{M}_{/O}(\vec{V_1}) = \vec{OA}\wedge\vec{V_1} = (-4t^4 - 9t).\vec{i} - (-2t^4 - 15t^3).\vec{j} + (3t - 10t^3).\vec{k}$

3) With respect to an axis (Δ) of unit vector $\vec{u}(1,0,-1)$

$$\vec{M}_{/\Delta}(\vec{V_1}) = -4t^4 - 9t - 3t + 10t^3 = -4t^4 + 10t^3 - 12t$$

Exercise n°14:

1- Compared to the origin O: $\overrightarrow{M}_{/O}\left(\overrightarrow{V}\right)=\overrightarrow{OA}\wedge\overrightarrow{V}=\begin{vmatrix} \overset{(+)}{\vec{i}} & \overset{(-)}{\vec{j}} & \overset{(+)}{\vec{k}} \\ 3 & 4 & 2 \\ 1 & 2 & 3 \end{vmatrix}$

$$\overrightarrow{M}_{/O}\left(\overrightarrow{V}\right)=\overrightarrow{OA}\wedge\overrightarrow{V} = 8.\vec{i}-7.\vec{j}+2.\vec{k}$$

2- In relation to the three main axes:

a- The ox axis: $\overrightarrow{M}_{/Ox}\left(\overrightarrow{V}\right)= \vec{i}.\overrightarrow{M}_{/O}\left(\overrightarrow{V}\right) = 8;$

b- The oy axis: $\overrightarrow{M}_{/Oy}\left(\overrightarrow{V}\right) = \vec{j}.\overrightarrow{M}_{/O}\left(\overrightarrow{V}\right) = -7$

c- The oz axis: $\overrightarrow{M}_{/Oz}\left(\overrightarrow{V}\right) = \vec{k}.\overrightarrow{M}_{/O}\left(\overrightarrow{V}\right) = 2$

3- Relative to an axis (Δ): $\overrightarrow{M}_{/\Delta}\left(\overrightarrow{V}\right)=\overrightarrow{M}_{/O}\left(\overrightarrow{V}\right).\vec{u}$,

$\vec{u}$ is the unit vector of the axis (Δ) and the direction cosines are its components.

$$\overrightarrow{M}_{/\Delta}\left(\overrightarrow{V}\right) = \begin{pmatrix} 8 \\ -7 \\ 2 \end{pmatrix} . \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{-8}{\sqrt{2}} - \frac{7}{2} + 1 = \frac{-8\sqrt{2}-5}{2}$$

Chapter3

KINEMATICS OF THE MATERIAL POINT

Learning Goals: After going through this chapter, students will be able to

- Learn about the concept of frames of reference in physics and specify a good frame of reference to use when describing the object's motion.
- Define position, velocity, and acceleration of a particle in rectilinear and curvilinear motion. Write the relationships between them and as a function of time. Velocity and acceleration depend on the choice of the reference frame.
- Identify and analyze special cases of rectilinear motion (uniform motion, uniformly accelerated or decelerated motion).
- Compute the derivative of a vector function and compute the components of vector in Cartesian, path, polar, cylindrical, and spherical coordinate systems. Use these concepts to analyze problems of projectile motion in both two-dimensions.

Exercise n°01:

A body moves on the x axis is according to the equation of motion:

$x = 2t^3 + 5t^2 + 5$, where x is the displacement at time t.

1) Give the expression of the speed v(t) and the acceleration a(t) at each instant t.
2) Calculate the positions of the body its instaneaous speeds and accelerations at t_1=2s and t_2= 3s.
3) Deduce the average speed and acceleration of the body between t_1 and t_2.

Exercise n°02:

The velocity-time graph of a point-like car moving along the *x* axis (*x*-direction is given in figure (shown below). We assume that at $t = t_0, x(t_0) = 0\ m$.

From the graph of velocity–time graph $v(t)$

1 - Plot the acceleration–time graph $a(t)$.

2- Deduce the nature of motion during the different phases of movement; justify your answers.

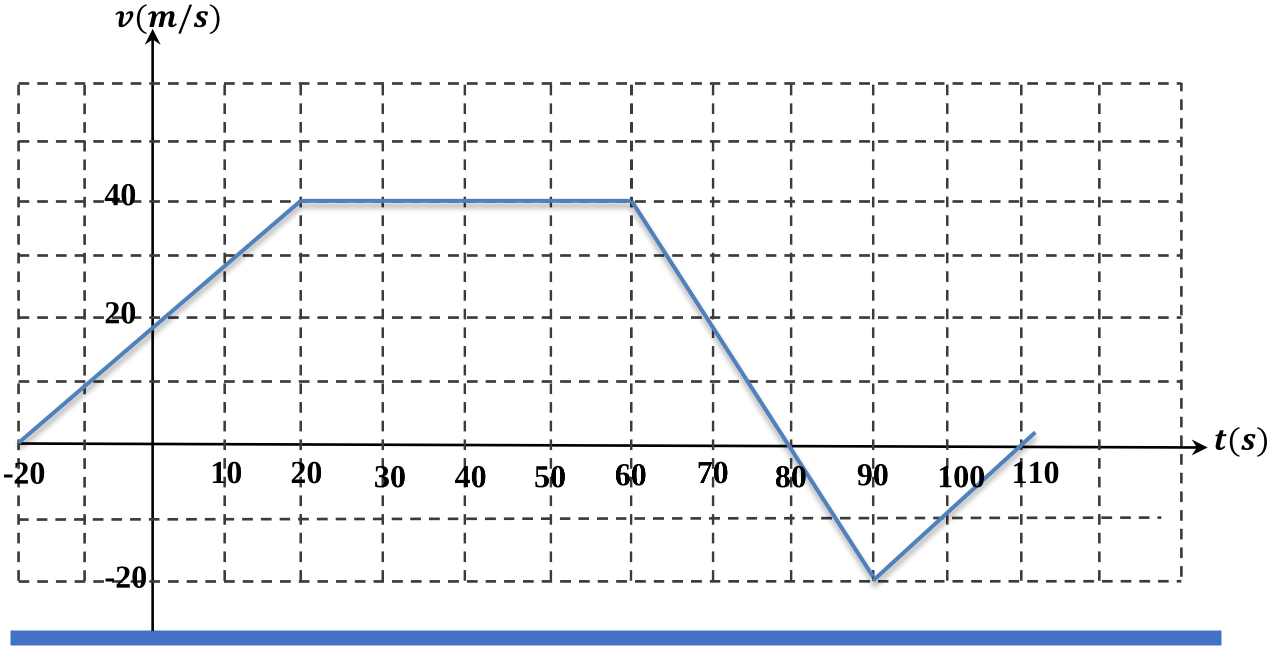
3- Draw the displacement–time graph $x = f(t)$.

4- Determine the expressions of the kinematic equations as a function of time $v(t)$ and $x(t)$ in the time interval [0, 110] (s).

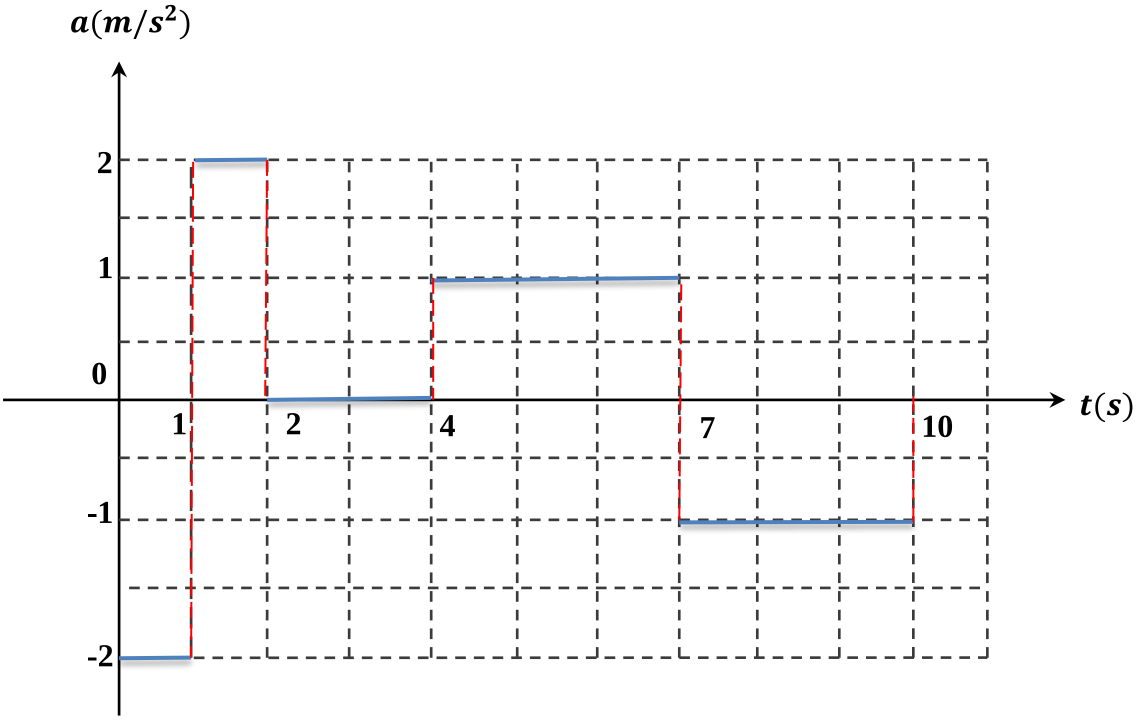
6-Calculate the average velocity during the time interval t = 0 (s) and t = 20 (s)

5- Use convenient scales to represent the, velocity and acceleration vectors at times t_1 = 10 (s), t_2 = 30 (s), t_3 = 65 (s), t_4 = 85 (s) and t_5 = 100 (s).

7-Calculate the total distance covered between t_0 and t = 110 (s), then compare it with the displacement at that time t = 110 (s).



Exercise n°03:



The diagram of the accelerations of a mobile moving in a rectilinear motion along the axis $(0X)$ is given by the figure (shown above).

We assume that at $t = 0s$, $x(t = 0) = 0m$ and $v(t = 0) = 0m/s$.

1. Deduce the nature of motion during the different phases of movement; justify your answers.

2. Write the expression for speed as a function of time $v(t)$ for each phase of movement. (Take the origin of times at the start of each phase).

3. Draw the speed diagram of the mobile.

Scale: $1m/s \rightarrow 1cm$, $1s \rightarrow 1cm$

4. Write the time equation of motion for each phase.

5. In what time intervals does the motion accelerate or decelerate? Justify.

6. Calculate the displacement of the mobile between instants $t = 0\ s$ and $t = 10s$

7. Calculate the average speed of the mobile between instants $t = 0\ s$ and $t = 10\ s$.

Exercise n°04:

The speed diagram of a mobile has a rectilinear motion along an axis (X'OX) in Figure (shown below).

1) Draw the diagram of the mobile's accelerations a (t).

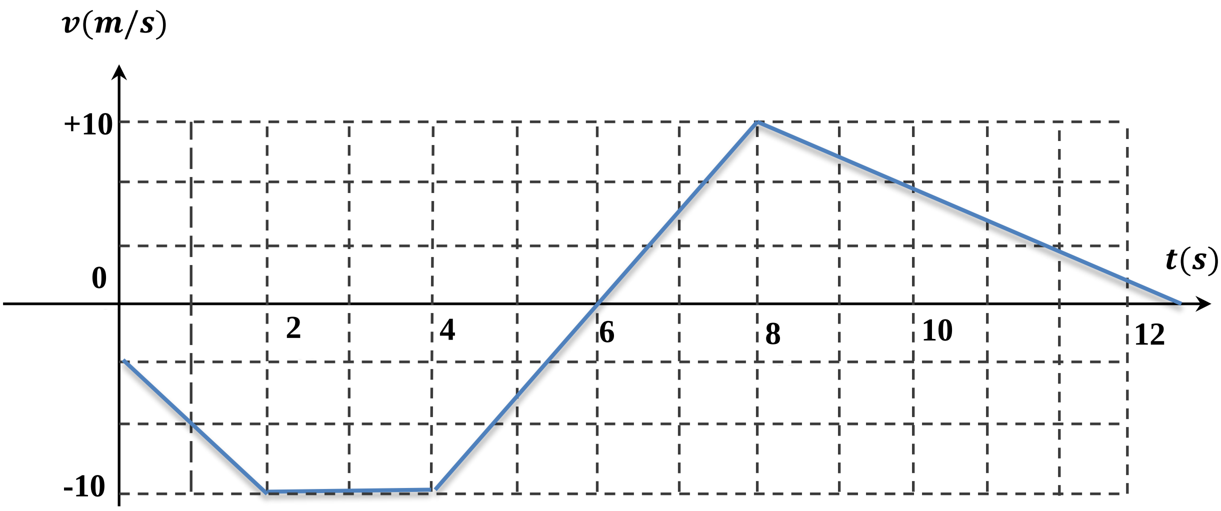
2) Determine the position of the mobile at instants t= 3s, t=7s and t=13 s.

We give x (t = 0)=30m.

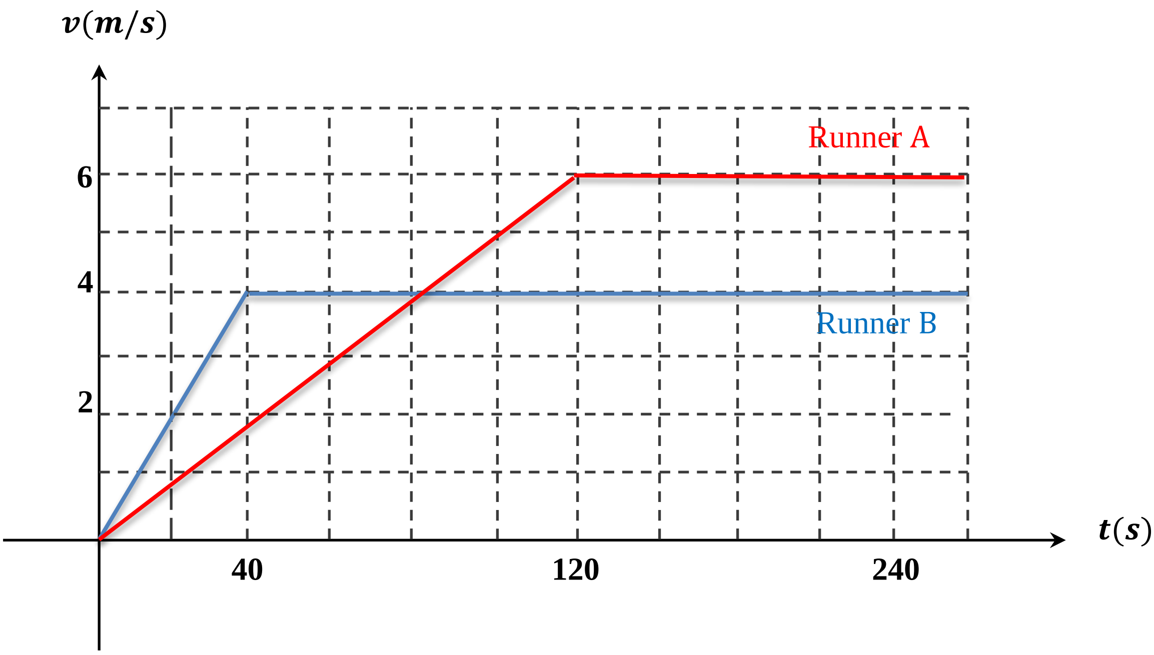
3) At what instant does the mobile turn back?

4) Deduce the nature of motion during the different phases of movement; justify your answers.

5°) Trace, on the trajectory, the vectors position, speed and acceleration at instants t= 3s and t=7s. Scale: $1cm \rightarrow 1m$, $1cm \rightarrow 5m/s$, $1cm \rightarrow 2\ m/s^2$



Exercise n°05:



In the figure (shown above), we have shown the speed diagrams of two runners ***A*** and ***B*** moving on the same track in a straight line. At the initial moment the two runners were on the starting line considered as the origin of the coordinates $(x_A(0) = x_B(0) = 0)$.

1. What is the nature of each runner's movement?
2. Calculate the instant t_1 when the two runners have the same speed.

Which runner is ahead at this instant t_1 and what distance D_1 separates him from the other runner?

3. Who is the runner ahead at t_2=120s and what distance D_2 separates him from the other runner?

4. At what time t_3 are the two runners side by side in the race?

5. Runner $\boldsymbol{A}$ crosses the finish line at time t_4=240s. At what time t_5 does runner $\boldsymbol{B}$ cross the finish line?

Exercise n°06:

An automobiles starts from rest and accelerates uniformly for 30 seconds to a speed of 72 km/h It then moves with a uniform velocity and it is finally brought to rest in 50 m with a constant retardation. If the total distance travelled is 950 m, find the acceleration, the retardation and total time taken.

Exercise n°07:

A bus accelerates uniformly from rest to the maximum speed of 20 m/s. It then drives at a constant speed and finally decelerates until it stops. It takes 21 s to cover 270 m. In absolute value, its acceleration is twice its deceleration.

1- Draw the graph of v as a function of t, v (t).
2- Find acceleration and distance traveled at maximum speed.

Exercise n°08:

A train traveling on a straight track, it begins with a constant acceleration a_1 starting with zero initial velocity to attain a speed of $v = 270\ km/h$ then keeps moving at this constant speed for a time t_2. Finally, it brakes its motion with a constant acceleration a_3=$-a_1$ to stop after having covered a total distance X=3 km (during the three phases of the movement).

1. What must be the train's acceleration a_1 so that the three stages have the same duration (t_1=t_2=t_3)?

2. What is the distance covered in each stage. (x_1, $x_{2,3}$)

3. Write the time equations of the three phases of movement considering the origin of the spaces the starting point of the train.

Exercise n°09:

A student is running at her top speed of 5 m/s to catch a bus, which is stopped at the bus stop. When the student is still 30 m from the bus, it starts to pull away moving with a constant acceleration of 0.15 m/s^2.

1. For how much time and what distance does the student? have to run at 5.0 m/s before she overtakes the bus?
2. When she reached the bus, how fast was the bus travelling?
3. Sketch an x-t graph for both the student and the bus.
4. The equations you used in part (1) to find the time have a second solution, corresponding to a later time for which the student and the bus are again at the same place if they continue their specified motions. Explain the significance of this second solution. How fast is the bus travelling at this point?
5. If the student's top speed is 3.5 m/s, will she catch the bus?
6. What is the minimum speed the student must catch up with the bus? For what does she have to run in that case?

Exercise n°10:

A car A is stopped on a straight horizontal road at a distance d_1=3 m from a red light. When the light turns green, at time t=0, the car starts with a constant acceleration a_1=3 m /s^2. At the same time a motorcyclist M traveling at a constant speed v_2=54 km/h is at a distance d_2=24 m rom the traffic light. We will choose as the origin O of the abscissa the position of the traffic light.

1. Determine the time equations $x_v(t)$ et $x_m(t)$ of the car and the biker respectively.
2. Determine the times of overtaking as well as the positions of the car and the biker at these times.
3. If the motorcyclist was traveling at speed v_2=36 km/h could he catch the car? Justify.
4. Determine, in this cases, the shortest distance that separates motorcyclis and car.

Exercise n°11:

The coordinates of a particle are given by the functions of time: $\begin{cases} x = 2.t \\ y = 4t(t-1) \end{cases}$

Find: 1- The equation of trajectory y=f(x).

2- The component of speed and acceleration.

3- The tangential acceleration a_T and normal acceleration a_N
4- The radius of curvature R of the particle's trajectory.

Exercise n°12:

The coordinates of a particle are given by the functions of time: $\begin{cases} x = 4t + 3 \\ y = t^2 - 5t + 2 \end{cases}$

Find:

1- The equation of trajectory y=f(x).
2- The component of speed and acceleration.
3- The tangential acceleration a_T and normal acceleration a_N
4- The radius of curvature R of the particle's trajectory

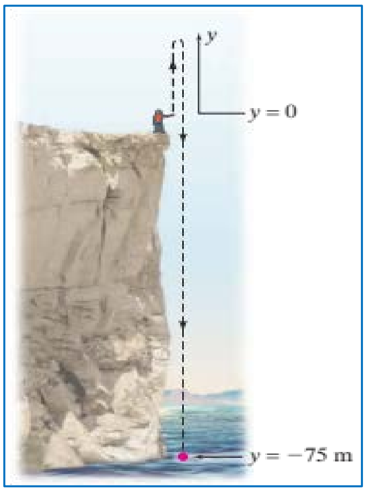
Exercise n°13:

The coordinates of an object moving in the x y plane vary with time according to the equations: $\begin{cases} x = 2a(\cos\omega t + 1) \\ y = a \sin\omega t \end{cases}$, where a is a positive constant

Find:

1- The equation of trajectory y=f(x). What is its nature?
2- The component of speed and acceleration.
3- The tangential acceleration a_T and normal acceleration a_N
4- The radius of curvature R of the particle's trajectory.

Exercise n°14:



A stone is thrown vertically upward with an initial speed of 15.5 ms^{-1} from the edge of a cliff y_0 = 75 m high.

- How much later does it reach the bottom of the cliff?
- What is its speed just before hitting?
- What total distance did it travel?

Exercise n°15:

A projectile is fired with an initial speed of 36.6 m/s at an angle of 42.2° above the horizontal on a long flat firing range. Determine

(a) The maximum height reached by the projectile,

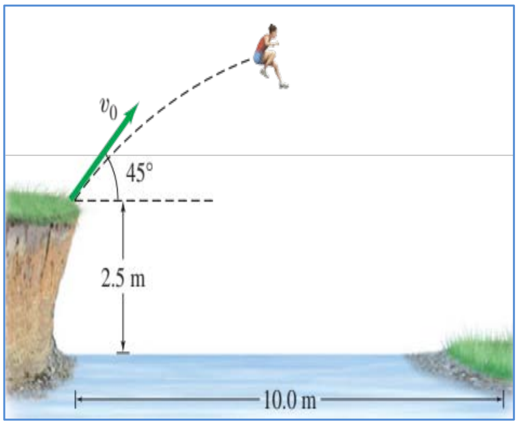
(b) The total time in the air,

(c) The total horizontal distance covered (that is the range),

(d) The speed of the projectile 1.50 s after firing.

Exercise n°16:

1- A long jumper leaves the ground at 45° above the horizontal and lands 8 m away. What is her “takeoff” speed v_0?



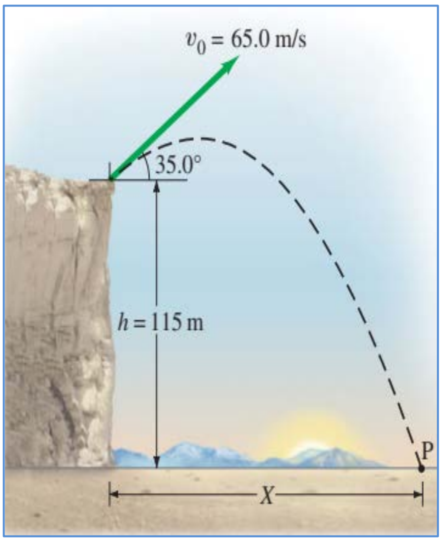
2- Now she is out on a hike and comes to the left bank of a river. There is no bridge and the right bank is 10 m away horizontally and 2.5 m vertically below.

If she long jumps from the edge of the left bank at 45° with the speed calculated in (1), how long, or short, of the opposite bank will she land

Exercise n°17:

A projectile is shot from the edge of a cliff 115 m above ground level with an initial speed of 65 m/s at an angle of 35° with the horizontal, as shown in Fig.

(*a*) Determine the time taken by the projectile to hit point P at ground level.



(*b*) Determine the distance X of point P from the base of the vertical cliff. At the instant just before the projectile hits point P,

(*c*) Find the horizontal and the vertical components of its velocity,

(*d*) The magnitude of the velocity,

(*e*) The angle made by the velocity vector with the horizontal.

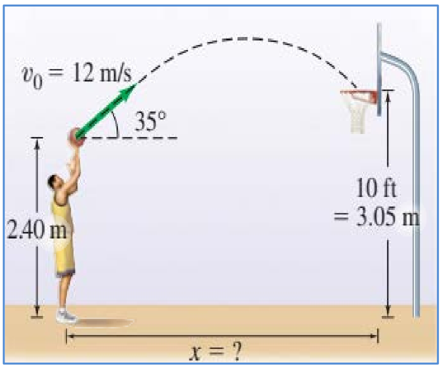
(f) Find the maximum height above the cliff top reached by the projectile.

Exercise n°18:

A basketball is shot from an initial height of 2.40m (as shown in Fig.) with an initial speed $v_0 = 12$ m/s directed at an angle ($\theta_0 = 35°$) above the horizontal.

(*a*) How far from the basket was the player if he made a basket?

(b) At what angle to the horizontal did the ball enter the basket?



Exercise n°19:

The curvilinear motion of a mobile is described by the following parametric equations:

$$r(t) = 2R_0 \sin(\omega t) \quad \text{and} \quad \theta = \omega t \qquad (0 \le \omega t \le \pi)$$

(t)in seconds; (r) in meter and ; (θ) in radians. (R_0) and (ω) are positive constants.

1. Find the equation of the trajectory in Cartesian coordinates. Represent this trajectory in an orthonormal coordinate system.

2. Calculate the expression for the velocity vector $\vec{v}(t)$ and its magnitude $v(t)$ in polar coordinates.

3. Calculate the expression for the acceleration vector $\vec{a}(t)$and its magnitude $a(t)$ in polar coordinates.

4. Calculate the tangential component a_T and the normal component a_N of the acceleration vector as a function of time.

5. Deduce the radius of curvature depending on time.

6. Find the expressions of the unit vectors; $\vec{e}_T$ tangent to the trajectory and $\vec{e}_N$ normal to the trajectory.

We give: $\sin(2\alpha) = 2\sin\alpha\cos\alpha$ and $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$

Exercise n°20:

A material point M describes, in a frame$\mathcal{R}(O, \vec{j}, \vec{k})$, a trajectory defined by the parametric equations:

$$\begin{cases} x = b\sin(\omega t) \\ y = b\,(1 - \cos(\omega t)) \end{cases}$$, Where b and ω are positive constants

1. Give the equation of the trajectory.

2. Give the polar coordinates ρ and θ of M.

3. Give the Cartesian, polar and intrinsic components of the velocity and acceleration vectors and calculate their magnitude.

Exercise n°21:

In the $x\,y$ plane of Cartesian coordinate system $\mathcal{R}(O;x,y,z)$, with an orthonormal set basis $(\vec{\imath},\vec{\jmath},\vec{k})$, consider a particle describing a curvilinear path where a point M has got polar coordinates ρ and θ given by,

$$\begin{cases}\rho = r_0 exp^{\theta(t)} \\ \theta(t) = \omega t\end{cases} \quad , \quad \text{Where } r_0 \text{ and } \omega \text{ positives and constants.}$$

1- Write the vector position in polar coordinates.

2- Find the velocity and acceleration in polar coordinates and calculate their magnitudes.

2- Calculate the tangential acceleration and the normal acceleration.

4- Deduce the radius of curvature.

5- Calculate the curvilinear abscissa S(t) as a function of time.

Exercise n°22:

The equation of spiral (helical) motion is given by:

$$x(t) = b\cos(\omega t)\ , \quad y(t) = b\sin(\omega t) \quad , \quad z(t) = c\omega t$$

1. Represent the movement in an orthonormal coordinate system.

2. Calculate the velocity and acceleration in the cylindrical coordinate system. What is the nature of the movement when $b = 0$; or when $c = 0$.

Exercise n°23:

The movement of a point in space is defined by:

$$x(t) = bcos(\gamma t^2)\ , \quad y(t) = b\sin(\gamma t^2) \quad , \quad z(t) = b\gamma t^2$$

1. Calculate vectors, speed $\vec{v}(t)$ and acceleration $\vec{a}(t)$ as a function of t.

2. Determine the tangential a_T and normala_N accelerations as a function of .

3. Deduce the radius of curvature $\rho(t)$ of the trajectory.

SOLUTIONS TO EXERCISES

Exercise n°01:

a - We have $x(t) = 2t^3 + 5t^2 + 5$ donc :

- The speed would be $v(t) = \frac{dx}{dt} = 6t^2 + 10t$

- The acceleration would be $a(t) = \frac{dv(t)}{dt} = 12t + 10$

b - The position of the body, at time $t_1 = 2s$, as well as its instantaneous speed acceleration

The position: $x(2) = 2(2)^3 + 5(2)^2 + 5 = 41m$

Instantaneous speed: $v(2) = 6(2)^2 + 10(2) = 44m/s$

Instantaneous acceleration: $a(2) = 12(2) + 10 = 34m/s^2$

- La position du corps, à l'instant $t_2 = 3s$, ainsi que sa vitesse et son accélération instantanée

The position: $x(3) = 2(3)^3 + 5(3)^2 + 5 = 104m$

Instantaneous speed: $v(3) = 6(3)^2 + 10(3) = 84m/s$

Instantaneous acceleration: $a(3) = 12(3) + 10 = 46m/s^2$

c - We deduce the average speed and acceleration of the body betweent_1 and t_2

Average speed:

$$v_{moy} = \frac{\Delta x}{\Delta t} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} v_{moy} = \frac{104 - 41}{3 - 2} = 63m/s$$

Average acceleration:

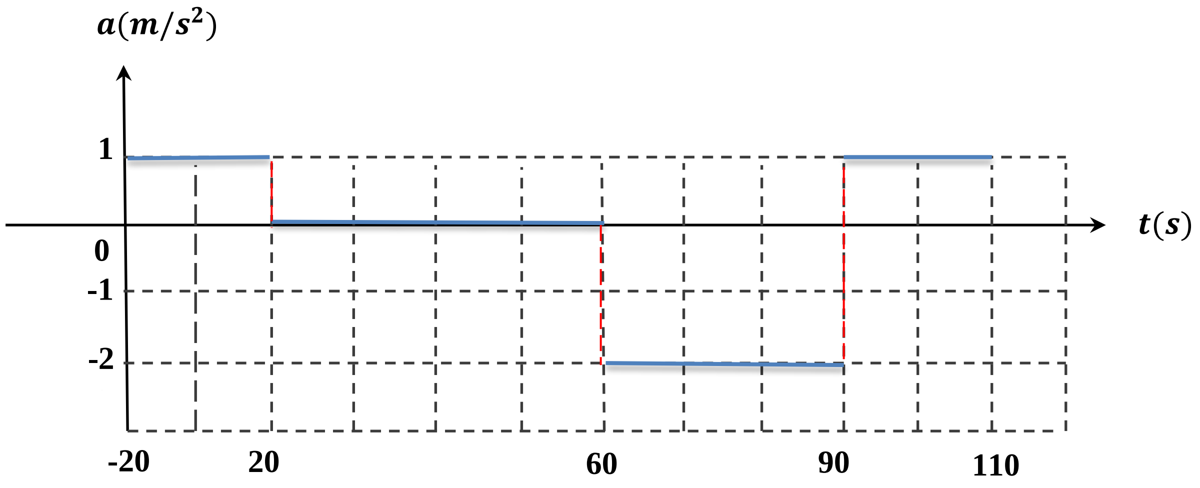
$$a_{moy} = \frac{\Delta v}{\Delta t} = \frac{v(t_2) - v(t_1)}{t_2 - t_1} a_{moy} = \frac{84 - 44}{3 - 2} = 40m/s^2$$

Exercise n°02:

1. From the speed diagram $v(t)$,

We deduces the initial conditions of the motion, (the initial time t_0 and the car's initial velocity $v(t_0)$). $t_0 = -20s$ et $v(t_0) = 0m/s$

(b) The acceleration of the mobile is defined by $a(t) = \frac{dv(t)}{dt}$, i.e, graphically, it is the slope of the tangent to the curve of $v(t)$. The calculation of the acceleration diagram $a(t)$ is shown in figure below.



2. The nature of movement of the mobile depends on the sign of product $\boldsymbol{v(t)} \times \boldsymbol{a(t)}$. The study of the two graphs $v(t)$ and $a(t)$ allows us easily on the one hand to identify the different phases of movement and the nature of movement on the other hand:$v(t) \times a(t)$

Phase 1 : [t ∈-20, 20]$\begin{cases} v > 0 \\ a > 0 \end{cases}$

$av > 0 \Rightarrow$ Uniformly Accelerated Rectilinear Motion (UARM)$(a = C^{st})$

Phase 2:$t \in [20,60]$ $\begin{cases} v > 0 \\ a = 0 \end{cases}$

$av = 0 \Rightarrow$ Uniform rectilinear motion (MRU) $(v = C^{st} \neq 0)$

Phase 3:$t \in [60,80]$ $\begin{cases} v > 0 \\ a < 0 \end{cases}$

$av < 0 \Rightarrow$ Uniformly decelerated Rectilinear Motion (UDRM)$(a = C^{st})$

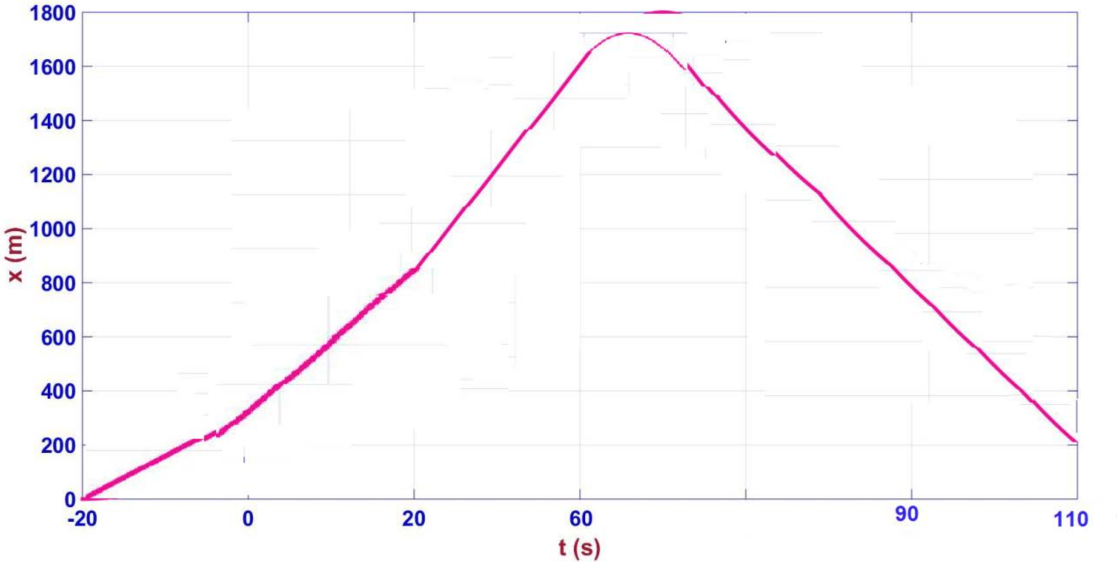
Phase 4:$t \in [80,90]$ $\begin{cases} v < 0 \\ a < 0 \end{cases}$

$av > 0 \Rightarrow$ Uniformly Accelerated Rectilinear Motion (UARM).$(a = C^{st})$

Phase 5:$t \in [90,110]$ $\begin{cases} v < 0 \\ a > 0 \end{cases}$

$av < 0 \Rightarrow$ Uniformly Accelerated Rectilinear Motion (UARM)$(a = C^{st})$

3. Figure shown below represents the space diagram obtained by calculating the area under the curve of $v(t)$



2. The kinematic equations $v(t)$ and $x(t)$ in the different phases of movement are calculated from the general expressions below which are valid whatever the initial conditions and depending on the nature of movement:

Case of uniform rectilinear motion:

$a = 0$ and $v = C^{st} \neq 0 \begin{cases} v = C^{st} = v(t_0) \\ x(t) = v(t_0)t + x(t_0) \end{cases}$

Case of a uniformly varied rectilinear motion:$(a = C^{st})$

$$a = C^{st} \neq 0 \begin{cases} v(t) = a(t - t_0) + v(t_0) \\ x(t) = \dfrac{a}{2}(t - t_0)^2 + v(t_0)(t - t_0) + x(t_0) \end{cases}$$

Phase 1 :

$t \in$[-20, 20]$\begin{cases} a = 1 \\ v(t) = t + 20 \\ x(t) = \frac{1}{2}t^2 + 20t + 200 \end{cases}$

Phase 2:

$t \in [20,60] \begin{cases} a = 0 \\ v = C^{st} = 40(m/s) \\ x(t) = 40t \end{cases}$

Phase 3:

$t \in [60,80] \begin{cases} a = -2 \\ v(t) = -2t + 160 \\ x(t) = -t^2 + 160t - 3600 \end{cases}$

Phase 4:

$t \in [80,90] \begin{cases} a = -2 \\ v(t) = -2t + 160 \\ x(t) = -t^2 + 160t - 3600 \end{cases}$

Phase 5:

$$t \in [90,110] \begin{cases} a = 1 \\ v(t) = t - 110 \\ x(t) = \dfrac{1}{2}t^2 - 110t + 8750 \end{cases}$$

3. The average speed between and is given by,$t = 0t = 20(s)$

$$v_{moy} = \frac{x(20) - x(0)}{20 - 0} = 30m/s$$

4. The speed and acceleration vectors at times $t_1 = 10(s), t_2 = 30(s), t_3 = 65(s), t_4 = 85(s) t_5 = 100(s)$, are given by:

$$t_1 = 10(s) \begin{cases} x(10) = 450(m) \\ \vec{v}(10) = 30\vec{\imath}(m/s) \\ a(10) = 1\,\vec{\imath}(m/s^2) \end{cases}$$

$$t_2 = 30(s) \begin{cases} x(30) = 1200(m) \\ \vec{v}(30) = 40\,\vec{\imath}(m/s) \\ \vec{a}(30) = 0\,\vec{\imath}(m/s^2) \end{cases}$$

$$t_3 = 65(s) \begin{cases} x(65) = 2575(m) \\ \vec{v}(65) = 30\vec{\imath}\ (m/s) \\ \vec{a}(65) = -2\,\vec{\imath}(m/s^2) \end{cases}$$

$$t_4 = 85(s) \begin{cases} x(85) = 2775(m) \\ \vec{v}(85) = -10\vec{\imath}(m/s) \\ \vec{a}(85) = -2\vec{\imath}(m/s^2) \end{cases}$$

$$t_5 = 100(s) \begin{cases} x(100) = 2750(m) \\ \vec{v}(100) = -10\,\vec{\imath}(m/s) \\ \vec{a}(100) = 1\vec{\imath}(m/s^2) \end{cases}$$

5. The total distance traveled by the mobile between t_0 and t = 110 (s), can be calculated by summing all the areas under the curve of $x(t)$. Either

$$d = \frac{40 \times 40}{2} + (60 - 20) \times 40 + \frac{(80 - 60) \times 40}{2} + \frac{(90 - 80) \times 20}{2} + \frac{(110 - 90) \times 20}{2}$$

$$\Rightarrow d = 3100m$$

On the other hand, the total displacement during this time interval is

$$x(t) = \frac{40 \times 40}{2} + (60 - 20) \times 40 + \frac{(80 - 60) \times 40}{2} - \frac{(90 - 80) \times 20}{2} - \frac{(110 - 90) \times 20}{2}$$

$$\Rightarrow x(t) = 2500m$$

Exercise n°03:

1. Nature of motion.

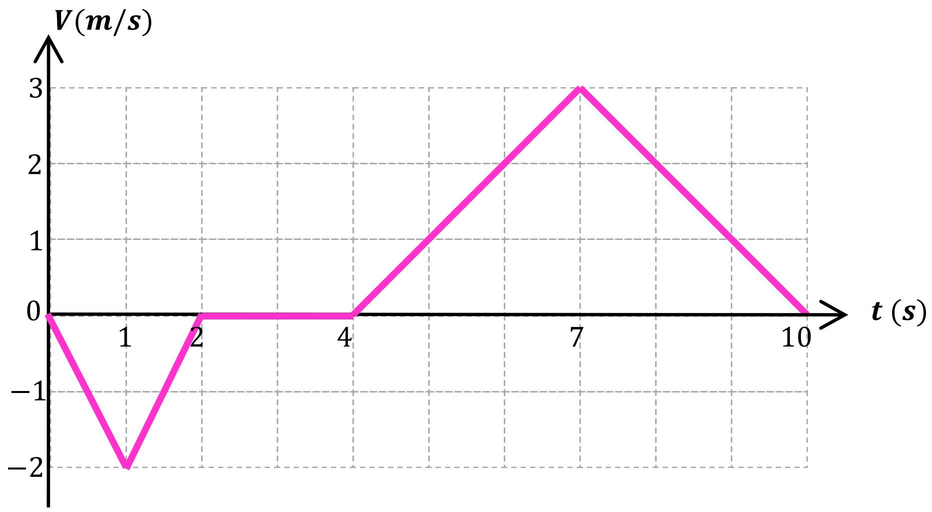
$0 \leq t \leq 1\,s$	$a = Cst$	Uniformly accelerated rectilinear
$1 \leq t \leq 2\,s$	$a = Cst$	Uniformly accelerated rectilinear motion
$2 \leq t \leq 4\,s$	$a = 0 \quad \Rightarrow \quad V = Cst$	Uniform rectilinear motion
$4 \leq t \leq 7\,s$	$a = Cst$	Uniformly accelerated rectilinear motion
$7 \leq t \leq 10\,s$	$a = Cst$	Uniformly accelerated rectilinear motion

2. Expression of speed.

$0 \leq t \leq 1\,s$	$a_1 = -2\ m/s^2$	$V_1(t) = a_1.t + V_{01}$	$V_1(t) = -2.t$
$1 \leq t \leq 2\,s$	$a_2 = +2\ m/s^2$	$V_2(t) = a_2.t + V_{02}$	$V_2(t) = +2.t - 2$
$2 \leq t \leq 4\,s$	$a_3 = 0\ m/s^2$	$V_3(t) = V_{03}$	$V_3(t) = 0$
$4 \leq t \leq 7\,s$	$a_4 = +1\ m/s^2$	$V_4(t) = a_4.t + V_{04}$	$V_4(t) = +1.t$
$7 \leq t \leq 10\,s$	$a_5 = -1\,m/s^2$	$V_5(t) = a_5.t + V_{05}$	$V_5(t) = -1.t + 3$

$V_{01} = 0\ m/s$ .The velocity at the end of each step is equal to the area under the acceleration diagram.

3. Speed diagram.



4. Kinematic equation of motion.

$0 \leq t \leq 1\,s$	$V_1(t) = -2.t$	$x_1(t) = \frac{1}{2}a_1.t^2 + V_{01}.t + x_{01}$	$x_1(t) = -1.t^2$
$1 \leq t \leq 2\,s$	$V_2(t) = +2.t - 2$	$x_2(t) = \frac{1}{2}a_2.t^2 + V_{02}.t + x_{02}$	$x_2(t) = +1.t^2 - 2.t - 1$
$2 \leq t \leq 4\,s$	$V_3(t) = 0$	$x_3(t) = V_{03}.t + x_{03}$	$x_3(t) = -2$
$4 \leq t \leq 7\,s$	$V_4(t) = +1.t$	$x_4(t) = \frac{1}{2}a_4.t^2 + V_{04}.t + x_{04}$	$x_4(t) = +0{,}5.t^2 - 2$
$7 \leq t \leq 10\,s$	$V_5(t) = -1.t + 3$	$x_5(t) = \frac{1}{2}a_5.t^2 + V_{05}.t + x_{05}$	$x_5(t) = -0{,}5.t^2 + 3.t + 2{,}5$

$x_{01} = 0\ m$ .The displacement at the end of each step is equal to the area under the velocity diagram.

$0 \leq t \leq 1\ s$	$V_1(t) \leq 0$	$a_1(t) < 0$	Accelerated motion.
$1 \leq t \leq 2\ s$	$V_2(t) \leq 0$	$a_2(t) > 0$	Decelerated motion.
$2 \leq t \leq 4\ s$	$V_3(t) = 0$	$a_3(t) = 0$	Stop from the material point.
$4 \leq t \leq 7\ s$	$V_4(t) \geq 0$	$a_4(t) > 0$	Accelerated motion.
$7 \leq t \leq 10\ s$	$V_5(t) \geq 0$	$a_5(t) < 0$	Decelerated motion.

5. Displacement = Area under the speed diagram between $t = 0\ s$ and $t = 10\ s$

$$D = x(10s) - x(0s) = \frac{2 \times (-2)}{2} + \frac{(10-4) \times 3}{2}$$

SO

$$\boxed{D = x(10s) - x(0s) = +7\ m}$$

6. The average speed between and is given by,$t = 0\ s$ and $t = 10\ s$

$$v_{moy} = \frac{x(10) - x(0)}{10 - 0} = 0.7 m/s$$

Exercise n°04:

1 - by definition, $a = \frac{dV}{ar}$, graphically represents the slope of the tangent to the graph $V(t)$.

As the graph of $V(t)$ is represented by line segments, the slope of the tangent is equal to that of these segments.

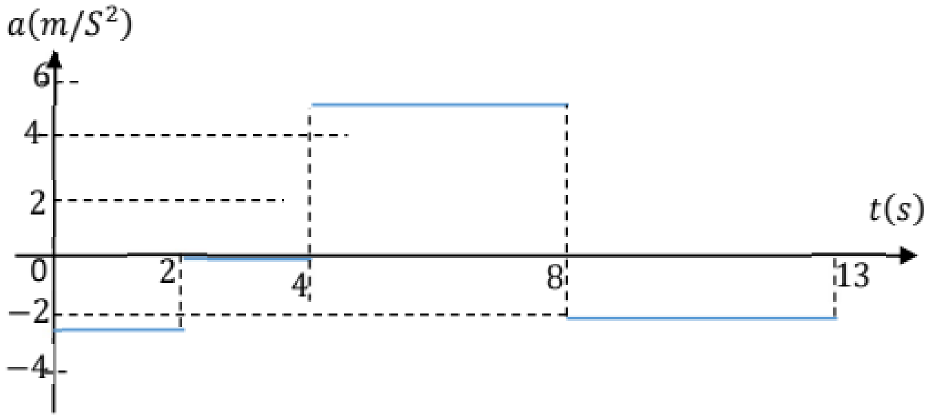
Phase :$t \in [0,2s]$; $a = \frac{V(2)-V(0)}{2-0} = \frac{-10-(-5)}{2} = -2{,}5 m/s^2$

Phase:$t \in [2,4s]$; $a = \frac{V(4)-V(2)}{4-2} = \frac{-10-(-10)}{2} = 0 m/s^2$

Phase:$t \in [4,8s]$; $= \frac{V(8)-V(4)}{8-4} = \frac{-10-(-10)}{4} = 5 m/s^2$

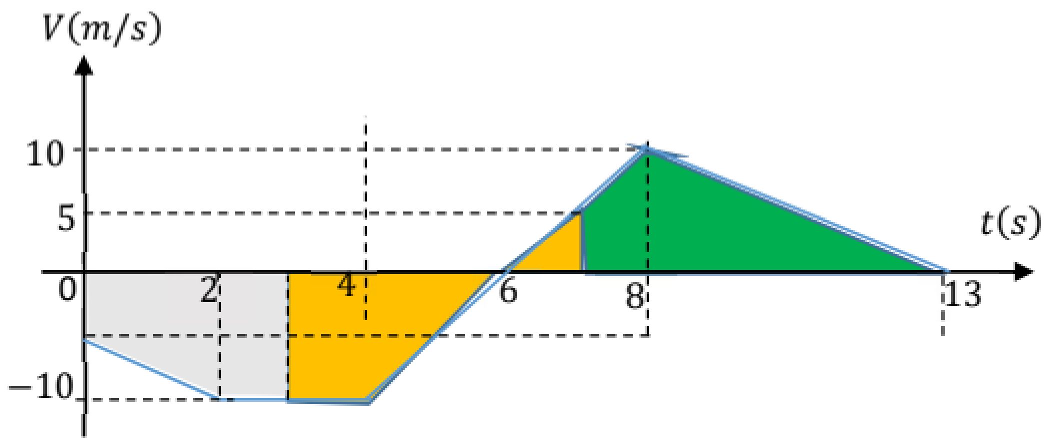
Phase:$t \in [8,13s]$; $= \frac{V(13)-V(8)}{13-8} = \frac{0-10}{5} = -2 m/s^2$

Pay attention at the end of the exercise, you have been given scales that must be used to draw the diagrams



2 - To determine the position of the mobile at a given time, we use the graph $V(t)$ to calculate the areas under the curve

$\int dx = \int V\, dt =$ Area under the graph $V(t)$



$$\Delta x_3 = x_3 - x_0 = \int_0^3 V\, dt = area\ in\ gray$$

$$\Delta x_3 = (x_2 - x_0) + (x_3 - x_2) = area\ of\ trapezoid + area\ of\ triangle$$

$$\Delta x_3 = \frac{(-5-10)2}{2} + 1 * (-10) = -15-10 = -25$$

$$\Delta x_3 = x_3 - x_0 \Rightarrow x_3 = x_0 - 25 = 30 - 25 = 5m$$

$$\Delta x_7 = x_7 - x_3 = \int_3^7 V\, dt = area\ in\ orange$$

$$\Delta x_7 = area\ of\ trapezoid + area\ of\ triangle$$

$$\Delta x_7 = \frac{(1+3)*(-10)}{2} + \frac{5*1}{2} = -20 + 2{,}5 = -17{,}5$$

$$\Delta x_7 = x_7 - x_3 \Rightarrow x_7 = \Delta x_7 + x_3 = 5 - 17{,}5 = -12{,}5m$$

$$\Delta x_{13} = x_{13} - x_7 = \int_7^{13} V\, dt = area\ in\ green$$

$$\Delta x_{13} = area\ of\ trapezoid + area\ of\ triangle$$

$$\Delta x_{13} = \frac{(5+10)*1}{2} + \frac{10*5}{2} = 32{,}5 = x_{13} - x_7 \Rightarrow x_{13} = x_7 + 20 = -12{,}5 + 32{,}5 = 20m$$

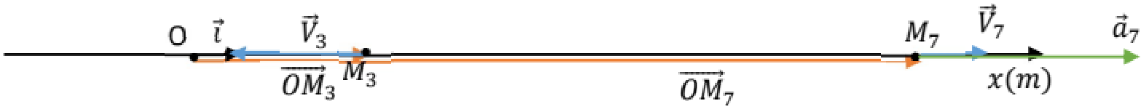
3- The mobile turns back when the speed changes sign passing through zero. From the graph $V(t)$, this position corresponds to t=6s.

4 - The nature of the movement is defined by the sign of the scalar product: $\vec{V}\vec{a} = V_X a_X$

Phase	Signs of $\vec{V}$ and $\vec{a}$	$a_X(m/s^2)$	Nature of motion
$t \in [0, 2]$	$V_X < 0;\ a_X < 0$	$-2{,}5$	Uniformly accelerated rectilinear motion
$t \in [2, 4]$	$V_X = \text{Cst}$	0	Uniform rectilinear movement
$t \in [4, 6]$	$V_X < 0;\ a_X > 0$	5	Uniformly decelerated rectilinear motion
$t \in [6, 8]$	$V_X > 0;\ a_X > 0$	5	Uniformly accelerated rectilinear motion
$t \in [8, 13]$	$V_X > 0;\ a_X < 0$	-2	Uniformly decelerated rectilinear motion

5 - Representation of position, speed and acceleration vectors

$t(S)$	$x(m)$	$V(\text{m/s})$	$a(\text{m}/s^2)$
3	5	-10	0
7	20	5	5



Exercise n°05:

1. Nature of motion.

Runner A	$0 \leq t \leq 120\ s$	Uniformly accelerated rectilinear motion	$a_A = \frac{6}{120} = 0{,}05\ m/s^2 = \text{Cst}$
	$120 \leq t \leq 240\ s$	Uniform rectilinear motion	$V_A = 6\ m/s = \text{Cst}$
Runner B	$0 \leq t \leq 40\ s$	Uniformly accelerated rectilinear motion	$a_B = \frac{4}{40} = 0{,}1\ m/s^2 = \text{Cst}$
	$t \geq 40\ s$	Uniform rectilinear motion	$V_B = 4\ m/s = \text{Cst}$

2. Both runners have the same speed $(t = t_1)$ at the intersection point of the two diagrams.

$$V_A(t_1) = V_B(t_1) = 4\ m/s \quad \Rightarrow \quad \tan(\alpha) = \frac{4}{t_1} = \frac{6}{120} \quad \Rightarrow \quad \boxed{t_1 = 80\ s}$$

The distance separating the two runners $D_1 = |x_A(t_1) - x_B(t_1)|$

The position at a time is calculated by the area under the velocity diagram $(x_A(0) - x_B(0) = 0)$ between times $t_0 = 0\ s$ and t .

$$x_A(t_1) = \frac{V_A(t_1) \times t_1}{2} = \frac{4 \times 80}{2} = 160\ m \qquad \text{et} \qquad x_B(t_1) = \frac{V_B(t_1) \times [(t_1 - 40) + t_1]}{2}$$

$$= \frac{4 \times (80 + 40)}{2} = 240\ m$$

Hence, the runner ***B*** is ahead of the runner ***A*** and the distance separating them is:

$$\boxed{D_1 = |x_A(t_1) - x_B(t_1)| = 80\ m}$$

3. $t_2 = 120\ s$:

$$x_A(t_2) = \frac{V_A(t_2) \times t_2}{2} = \frac{6 \times 120}{2} = 360\ m \quad \text{et} \quad x_B(t_2) = \frac{V_B(t_2) \times [(t_2 - 40) + t_2]}{2}$$

$$= \frac{4 \times (80 + 40)}{2} = 400\ m$$

Hence, the runner ***B*** is ahead of the runner ***A*** and the distance separating them is:

$$\boxed{D_2 = |x_A(t_2) - x_B(t_2)| = 40\ m}$$

4. Since the runner ***A*** has not yet caught up with the runner ***B*** at$(t = t_2)$, then the time when the two runners are side by side is$(t_3 > t_2)$.

$$x_A(t_3) = \frac{V_A(t_3) \times [(t_3 - 120) + t_3]}{2} = 6.t_3 - 360$$

$$\text{and} \quad x_B(t_3) = \frac{V_B(t_3) \times [(t_3 - 40) + t_3]}{2} = 4.t_3 - 80$$

The two runners are side by side:

$$x_A(t_3) = x_B(t_3) \Rightarrow 6.t_3 - 360 = 4.t_3 - 80 \Rightarrow t_3 = 140\ s$$

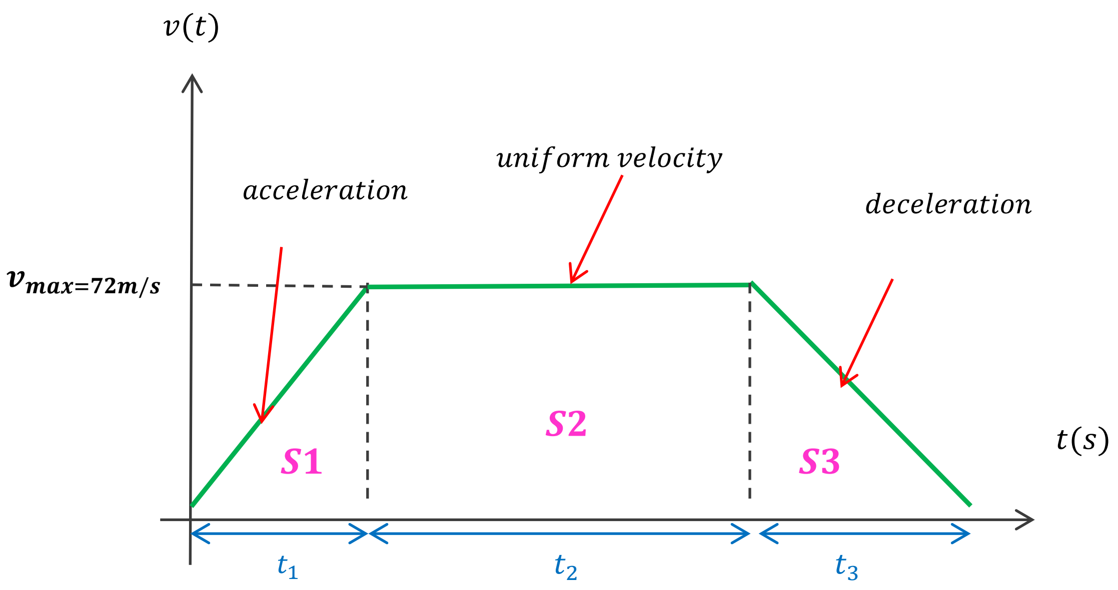
5. $t_4 = 240\ s$: The length of the track is

$$L = x_A(t_4) = \frac{V_A(t_4) \times [(t_4 - 120) + t_4]}{2} = \frac{6 \times (120 + 240)}{2} \Rightarrow L = 1080\ m$$

The runner ***B*** reaches the finish line at$(t = t_5)$ $\Rightarrow$ $x_B(t_5) = L$. SO :

$$x_B(t_5) = \frac{V_B(t_5) \times [(t_5 - 40) + t_5]}{2} = 4.t_5 - 80 = 1080 \Rightarrow \boxed{t_5 = 290\ s}$$

Exercise n°06:



t_1= 30 second,

v = 72Km/h =72 $\times \frac{5}{18} = 20m/s$

$a = \frac{v}{t_1} = \frac{20}{30} \Rightarrow a = \frac{2}{3} m/s^2$

Let distance travelled in acceleration be S1

$S1 = \frac{1}{2}at^2 = \frac{1}{2}\frac{2}{3} \times 30^2 = 300m$

Let distance travelled in uniform velocity and retardation be S2 and S3 respectively

S3=50m

Total distance travelled is $d = 950m$

$\Rightarrow S2 = d - (S1 + S3) = 950 - (300 + 50) = 600m$

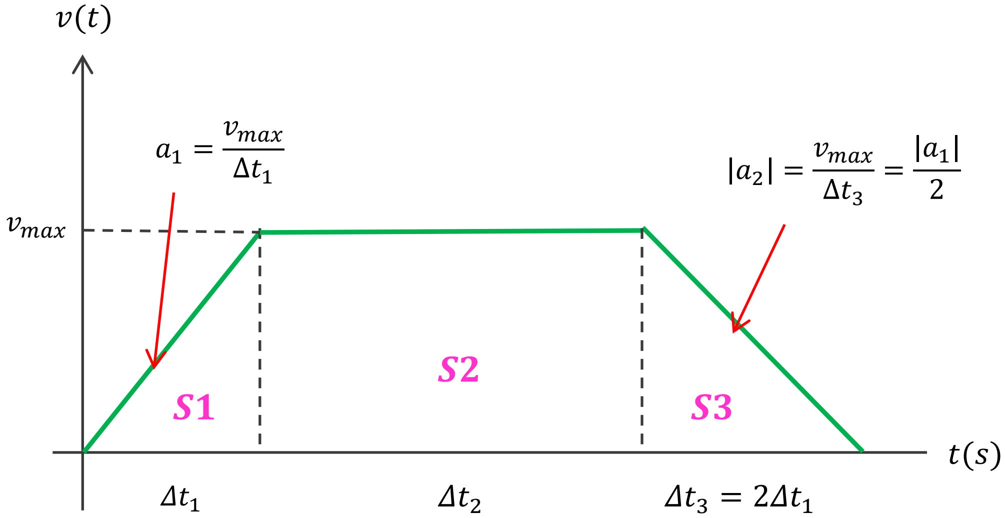
$t_2 = \frac{S2}{v} = \frac{600}{20} = 30s$

Retardation $r = \frac{v^2}{2S3} = \frac{20^2}{20\times 50} = 4m/s^2$

$t_3 = \frac{v}{r} = \frac{20}{4} = 5s$

Total time $t_1 + t_2 + t_3 = 30 + 30 + 5 = 65s$

Exercise n°07:



Maximum speed is 20 m/s, time (t) = 21 second, d= 270 m is the total distance traveled

1.According to the variation graph $v = f(t)$, we can find the distances Δx_1, Δx_2et Δx_3 from the areas (S1), (S2) et (S3)

$$\Delta x = \quad v = \frac{dx}{dt} \Rightarrow x = vdt \Rightarrow \Delta x = \int vdt$$

Ou Δx = ($S_{i=1to3}$)

$$\Rightarrow \begin{cases} \Delta x_1 = v_{max}\frac{\Delta t_1}{2} \\ \Delta x_2 = v_{max}\Delta t_2 \\ \Delta x_3 = v_{max}\frac{\Delta t_3}{2} \end{cases}$$

$d = \Delta x_1 + \Delta x_2 + \Delta x_3$

$$\Rightarrow \begin{cases} 270 = v_{max}\left(\frac{\Delta t_1}{2} + \Delta t_2 + \frac{\Delta t_3}{2}\right) \\ 21 = \Delta t_1 + \Delta t_2 + \Delta t_3 \\ \Delta t_3 = 2\Delta t_1 \end{cases}$$

$$\Rightarrow \begin{cases} 270 = 20\left(\frac{\Delta t_1}{2} + \Delta t_2 + \Delta t_1\right) \\ 21 = 3\Delta t_1 + \Delta t_2 \end{cases}$$

$$\Rightarrow \begin{cases} 270 = 30\Delta t_1 + 20\Delta t_1 \\ 21 = 3\Delta t_1 + \Delta t_2 \end{cases}$$

$$\Rightarrow \begin{cases} \Delta t_1 = 5s \\ \Delta t_2 = 6s \\ \Delta t_3 = 10s \end{cases}$$

$v_{max} = a_1 \Delta t_1 \Rightarrow a_1 = \frac{v_{max}}{\Delta t_1} \Rightarrow a_1 = 4m/s^2$

2. Next, let's find the distance traveled at maximum speed

$\Delta x_2 = v_{max}\Delta t_2 \Rightarrow \Delta x_2 = 120m.$

Exercise n°08:

1. Kinematic equation of the three steps (considering the origin of times and spaces at the start of each step)

$$\begin{cases} x_1(t) = \frac{1}{2}a_1.t^2 \\ V_1(t) = x^{\bullet} = a_1.t \\ a_1(t) = x^{\bullet\bullet} = a_1 \end{cases} \qquad \begin{cases} x_2(t) = V.t \\ V_2(t) = V \\ a_2(t) = 0 \end{cases} \qquad \begin{cases} x_3(t) = -\frac{1}{2}a_1.t^2 + V.t \\ V_3(t) = -a_1.t + V \\ a_3(t) = -a_1 \end{cases}$$

The three steps have the same duration $\Rightarrow t_1 = t_2 = t_3 = T$

The total distance traveled is: $X_1 + X_2 + X_3 = X$

Substituting into the kinematic equation of motion

$$\frac{1}{2}a_1.t^2 + V.t + \left(-\frac{1}{2}a_1.t^2 + V.t\right) = X \qquad \Rightarrow \qquad T = \frac{X}{2V}$$

On the other hand and when then$V_1(t) = a_1.tt = t_1 = TV_1(T) = a_1.T = V \Rightarrow T = V/a_1$

Comparing the two equations we obtain

$$\frac{X}{2V} = \frac{V}{a_1} \qquad \Rightarrow \qquad \boxed{a_1 = \frac{2.V^2}{X}}$$

NC: $V = 270\ km/h = 75\ m/s$ and $X = 3\ km = 3000\ m$

$$\boxed{a_1 = 3{,}75\ m/s^2} \text{ and } \boxed{t_1 = t_2 = t_3 = T = 20\ s}$$

2. The distance traveled at each step

$$\boxed{\begin{cases} X_1 = \frac{1}{2}a_1.T^2 = 750\ m \\ X_2 = V.T = 1500\ m \\ X_3 = -\frac{1}{2}a_1.T^2 + V.T = 750\ m \end{cases}}$$

3. The kinematic equations of motion (considering the origin of the spaces at the start of the train's motion):

$$\begin{cases} x_1(t) = \frac{1}{2}a_1.t^2 + x_1(0) = 1{,}875.t^2 \\ x_2(t) = V.t + x_2(0) = 75.t + 750 \\ x_3(t) = -\frac{1}{2}a_1.t^2 + V.t + x_3(0) = -1{,}875.t^2 + 75.t + 2250 \end{cases}$$

Exercise n°09:

1. For the student speed- cst (URM):$x_1(t) = v_1 t + x_{01} = 5t$, $x_{01} = 0$ initial position for the student

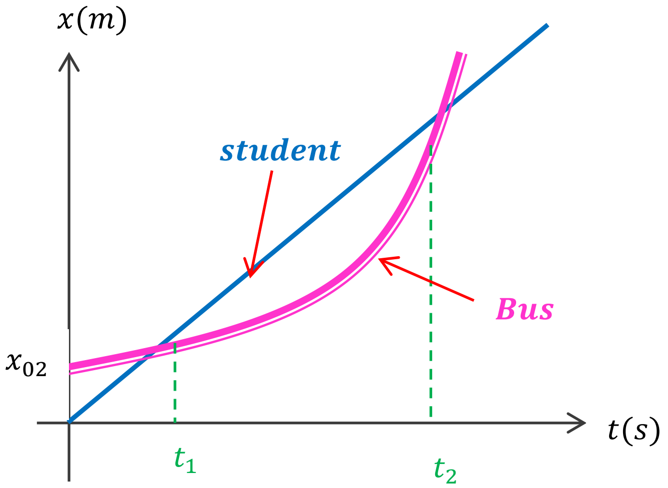
   For the bus acceleration- cst (URM V):$x_2(t) = \frac{1}{2}\ a_2 t^2 +\ v_{02} t + x_{02} = 0.075t^2 + 30$ with $v_{02} = 0$

The student catches up with the cad bus: $x_1(t) = x_2(t)\ \Rightarrow\ 0.075t^2 + 30 = 5t$

$\Rightarrow\ 0.075t^2 - 5t + 30$= 0 and solvint for the time t give: $t_1 = 6.66\ s\ , t_2 = 60\ s$

Distance run by the student to catch the bus is $x_1(t_1)$= $5t_1 = 33{,}33\ m$

Bus speed: $v_2(t) = a_2(t) + v_{02} = 0.15(6{,}66) = 0{,}99$



$x_1(t) = 2{,}6t$, $x_2(t) = 0.075t^2 + 30$

$x_1(t) = x_2(t) \Rightarrow 0.075t^2 + 30 = 2{,}6t$

$\Delta < 0$ No solutions i.e. with this speed, the student never catches up with the bus.

2. Determining the minimum speed for the student to catch up with the bus:

The student catches up with the bus if $\frac{1}{2}a_2t^2 + v_1t + x_{02} = 0$ admits solutions $v_1{}^2 - 2.a_2.x_{02} > 0$

And so that the minimum speed if $v_1{}^2 - 2.a_2.x_{02} = 0 \Rightarrow v_1 = \sqrt{2.0{,}75.30} = 3\ m/s$

Time: $t = \frac{3}{0{,}15} = 20\ s$

Distance: $x_1 = 3.20 = 60m$

Exercise n°10:

1- For the car: $x_1(t) = \frac{a_1}{2}t^2 + d_1 = 1.5t^2 - 3$

For the motorcycle: $x_2(t) = v_2t - d_2 = 15t - 24$

2- The student will be likely to hop on the bus the first time she passes it :

$x_1(t) = x_2(t) \Rightarrow 1.5t^2 - 3 = 15t - 24$

$\Rightarrow 1.5t^2 - 15t + 21 = 0$ and solvint for the time t give:

$$\begin{cases} t_1 = 1{,}68\ s \\ x_1(2) = 1{,}2\ m \end{cases} \quad \text{et} \quad \begin{cases} t_1 = 8{,}32\ s \\ x_1(2) = 100{,}8\ m \end{cases}$$

3- If : $v_2 = 36$ km/h = 10 m/s we solving the equation: $1{,}5t^2 - 10t + 21 = 0$ which has no solution because $\Delta' = -6{,}5$ is negative so they are not going to meet.

4- Determining the minimum distance:

a- $\Delta x = x_2 - x_1 = 1{,}5t^2 - 10t + 21$; Δx $x$is minimal if its derivative is zero

i .e : $\Delta x' = 0 \Rightarrow 3t - 10 = 0$

$$\Rightarrow t = \frac{10}{3}\ s$$

b- Δx_{min} =4.2 m

Exercise n°11:

1. $\begin{cases} x = 2.t \ \ldots\ldots\ldots\ldots\ldots (1) \\ y = 4.t(t-1) \ldots\ldots\ldots\ldots\ldots\ldots (2) \end{cases}$

$(1) \Rightarrow t = \frac{x}{2}$ ……(3) we replace (3) in (2)

The result is $\boldsymbol{y} = = \boldsymbol{x^2 - 2x}$ it's the equation of a parabola

2. $v = ?$

$$\begin{cases} v_x(t) = \dfrac{dx(t)}{dt} = 2 \\ v_y(t) = \dfrac{dy(t)}{dt} = 8t - 4 \end{cases}$$

$v = |\vec{v}| = \sqrt{{v_x}^2 + {v_y}^2}$ $\Rightarrow$ $v = \sqrt{2^2 + (8t-4)^2} = 2\sqrt{16t^2 - 16t + 5}$

$a = ?$

$$\begin{cases} a_x(t) = \dfrac{dv_x(t)}{dt} = \dfrac{dv_0}{dt} = 0. \\ a_y(t) = \dfrac{dv_y(t)}{dt} = \dfrac{d(8t-4)}{dt} = 8 \end{cases} \Rightarrow a = |\vec{a}| = \sqrt{0^2 + (8)^2} \Rightarrow a = +8\ m/s^2$$

$a_T = ?$

$$a_T = \frac{dv}{dt} = \frac{d(2\sqrt{16t^2-16t+5})}{dt} \Rightarrow a_T = \frac{16.t - 8}{\sqrt{16t^2-16t+5}}$$

$a_N = ?$

We know that $\vec{a} = \vec{a_T} + \vec{a_N} \Rightarrow a^2 = {a_T}^2 + {a_N}^2 \Rightarrow a_N = \sqrt{a^2 - {a_T}^2}$

So $a_N = \dfrac{16}{2\sqrt{16t^2-16t+5}} = \dfrac{16}{v}$

R= ?

We know that $a_N = \dfrac{v^2}{R} \Rightarrow$ R$= \dfrac{v^2}{a_N}$ $\Rightarrow$ R $= \dfrac{v^3}{16}$

Exercise n°12:

1. $\begin{cases} x = 4.t + 3 \ldots\ldots\ldots\ldots\ldots (1) \\ y = t^2 - 5t + 2 \ldots\ldots\ldots\ldots\ldots (2) \end{cases}$

$(1) \Rightarrow t = \dfrac{x-3}{4}$ ……(3) we replace (3) in (2)

The result is $\boldsymbol{y = \frac{1}{16}x^2 - \frac{13}{8}x + \frac{101}{16}}$ it's the equation of a parabola

2. $v = ?$

$$\begin{cases} v_x(t) = \dfrac{dx(t)}{dt} = 4 \\ v_y(t) = \dfrac{dy(t)}{dt} = 2t - 5 \end{cases}$$

$v = |\vec{v}| = \sqrt{{v_x}^2 + {v_y}^2}$ $\Rightarrow$ $v = \sqrt{4^2 + (2t-5)^2} = \sqrt{4t^2 - 20t + 41}$

$a = ?$

$$\begin{cases} a_x(t) = \dfrac{dv_x(t)}{dt} = \dfrac{dv_0}{dt} = 0. \\ a_y(t) = \dfrac{dv_y(t)}{dt} = \dfrac{d(8t-4)}{dt} = 2 \end{cases} \Rightarrow a = |\vec{a}| = \sqrt{0^2 + (2)^2} \Rightarrow a = +2\ m/s^2$$

$a_T = ?$

$$a_T = \frac{dv}{dt} = \frac{d(\sqrt{4t^2-20t+41})}{dt} \Rightarrow a_T = \frac{8.t - 20}{2\sqrt{4t^2-20t+41}} = \frac{4.t - 10}{\sqrt{4t^2-20t+41}}$$

$a_N = ?$

We know that $\vec{a} = \vec{a_T} + \vec{a_N} \Rightarrow a^2 = {a_T}^2 + {a_N}^2 \Rightarrow a_N = \sqrt{a^2 - {a_T}^2}$

So $a_N = \dfrac{64}{\sqrt{4t^2-20t+41}} = \dfrac{8}{v}$

R= ?

We know that $a_N = \frac{v^2}{R} \Rightarrow$ R=$\frac{v^2}{a_N}$ $\Rightarrow$ R $= \frac{v^3}{8}$

Exercise n°13:

$x = 2a(\cos\omega t + 1)$, $y = a\sin\omega t$, $z = o$

1. $\begin{cases} x = 2a(\cos\omega t + 1) \Rightarrow \cos\omega t = \frac{x}{2a} - 1 \Rightarrow sin^2(\omega t) = \left(\frac{x}{2a} - 1\right)^2 \\ y = a\sin\omega t \Rightarrow \sin\omega t = \frac{y}{a} \Rightarrow (\sin\omega t)^2 = \left(\frac{y}{a}\right)^2 \end{cases}$

$sin^2(\omega t) + cos^2(\omega t) = 1 = \left(\frac{x}{2a} - 1\right)^2 + \left(\frac{y}{a}\right)^2$, This is the equation of an ellipse with center (1.0) and major axis $2a$, small axis a.

2. $\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k} = 2a(\cos\omega t + 1)\vec{\imath} + a\sin\omega t\,\vec{\jmath}$

$$\vec{V} = \frac{d\vec{r}}{dt} = -2a\omega\sin\omega t\,\omega t\,\vec{\imath} + a\omega\cos\omega t\,\vec{\jmath}$$

$$\|\vec{V}\| = \sqrt{4a^2\omega^2\sin^2\omega t + a^2\omega^2\cos^2\omega t} = a\omega(1 + 3\sin^2\omega t)^{1/2}$$

$$\vec{V}\begin{cases} V_x = -2a\omega\sin\omega t\,\omega t \\ V_y = a\omega\cos\omega t \\ V_z = 0 \end{cases} \Rightarrow \vec{\gamma}\begin{cases} \gamma_x = -2a\omega^2\cos\omega t \\ \gamma_y = -a\omega^2\sin\omega t \\ \gamma_z = 0 \end{cases}$$

γ_T =?

$$a_T = \frac{dv}{dt} = \frac{d}{dt}[a\omega(1 + 3\sin^2\omega t)^{1/2}] \Rightarrow \gamma_T = \frac{\frac{1}{2}a\omega^2 6\cos\omega t\sin\omega t}{(1+3\sin^2\omega t)^{1/2}}$$

$$\Rightarrow \gamma_T = \frac{3a\omega^2\cos\omega t\sin\omega t}{(1+3\sin^2\omega t)^{1/2}}$$

γ =?

$$\gamma^2 = 4a^2\omega^4\cos^2\omega t + a^2\omega^4\sin^2\omega t = 4a^2\omega^4 - 3a^2\omega^4\sin^2\omega t$$

γ_N =?

We know that $\vec{\gamma} = \vec{\gamma_T} + \vec{\gamma_N} \Rightarrow \gamma^2 = {\gamma_T}^2 + {\gamma_N}^2 \Rightarrow \gamma_N = \sqrt{\gamma^2 - {\gamma_T}^2}$

So $\gamma_N = \frac{2a\omega^2}{\sqrt{1+3\sin^2\omega t}}$

R= ?

We know that $a_N = \frac{v^2}{R} \Rightarrow$ R=$\frac{v^2}{a_N}$ $\Rightarrow$ R $= \frac{a}{2}(1 + 3\sin^2\omega t)^{1/2}$

Exercise n°14:

Choose upward to be the positive direction and $y_0 = 0$ to be at the throwing location of the stone. The initial velocity is $v_0 = 15.5m/s$, the acceleration is $a = -9.80m/s^2$, and the final location is $y = -75m$.

The kinematic equations for constant acceleration $a = -9.80m/s^2$:

$$\left| \begin{array}{l} \boldsymbol{y = \frac{1}{2}at^2 + v_0 t + y_0 \ldots (1)} \\ \boldsymbol{v = at + v_0 \ldots (2)} \\ \boldsymbol{v^2 - v_0{}^2 = 2a\left(y - y_0\right) \ldots (3)} \end{array} \right.$$

(a) Using kinematic equation (1) and substituting y for x, we have the following:

$$y = \frac{1}{2}at^2 + v_0 t + y_0 \quad \rightarrow (4.9m/s^2)t^2 - (15.5m/s)t - 75m = 0 \rightarrow$$

To solve any quadratic equation of the form:

$$at^2 + bt + c = 0$$

Where *a*, *b*, and *c* are constants (*a* is not acceleration here), we use the quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We rewrite our *y* equation just above in standard form, $at^2 + bt + c = 0$

$$(4.9m/s^2)t^2 - (15.5m/s)t - 75m = 0$$

Using the quadratic formula, we find as solutions

$$t = \frac{15.5 \pm \sqrt{(155)^2 - 4(49)(-75)}}{2(4.9)} = \begin{cases} t = 5.802\text{s} \\ t = -2.638\text{s} \end{cases}$$

The positive answer is the physical answer: $\boxed{= 5.80\text{s}}$.

(b) Use kinematic equation(2) to find the velocity just before hitting.

$$v = at + v_0 = (-9.80\ m/s^2)(5.802\ \text{s}) + 15.5m/s = -41.4m/s \rightarrow \boxed{|v| = 41.4\ m/s}$$

(c)

The total distance traveled will be the distance up plus the distance down. The distance down will be 75 m more than the distance up. To find the distance up, use the fact that the speed at the top of the path will be 0. Then using kinematic equation(3) we have the following:

$$v^2 = 2a\left(y - y_0\right) + v_0{}^2 \Rightarrow y = y_0 + \frac{v^2 - v_0^2}{2a} = 0 + \frac{0-(15.5m/s)^2}{2(-9.80m/s^2)} = 12.26m$$

Thus the distance up is 12.26 m, the distance down is 87.26 m, and the total distance traveled is $\boxed{99.5\ m.}$

Exercise n°15:

Choose the origin to be where the projectile is launched and upward to be the positive y direction. The initial velocity of the projectile is v_0, the launching angle isθ_0, $a_y = -g$, and $v_{0y} = v.\sin\theta_0$.

(a) The maximum height is found from Eq. (3) with $v_y = 0$ at the maximum height.

$$y_{\max} = 0 + \frac{v_y^2 - v_{y0}^2}{2a_y} = \frac{-v_0^2 \sin^2\theta_0}{-2g} = \frac{v_0^2 \sin^2\theta_0}{2g} = \frac{(36.6m/s)^2 \sin^2 42.2°}{2(980m/s^2)} = 30{,}8m$$

(b) The total time in the air is found from Eq. (1) , with a total vertical displacement of 0 for the ball to reach the ground.

$$y = \frac{1}{2}a_y t^2 + v_{0y}t + y_0 \quad \rightarrow 0 = v_0 \sin\theta_0 \, t - \frac{1}{2}gt^2$$

$$\Rightarrow t = \frac{2v_0 \sin\theta_0}{g} = \frac{2(36.6\text{m/s}) \sin 42.2^o}{9.80\text{m/s}^2} = 5.0173\text{s} = 5{,}02\text{s and } t = 0$$

The time of 0 represents the launching of the ball.

(c) The total horizontal distance covered is found from the horizontal motion at constant velocity. $\Delta x = v_x\, t = (v_0 \cos\theta_0)\, t = (36.6\, m/s)(\cos 42.2°)\,(5.0173\text{s}) = 136m$

(d) The velocity of the projectile 1.50 s after firing is found as the vector sum of the horizontal and vertical velocities at that time. The horizontal velocity is a constant $v_0 \cos\theta_0$ = (36.6 m/s)(cos 42.2^o) = $27{,}11 m/s$. The vertical velocity is found from Eq. (2).

$$v_y = at + v_{y0} = v_0 \sin\theta_0 - gt = (36.6m/s)\sin 42.2° - (9.80m/s^2)(1.50\text{s})$$

$$\Rightarrow v_y = 9{,}885 m/s$$

Thus the speed of the projectile is as follows:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(27{,}11m/s)^2 + (9{,}885m/s)^2} = 28{,}9\, m/s$$

Exercise n°16:

(a) Use the level horizontal range formula from the text to find her takeoff speed.

$$R = \frac{v_0^2 \sin 2\theta_0}{g} \rightarrow v_0 = \sqrt{\frac{gR}{\sin 2\theta_0}} = \sqrt{\frac{(980\text{m/s}^2)(80\text{m})}{\sin 90°}} = 8.854\text{m/s} = 8.9\text{m/s}$$

(b) Let the launch point be at the $y = 0$ level, and choose upward to be positive. Use Eq. (1) to solve for the time to fall to 2.5 m below the starting height, and then calculate the horizontal distance traveled.

$$y = \frac{1}{2}a_y t^2 + v_{0y}t + y_0 \quad \rightarrow -2.5\text{m} = \frac{1}{2}(-9.80\text{ m/s}^2)t^2 + (8.854\text{m/s})\sin 45° t$$

$$4.9t^2 - 6.261t - 2.5\text{m} = 0 \rightarrow$$

$$t = \frac{6{,}261 \pm \sqrt{(6{,}261)^2 - 4(4{,}9)(-2.5)}}{2(4{,}9)} = \frac{6.261 \pm 9.391}{2(4{,}9)} = \begin{cases} t = -0.319\text{s} \\ t = 1.597\text{ s} \end{cases}$$

Use the positive time to find the horizontal displacement during the jump.

$$\Delta x = v_{0x}\, t = (v_0 \cos 45°)\, t = (8.854\text{ m/s})\cos 45°(1.597\text{s}) = 10.0\text{m}$$

She will land exactly on the opposite bank, neither long nor short.

Exercise n°17:

Choose the origin to be at ground level, under the place where the projectile is launched, and upward to be the positive y direction. For the projectile, $v_0 = 65.0\text{m/s}$, $\theta_0 = 35.0°$, $a_y = -g$, $y_0 = 115\text{m}$, and $v_{y0} = v_0 \sin\theta_0$

(a) The time taken to reach the ground is found from Eq. (1), with a final height of 0.

$y = \frac{1}{2}a_y t^2 + v_{0y}t + y_0 \quad \rightarrow 0 = y_0 + \upsilon \sin\theta\, t - \frac{1}{2}gt^2 \rightarrow$

$$t = \frac{-v_0 \sin\theta_0 \pm \sqrt{v_0^2 \sin^2\theta_0 - 4\left(-\frac{1}{2}g\right)y_0}}{2\left(-\frac{1}{2}g\right)} = 9.964\text{s}, -2.3655\text{s} = 9.96\text{s}$$

(b) Choose the positive time since the projectile was launched at time $t = 0$.

The horizontal range is found from the horizontal motion at constant velocity.

$\Delta x = v_x\, t = (v_0 \cos\theta_0)\, t = 65.0\ \text{m/s} \cos 35.0°\ 9.964\ \text{s} = 531\text{m}$

(c) At the instant just before the particle reaches the ground, the horizontal component of its velocity is the constant $v_x = v_0 \cos\theta_0 = 65.0\ \text{m/s}\ \cos 35.0° = 53{,}2\text{m/s}$.The vertical component is found from Eq. (2).

$v_y = at + v_{y0} = v_0 \sin\theta_0 - gt = 65.0\ \text{m/s} \sin 35.0^o - (9.80\text{m/s}^2)\ (9.964\text{s})$

$\Rightarrow v_y = -60{,}4\text{m/s}$

(d) The magnitude of the velocity is found from the x and y components calculated in part (c) above. $\upsilon = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{(53{,}2\text{m/s})^2 + (-60{,}4\text{m/s})^2} = 80{,}5\text{m}$

(e) The direction of the velocity is $\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{-60.4}{53.2} = -48.6°$, so the object is moving 48.6^o below the horizontal .

(f) The maximum height above the cliff top reached by the projectile will occur when the y velocity is 0 and is found from Eq.(3).

${v_y}^2 = 2a\left(y - y_0\right) + {v_{y0}}^2 \ \rightarrow 0 = v_0 \sin^2\theta_0 - 2gy_{\max}$

$$y_{\max} = \frac{v_0^2 \sin^2\theta_0}{2g} = \frac{(65.0\text{m/.s})^2 \sin^2 35^o}{2(9{,}80\text{m/s}^2)} = 70{,}9\text{m}$$

Exercise n°18:

Find the time of flight from the vertical data, using Eq. (1). Call the floor the $y = 0$ location, and choose upward as positive.

$y = \frac{1}{2}a_y t^2 + v_{0y}t + y_0 \rightarrow 3.05\text{m} = \frac{1}{2}(-9.80\text{m/s}^2)t^2 + (12\text{m/s}) \sin 35°t + 2.40\text{m}$

$$4{,}90t^2 - 6.883t + 0.65\text{m} = 0 \rightarrow$$

$$t = \frac{6.883 \pm \sqrt{6,883^2 - 4(4,90)(0.65)}}{2(4.90)} = \begin{cases} t = 1.303\text{s} \\ t = 0\,.102\text{ s} \end{cases}$$

(a) Use the longer time for the time of flight. The shorter time is the time for the ball to rise to the basket height on the way up, while the longer time is the time for the ball to be at the basket height on the way down.

$x = v_x\, t = (v_0 \cos 35°)\, t = (12\text{ m/s})(\cos 35°)\,(1.303\text{s}) = 12.81\text{m} \approx 13\text{m}$

(b) The angle to the horizontal is determined by the components of the velocity.

$v_x = v_0 \cos 35° = 12 \cos 35° = 9.830\text{m/s}$

$v_y = at + v_{y0} = v_0 \sin \theta_0 - gt = 12 \sin 35° - 9.80(1.303) = -5.886\text{m/s}$

$\theta = \tan^{-1} \frac{v_y}{v_x} = \tan^{-1} \frac{-5.886}{9.830} = -30.9° = -31°$

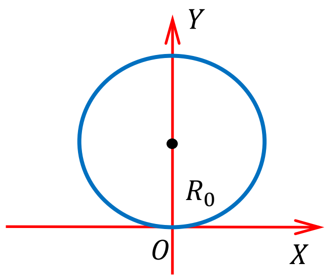
The negative angle means it is below the horizontal.

Exercise n°19:

$$\begin{cases} r(t) = 2R_0.\sin(\omega.t) \\ \theta(t) = \omega.t \end{cases} \qquad \begin{cases} r^{\bullet} = 2R_0\omega.\cos(\omega.t) \\ \theta^{\bullet} = \omega \end{cases} \qquad \begin{cases} r^{\bullet\bullet} = -2R_0\omega^2.\sin(\omega.t) \\ \theta^{\bullet\bullet} = 0 \end{cases}$$

1. Trajectory:

$$\begin{cases} x = r.\cos(\theta) \\ y = r.\sin(\theta) \end{cases} \Rightarrow \begin{cases} x = 2R_0.\sin(\omega.t)\,.\cos(\omega.t) = R_0.\sin(2\omega.t) \\ y = 2R_0.\sin(\omega.t)\,.\sin(\omega.t) = R_0.[1 - \cos(2\omega.t)] \end{cases}$$



$$\left(\frac{x}{R_0}\right)^2 + \left(1 - \frac{y}{R_0}\right)^2 = 1 \quad \Rightarrow \quad x^2 + (y - R_0)^2 = R_0^2$$

It is the equation of a circle with center and radius.$(0, R_0)R_0$

2. Velocity vector in polar coordinates:

$$\boxed{\vec{V} = r^{\bullet}.\vec{e}_r + r.\theta^{\bullet}.\vec{e}_\theta}$$

SO

$$\boxed{\vec{V} = 2R_0\omega.\cos(\omega.t)\,.\vec{e}_r + 2R_0\omega.\sin(\omega.t)\,.\vec{e}_\theta = 2R_0\omega.[\cos(\omega.t)\,.\vec{e}_r + \sin(\omega.t)\,.\vec{e}_\theta]}$$

And its magnitude

$$|\vec{V}| = V = 2R_0\omega.\sqrt{\cos(\omega.t)^2 + \sin(\omega.t)^2}$$

$$\Rightarrow \boxed{V = 2R_0\omega}$$

3. Acceleration vector:

$$\boxed{\vec{a} = (r^{\bullet\bullet} - r.\theta^{\bullet 2})\vec{e}_r + (2.r^{\bullet}\theta^{\bullet} + r.\theta^{\bullet\bullet})\vec{e}_\theta}$$

And

$$\boxed{\vec{a} = -4R_0\omega^2.\sin(\omega.t)\,.\vec{e}_r + 4R_0\omega^2.\cos(\omega.t)\,.\vec{e}_\theta = 4R_0\omega^2.[-\sin(\omega.t)\,.\vec{e}_r + \cos(\omega.t)\,.\vec{e}_\theta]}$$

And its magnitude

$$\boxed{|\vec{a}| = a = 4R_0\omega^2}$$

Tangential acceleration:

$$\boxed{a_T = \frac{dV}{dt} = 0}$$

Normal acceleration:

$$a^2 = a_T^2 + a_N^2 \Rightarrow a_N = \sqrt{a^2 - a_T^2} = a$$

Replacing

$$\boxed{a_N = 4R_0\omega^2}$$

4. Radius of curvature :

$$a_N = \frac{V^2}{\rho} \qquad \Rightarrow \qquad \rho = \frac{V^2}{a_N}$$

Replacing

$$\boxed{\rho = R_0}$$

Conclusion the motion is circular ($\rho = \text{Cte}$) uniform ($V = \text{Cte}$).⇒ Uniform circular motion

5. Tangent unit vector:

$$\vec{e}_T = \frac{\vec{V}}{V} \qquad \Rightarrow \qquad \boxed{\vec{e}_T = \cos(\omega.t)\,.\vec{e}_r + \sin(\omega.t)\,.\vec{e}_\theta}$$

Normal unit vector:

$$\vec{e}_N = \frac{\vec{a}_N}{a_N} = \frac{\vec{a}}{a} \qquad \Rightarrow \qquad \boxed{\vec{e}_N = -\sin(\omega.t)\,.\vec{e}_r + \cos(\omega.t)\,.\vec{e}_\theta}$$

Exercise n°20:

1. The trajectory equation

$$\begin{cases} x = bsin(\omega t) \\ y = b\big(1 - cos(\omega t)\big) \end{cases}$$

$$\Rightarrow \begin{cases} x = bsin(\omega t) \\ y - b = -bcos(\omega t) \end{cases}$$

$$\Rightarrow (1)^2 + (2)^2 = x^2 + (y - b)^2 = b^2$$

The trajectory is a circle with center $C(0.b)$ and de radius b.

2. The polar coordinates ρ *et* θ of M.

$$\rho = \sqrt{x^2 + y^2} = \sqrt{\big(bsin(\omega t)\big)^2 + \big(b\big(1 - cos(\omega t)\big)^2}$$

$$\rho = b\sqrt{sin^2(\omega t) + 1 - 2cos(\omega t) + cos^2(\omega t)}$$

$$\rho = b\sqrt{2 - 2cos(a)t)}$$

Using the relationship: $\cos(2\alpha) = cos^2(\alpha) - sin^2(\alpha)$

After simplification:

$$\rho = 2bsin\left(\frac{\omega t}{2}\right)$$

Knowing that: $tg\phi = \frac{y}{x} = \frac{b\big(1-cos(\omega t)\big)}{bsin(\omega t))}$,

Using the relationship: $\cos(2\alpha) = cos^2(\alpha) - sin^2(\alpha)$ *and* $\sin(2\alpha) = 2cos(\alpha)sin(\alpha)$, We find: $tg\theta = tg\left(\frac{\omega t}{2}\right) \Rightarrow \theta = \frac{\omega t}{2}$

3. The Cartesian, polar and intrinsic components of the velocity and acceleration vectors of M

a. In Cartesian coordinates

$$\begin{cases} \overrightarrow{OM} = bsin(\omega t)\vec{\imath} + \left(1 - cos(\omega t)\right)\vec{\jmath} \\ \vec{v} = b\omega cos(\omega t)\vec{\imath} + b\omega sin(\omega t)\vec{\jmath} \\ \vec{a} = -b\omega^2 sin(\omega t)\vec{\imath} + b\omega^2 cos(\omega t)\vec{\jmath} \end{cases}$$

b. In polar coordinates

$\overrightarrow{0M} = 2bsin\left(\frac{\omega t}{2}\right)\overrightarrow{u_\rho}$, $\theta = \frac{\omega t}{2}$

With: $\dot{\rho} = b\omega \cos\left(\frac{\omega t}{2}\right), \ddot{\rho} = -\frac{b\omega^2}{2} \sin\left(\frac{\omega t}{2}\right), \dot{\theta} = \frac{\omega}{2}$ and $\theta \ddot{=} 0$

$\vec{v} = \dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta = b\omega \cos\left(\frac{\omega t}{2}\right)\vec{u}_\rho + b\omega\,)\sin\left(\frac{\omega t}{2}\right)\vec{u}_\theta$

$\vec{a} = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{u}_\beta + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\vec{u}_\theta = b\omega^2\left(-\sin\left(\frac{\omega t}{2}\right)\vec{u}_\rho + \cos\left(\frac{\omega t}{2}\right)\vec{u}_\theta\right)$

c. In polar coordinates

Calculation of the curvilinear abscissa, S:

Since the trajectory is circular we can write

$S = R\theta \Leftrightarrow S = b\omega t$

$\overrightarrow{v''} = \frac{ds}{at} = b\omega \Rightarrow \overrightarrow{v''} = b\omega t\,\vec{u}_T$

$\overrightarrow{a''} = a_T\vec{u}_T + a_N\vec{u}_N$

$$\begin{cases} a_T = \frac{dv}{dC} = 0 \text{ Uniform circular motion} \\ a_N = \frac{v^2}{R} = bt^2 \end{cases}$$

$\Rightarrow \vec{a}'' = b\omega^2\vec{u}_N$

Calculation of magnitudes:

$v = b\omega\, m/s$ and $a = b\omega^2\, m/s^2$

$v' = b\omega\, m/s$ and $a' = b\omega^2\, m/s^2$

$v'' = b\omega\, m/s$ and $a'' = b\omega^2\, m/s^2$

Exercise n°21:

The polar coordinates of a material point are: $\rho(t) = r_0 e^{\theta}$, $\theta = \omega t, \omega, r_0$ constants

1 - The vector position in polar coordinates.

$$\overrightarrow{OM} = \rho\vec{u}_p \Rightarrow \overrightarrow{OM} = r_0 e^{\omega t}\vec{u}_\rho$$

2 - The velocity and acceleration in polar coordinates and their magnitudes.

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(r_0 e^{\omega t}\vec{u}_\rho)}{dt} = r_0\omega e^{\omega t}\vec{u}_\rho + r_0\omega e^{\omega t}\vec{u}_\theta$$

$$\Rightarrow \vec{v} = r_0\omega e^{\omega t}(\vec{u}_\rho + \vec{u}_\theta)$$

$$v = \sqrt{2}r_0\omega e^{\omega t}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left(r_0\omega e^{\omega t}(\vec{u}_\rho + \vec{u}_\theta)\right)}{dt} = r_0\omega^2 e^{\omega t}(\vec{u}_\rho + \vec{u}_\theta) + r_0\omega e^{\omega t}(\omega\,\vec{u}_\theta - \omega\,\vec{u}_\rho)$$

$$\Rightarrow \vec{a} = 2r_0\omega^2 e^{\omega t}\,\vec{u}_\theta$$

$$a = 2r_0\omega^2 e^{\omega t}$$

2 - Calculate the tangential acceleration and the normal acceleration.

$$\begin{cases} a_T = \dfrac{dV}{dt} \\ a_N = \dfrac{V^2}{\Re} = \sqrt{a^2 - {a_T}^2} \end{cases} \Rightarrow \begin{cases} a_T = \sqrt{2}r_0\omega^2 e^{\omega t} \\ a_N = \sqrt{(2r_0\omega^2 e^{\omega t})^2 - \left(\sqrt{2}r_0\omega^2 e^{\omega t}\right)^2} = \sqrt{2}\,r_0\omega^2 e^{\omega t} \end{cases}$$

4 - Deduce the radius of curvature.

$$a_N = \frac{V^2}{\Re} \Rightarrow \Re = \frac{V^Z}{a_N} \Rightarrow \Re = \frac{\left(a\omega e^{\omega t}\sqrt{2}\right)^2}{\sqrt{2}a\omega^2 e^{wt}} \Rightarrow \Re = \sqrt{2}\,r_0 e^{\omega t}$$

5 - Calculate the curvilinear abscissa S t as a function of time.

$$v = \frac{dS}{at} \Rightarrow dS = v.dt \Rightarrow \int dS = \sqrt{2}\,r_0 \int \omega e^{\omega t}dt \Rightarrow S = \sqrt{2}r_0 e^{\omega t} + C$$

Exercise n°22:

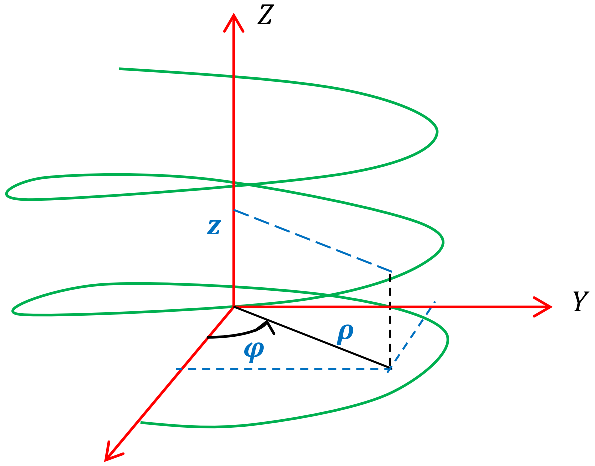
1.

$$\begin{cases} x(t) = b.\cos(\omega.t) \\ y(t) = b.\sin(\omega.t) \\ z(t) = c.\omega.t \end{cases}$$

In cylindrical coordinates

$$\begin{cases} x = \rho.\cos(\varphi) \\ y = \rho.\sin(\varphi) \\ z = z \end{cases}$$

$$\begin{cases} \rho = b = Cte \\ \varphi = \omega.t \\ z = c.\omega.t \end{cases}$$



Helical motion

2.

Velocity vector : $\vec{V} = \rho^\bullet.\vec{e}_\rho + \rho.\varphi^\bullet.\vec{e}_\varphi + z^\bullet.\vec{e}_z$

$$\Rightarrow \boxed{\vec{V} = b.\omega.\vec{e}_\varphi + c.\omega.\vec{e}_z}$$

Acceleration vector:$\vec{a} = \left(\rho^{\bullet\bullet} - \rho.\varphi^{\bullet 2}\right).\vec{e}_\rho + (2\rho^{\bullet}.\varphi^{\bullet} + \rho.\varphi^{\bullet\bullet}).\vec{e}_\varphi + z^{\bullet\bullet}.\vec{e}_z$
From where

$$\boxed{\vec{a} = -b.\omega^2.\vec{e}_\rho}$$

$b = 0 \quad \Rightarrow \quad (x(t) = 0\ ;\ y(t) = 0\ ; z(t) = c.\omega.t)$: Uniform rectilinear motion along the OZ axis.
$c = 0 \quad \Rightarrow \quad (x(t) = b.\cos(\omega.t)\ ;\ y(t) = b.\sin(\omega.t)\ ;\ z(t) = 0)$: Uniform circular motion in the plane OXY.

Exercise n°23:

$$x(t) = b.\cos(\gamma.t^2) \quad ; \quad y(t) = b.\sin(\gamma.t^2) \quad ; \quad z(t) = b.\gamma.t^2$$

In cylindrical coordinates

$$\begin{cases} x = \rho.\cos(\varphi) \\ y = \rho.\sin(\varphi) \\ z = z \end{cases} \Rightarrow \boxed{\begin{cases} \rho = b = \text{Cte} \\ \varphi = \gamma.t^2 \\ z = b.\gamma.t^2 \end{cases}}$$

1. Velocity vector :$\vec{V} = \rho^{\bullet}.\vec{e}_\rho + \rho.\varphi^{\bullet}.\vec{e}_\varphi + z^{\bullet}.\vec{e}_z$

$$\Rightarrow \boxed{\vec{V} = 2b\gamma.t.\left(\vec{e}_\varphi + \vec{e}_z\right)}$$

Acceleration vector:$\vec{a} = \left(\rho^{\bullet\bullet} - \rho.\varphi^{\bullet 2}\right).\vec{e}_\rho + (2\rho^{\bullet}.\varphi^{\bullet} + \rho.\varphi^{\bullet\bullet}).\vec{e}_\varphi + z^{\bullet\bullet}.\vec{e}_z$

$$\boxed{\vec{a} = -4b\gamma^2.t^2.\vec{e}_\rho + 2b\gamma.\vec{e}_\varphi + 2b\gamma.\vec{e}_z} \text{Or} \boxed{\vec{a} = 2b\gamma.\left(-2\gamma.t^2.\vec{e}_\rho + \vec{e}_\varphi + \vec{e}_z\right)}$$

2. Speed magnitude: $V = \left|\vec{V}\right| = |2b\gamma.t|.\sqrt{1+1} \quad \Rightarrow \quad V = 2\sqrt{2}.b\gamma.t$

Tangential acceleration:

$$\boxed{a_T = \frac{dV}{dt} = 2\sqrt{2}.b\gamma}$$

Acceleration magnitude:

$$a = |\vec{a}| = |2b\gamma|.\sqrt{(-2\gamma.t^2)^2 + 1 + 1} \ \Rightarrow\ a = 2\sqrt{2}.b\gamma.\sqrt{2\gamma^2.t^4 + 1}$$

Normal acceleration:

$$a^2 = a_T^2 + a_N^2 \quad \Rightarrow \quad a_N^2 = a^2 - a_T^2$$

$$a_N^2 = 8.b^2\gamma^2(2\gamma^2.t^4 + 1) - 8.b^2\gamma^2 = 16.b^2\gamma^4.t^4$$

And

$$\boxed{a_N = 4.b\gamma^2.t^2}$$

3. Radius of curvature :

$$a_N = \frac{V^2}{\rho} \quad \Rightarrow \quad \rho = \frac{V^2}{a_N} = \frac{8.b^2\gamma^2.t^2}{4.b\gamma^2.t^2}$$

$$\Rightarrow \boxed{\rho = 2b}$$

Chapter4

RELATIVE MOTION

Learning Goals: After going through this chapter, students will be able to

- Describe the concept of relative motion and compute position, velocity, and acceleration of particles in relative motion and dependent relative motion.
- Write the position and velocity vector equations for relative motion.
- Apply the relationship between a particle's position, velocity, and acceleration as measured from two reference frames that move relative to each other at constant velocity and along a single axis.

Exercise n°01:

In the plane (Oxy), we consider a system of mobile axes (OXY) of the same origin O and such that Ox makes a variable angle θ with OX. A moving point M on the axis OX is marked by $OM = r$. We call relative movement of M its movement with respect to (OXY), and the absolute movement with respect to (Oxy).

Calculate in the moving frame (polar coordinates)

1- Relative velocity and acceleration of M.

2- Training velocity and acceleration of M.

3- Coriolis acceleration.

4- Deduce its absolute velocity and acceleration.

Exercise n°02:

Consider a fixed reference frame $(Oxyz)$, a point O' moves on the axis (Oy) with a constant acceleration a. We connect to the point O' a mobile reference frame $(O'XYZ)$ which rotates around $(O'Z)$ with a constant angular velocity ω. a point M moves in the moving frame with coordinates: $X = t^2$, .$Y = t$

At the initial time $t = 0$, $(O'X)$ is confused with (Ox)

Determine in the mobile reference of basic $(\overrightarrow{u_X}, \overrightarrow{u_Y}, \overrightarrow{u_Z})$:

1. Relative velocity $\overrightarrow{V_r}$ and training velocity $\overrightarrow{V_e}$. Deduce the absolute velocity $\overrightarrow{V_a}$.
2. Relative acceleration $\overrightarrow{a_r}$, training acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$. Deduce the absolute acceleration $\overrightarrow{a_a}$.

Exercise n°03:

Consider a fixed reference $(Oxyz)$. A moving frame $(O'XYZ)$ rotates around (Oz) with a constant angular velocity ω. The point O' moves with a constant acceleration on the axis (Oy) : $\overrightarrow{OO'} = \frac{1}{2} at\vec{i}$. A point M moves on the axis $(O'X)$ with a constant velocity :

$\overrightarrow{O'M} = v\, t\vec{i'}$.

Determine in the mobile reference $(O'XYZ)$:

1. Relative velocity $\overrightarrow{V_r}$ and training velocity $\overrightarrow{V_e}$. Deduce the absolute velocity $\overrightarrow{V_a}$.
2. Relative acceleration $\overrightarrow{a_r}$, training acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$. Deduce the absolute acceleration $\overrightarrow{a_a}$.

Exercise n°04:

Consider a fixed reference $(Oxyz)$. A moving frame $(O'XYZ)$ rotates around (Oz) with a constant angular velocity ω. The point O' moves on the axis (Ox) . A point M moves on the axis $(O'Y)$: $\overrightarrow{OO'} = at\vec{\imath}$ such that $\overrightarrow{O'M} = bt^2\vec{\jmath}'$ with a and b positive constants.

Determine in the fixed reference $(Oxyz)$:

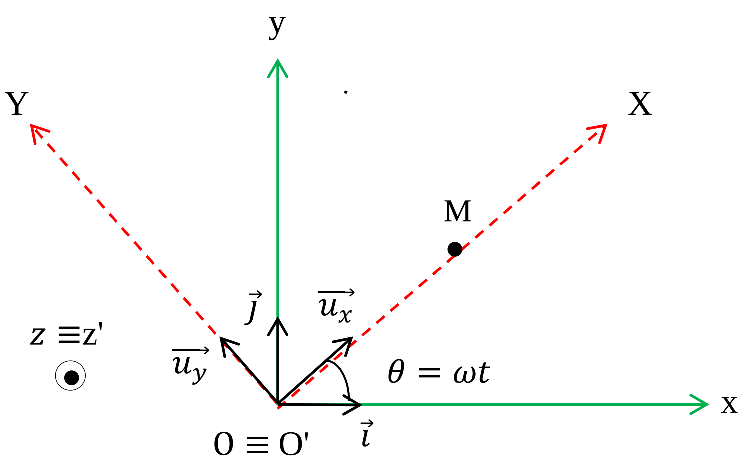
3. Relative velocity $\overrightarrow{V_r}$ and training velocity $\overrightarrow{V_e}$. Deduce the absolute velocity $\overrightarrow{V_a}$.
4. Relative acceleration $\overrightarrow{a_r}$, training acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$. Deduce the absolute acceleration $\overrightarrow{a_a}$.

Exercise n°05:

Consider a fixed frame $(Oxyz)$ and a mobile frame $(O'XYZ)$ rotating around (Oz) with a constant angular velocity ω. The point O' moves with a constant acceleration on the axis (Oy). A point M moves on the axis (O'X) with a constant velocity v : $\overrightarrow{OM} = vt\,\vec{\imath}'$

In the moving reference frame $(O'XYZ)$:

1. Determine the relative velocity $\overrightarrow{V_r}$ and training velocity $\overrightarrow{V_e}$. Deduce the absolute velocity $\overrightarrow{V_a}$
2. Determine the relative acceleration $\overrightarrow{a_r}$, training acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$. Deduce the absolute acceleration $\overrightarrow{a_a}$.



Exercise n°06:

Consider a fixed frame of reference$(0xyz)$ and a mobile frame$(0x'y'z')$ of reference which rotating around (Oz) with a constant angular velocity ω see figure. A point M $(OM = r)$ moves on the Ox axis following the time equation:

$(r = r_0 \sin\omega t)$ with r_0 constant

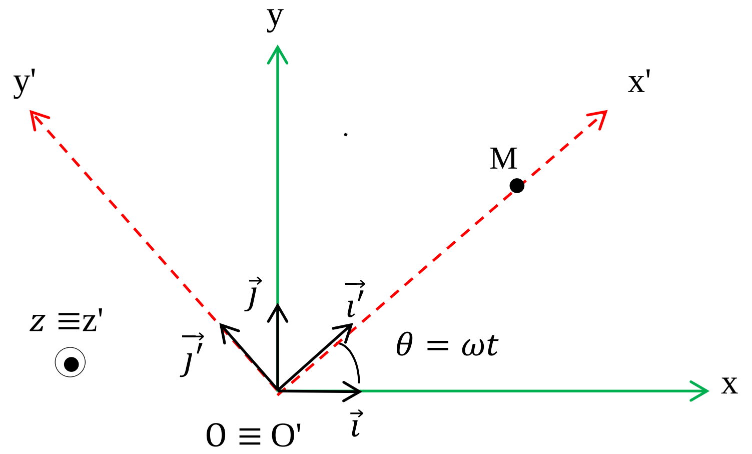
Determine for the point M, in the base):$(\vec{i'},\vec{j'})$

1- Relative velocity and acceleration of M.

2- Training velocity and acceleration of M.

3- Coriolis acceleration.

4- Deduce its absolute velocity and acceleration.



Exercise n°07:

Consider a fixed frame of reference$(0xyz)$ and a mobile frame$(0x'y'z')$ of reference which rotating around (Oz) with a constant angular velocity ω. A point M $(OM = r)$ moves on the Ox axis following the time equation:

$r = r_0(\cos\omega t + \sin\omega t)$ with r_0 constant

Determine in the mobile reference (OX'Y'Z'):

1. Relative velocity $\vec{V_r}$ and training velocity $\vec{V_e}$. Deduce the absolute velocity $\vec{V_a}$.
2. Relative acceleration $\vec{a_r}$, training acceleration $\vec{a_e}$ and Coriolis acceleration $\vec{a_c}$. Deduce the absolute acceleration $\vec{a_a}$.

Exercise n°08:

Consider the frame R (Oxyz) where the point O' moves on the axis (Ox) with a constant acceleration a and with an initial velocity$(v_0 \neq 0)$.We link to O' the frame (O'XYZ) which rotates around (Oz) with a constant angular velocity ω. The coordinates of a mobile M in the mobile frame are $x' = (t + 1)$ and $y' = t^2$. At the instant $t = 0$, the point O' coincides with the point O.

Calculate in the mobile reference:

1. Relative velocity $\vec{V_r}$ and training velocity $\vec{V_e}$. Deduce the absolute velocity $\vec{V_a}$.

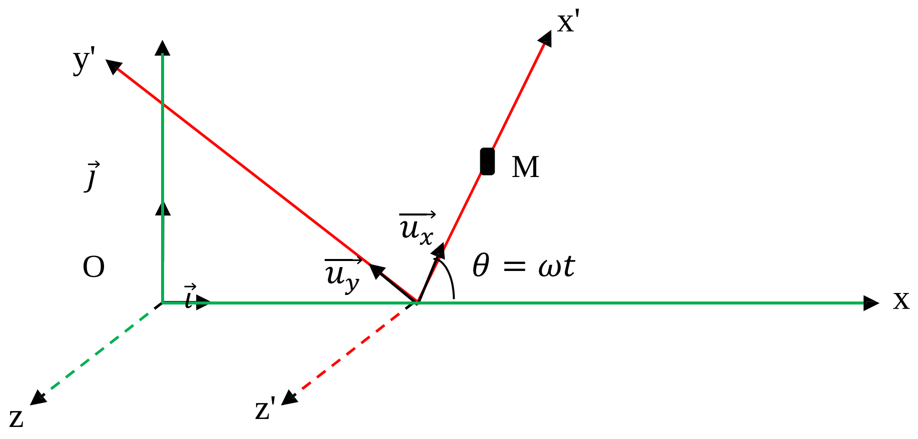
2. Relative acceleration $\overrightarrow{a_r}$, training acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$. Deduce the absolute acceleration $\overrightarrow{a_a}$.

Exercise n°09:

Consider the frame R(Oxyz) where the point O' moves on the axis (Ox) with a constant velocity v. We link to O' the reference frame (O'x'y'z') which rotates around (Oz) with a constant angular velocity ω see figure above. A moving point M moves on the axis O'x' such that $|OM'| = t^2$

At the instant $t = 0$, the axes (Ox) and (O'x') coincide and M is at O.

1. Relative velocity $\overrightarrow{V_r}$ and training velocity $\overrightarrow{V_e}$. Deduce the absolute velocity $\overrightarrow{V_a}$.
2. Relative acceleration $\overrightarrow{a_r}$, training acceleration $\overrightarrow{a_e}$ and Coriolis acceleration $\overrightarrow{a_c}$. Deduce the absolute acceleration $\overrightarrow{a_a}$.



Exercise n°10:

In the plane (Oxy), of a reference frame (Oxyz), a point O', to which the reference frame (O'XYZ) is linked, describes a circle of center O and radius r, and rotates with a constant angular velocity ω see figure above.

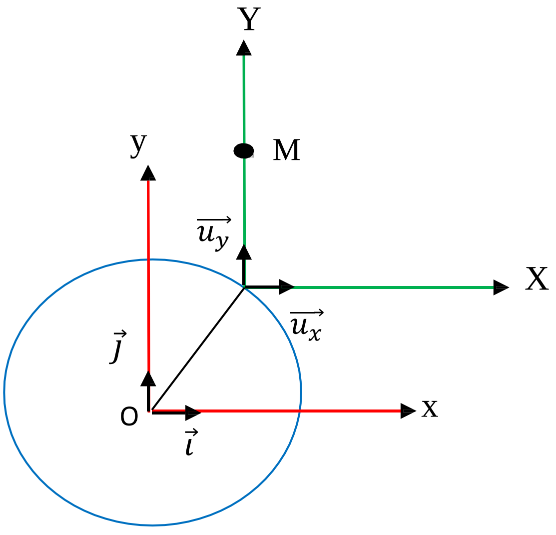
A point M moves along the axis (O'Y) parallel to Oy with constant acceleration a (at time $t = 0$, M is merged with M_0 $(r, 0,0)$ and its initial velocity is positive).

1- Calculate in the (Oxyz) reference frame the position vector $\overrightarrow{OM}$, the absolute velocity $\overrightarrow{V_a}$ and the absolute acceleration $\overrightarrow{a_a}$.

2- Knowing that O'X//Ox, O'Y//Oy and O'Z//Oz, calculate:

a- Relative velocity $\overrightarrow{V_r}$ and training velocity $\overrightarrow{V_e}$, check that $\overrightarrow{V_a} = \overrightarrow{V_r} + \overrightarrow{V_e}$

b- A relative acceleration $\overrightarrow{a_r}$, training acceleration $\overrightarrow{a_e}$ and the Coriolis acceleration $\overrightarrow{a_c}$, check that $\overrightarrow{a_a} = \overrightarrow{a_r} + \overrightarrow{a_e} + \overrightarrow{a_c}$



SOLUTIONS TO EXERCISES

Exercise n°01:

$\overrightarrow{OM} = r\,\overrightarrow{U_x}$

$\overrightarrow{OO'} = \vec{0} \Rightarrow \overrightarrow{O'M} = r.\overrightarrow{U_x}$

Relative velocity : $\overrightarrow{v_r} = \frac{d\overrightarrow{O\prime M}}{dt}/(OXY) = \frac{d\,r}{dt}\,\overrightarrow{U_x} = \dot{r}\,\overrightarrow{U_x}$

Training velocity: $\overrightarrow{v_e} = \frac{d\overrightarrow{OO\prime}}{dt}\,| + (\vec{\omega}\wedge\overrightarrow{O'M}) \Rightarrow \overrightarrow{v_e} = (\vec{\omega}\wedge\overrightarrow{O'M})$

$$\left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ r & 0 & 0 \end{vmatrix} = \omega\,r\,\overrightarrow{U_y} \Rightarrow \overrightarrow{v_e} = \omega\,r\,\overrightarrow{U_y}$$

Absolute velocity: $\overrightarrow{v_a} = \overrightarrow{v_e} + \overrightarrow{v_r} \Rightarrow \overrightarrow{v_a} = \dot{r}\,\overrightarrow{U_x} + \omega\,r\,\overrightarrow{U_y}$

Relative acceleration: $\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}\,|/(OXY) = \frac{d^2\,r}{dt^2}\,\overrightarrow{U_x} = \ddot{r}\,\overrightarrow{U_x}$

Training acceleration: $\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO\prime}}{dt^2} + (\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}) + \vec{\omega}\wedge(\,\vec{\omega}\wedge\overrightarrow{O'M})$

$\left(\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}\right) = 0$ because ω is constant and $\overrightarrow{OO'} = \vec{0}$

$$\Rightarrow \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = \begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ o & r\omega & 0 \end{vmatrix} = -\omega^2\,r\,\overrightarrow{U_x} \quad \Rightarrow \overrightarrow{a_e} = -\omega^2\,r\,\overrightarrow{U_x}$$

Coriolis acceleration : $\overrightarrow{a_c} = 2\vec{\omega}\wedge\overrightarrow{v_r} = 2\begin{vmatrix} \overrightarrow{U_x} & \overrightarrow{U_y} & \overrightarrow{U_z} \\ 0 & 0 & \omega \\ \dot{r} & 0 & 0 \end{vmatrix} = 2\dot{r}\omega\,\overrightarrow{U_y}$

Absolute acceleration : $\overrightarrow{a_a} = \overrightarrow{a_e} + \overrightarrow{a_r} + \overrightarrow{a_c} = (\ddot{r} \;= -\omega^2\,r)\,\overrightarrow{U_x} + 2\dot{r}\omega\,\overrightarrow{U_y}$

Exercise n°02:

$\overrightarrow{OO'} = \frac{1}{2}at^2\vec{j}$

The angular velocity is constant around the axis Oz $\vec{\omega} = \omega\vec{k} = \omega\overrightarrow{u_z}$

$\overrightarrow{O'M}\begin{cases}\boldsymbol{X = t^2}\\ \boldsymbol{Y = t}\end{cases} \Rightarrow \overrightarrow{O'M} = t^2\overrightarrow{u_x} + t\overrightarrow{u_y}$

Determine in the moving frame of basic $(\overrightarrow{u_x}, \overrightarrow{u_y}, \overrightarrow{u_z})$

Relative velocity : $\overrightarrow{v_r} = \frac{d\overrightarrow{O\prime M}}{dt} = \boldsymbol{2t\overrightarrow{u_x} + \overrightarrow{u_y}}$

Training velocity: $\overrightarrow{v_e} = \frac{d\overrightarrow{OO\prime}}{dt}\,| + \left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = at\vec{j} + \omega t^2\overrightarrow{u_y}$ - $\omega t\overrightarrow{u_x}$

$$\left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = \begin{vmatrix} \overrightarrow{u_x} & \overrightarrow{u_y} & \overrightarrow{u_z} \\ 0 & 0 & \omega \\ t^2 & t & 0 \end{vmatrix} = -\,\omega t\,\overrightarrow{u_x} + \omega t^2\overrightarrow{u_y}$$

$\Rightarrow \overrightarrow{v_e} = at\vec{j} + \omega t^2\overrightarrow{u_y}$ - $\omega t\overrightarrow{u_x}$ = $at[sin(\omega t)\overrightarrow{u_x}$+$\cos(\omega t)\overrightarrow{u_y}] + \omega t^2\overrightarrow{u_y}$ - $\omega t\overrightarrow{u_x}$

$\Rightarrow \boldsymbol{\overrightarrow{v_e} = [atsin(\omega t) - \omega t]\overrightarrow{u_x}}$ + $\boldsymbol{[atcos(\omega t) + \omega t^2]\overrightarrow{u_y}}$

Absolute velocity: $\overrightarrow{v_a} = \overrightarrow{v_e} + \overrightarrow{v_r}$

$\Rightarrow\ \ \boldsymbol{\overrightarrow{v_a} = [atsin(\omega t) - \omega t + 2t]\overrightarrow{u_x}}$ + $\boldsymbol{[atcos(\omega t) - \omega t^2 + 1]\overrightarrow{u_y}}$

Relative acceleration: $\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}\,|= 2\overrightarrow{u_x}$

Training acceleration: $\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO\prime}}{dt^2} + (\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}) + \vec{\omega}\wedge(\vec{\omega}\wedge\overrightarrow{O'M})$

$$\Rightarrow \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = \begin{vmatrix} \overrightarrow{u_x} & \overrightarrow{u_y} & \overrightarrow{u_z} \\ 0 & 0 & \omega \\ -\omega t & \omega t^2 & 0 \end{vmatrix} = -\omega^2 t^2\,\overrightarrow{u_x} - \omega^2 t\,\overrightarrow{u_y}$$

$\Rightarrow \overrightarrow{a_e} = a[sin(\omega t)\overrightarrow{u_x}$+$\cos(\omega t)\overrightarrow{u_y}] - \omega^2 t^2\overrightarrow{u_x}$ - $\omega^2 t\overrightarrow{u_y}$

$\Rightarrow \boldsymbol{\overrightarrow{a_e} = [asin(\omega t) - \omega^2 t^2]\overrightarrow{u_x} + [acos(\omega t) - \omega^2 t]\overrightarrow{u_y}}$

Coriolis acceleration : $\overrightarrow{a_c} = 2\vec{\omega}\wedge\overrightarrow{v_r} = \begin{vmatrix} \overrightarrow{u_x} & \overrightarrow{u_y} & \overrightarrow{u_z} \\ 0 & 0 & 2\omega \\ 2t & 1 & 0 \end{vmatrix} = -2\omega\overrightarrow{u_x} + 4\omega t\overrightarrow{u_y}$

$\Rightarrow \boldsymbol{\overrightarrow{a_c} = 2\omega\overrightarrow{u_z}\wedge\left(2t\overrightarrow{u_x} + \overrightarrow{u_y}\right) = 4\omega t\overrightarrow{u_y} - 2\omega\overrightarrow{u_x}}$

Absolute acceleration : $\overrightarrow{a_a} = \overrightarrow{a_e} + \overrightarrow{a_r} + \overrightarrow{a_c}$

$\Rightarrow \boldsymbol{\overrightarrow{a_a} = asin(\omega t) - \omega^2 t^2 + 2 - 2\omega]\overrightarrow{u_x}}$+$\boldsymbol{[acos(\omega t) - \omega^2 t + 4\omega t]\overrightarrow{u_y}}$

Exercise n°03:

$\overrightarrow{OO'} = \frac{1}{2}at^2\vec{j}$

The angular velocity is constant around the axis Oz $\vec{\omega} = \omega\vec{k} = \omega\overrightarrow{u_z}$

$\overrightarrow{O'M} = v\;t\;\vec{\imath}'$

Determine in the mobile frame of basic $(\vec{\imath}', \vec{\jmath}', \vec{k}')$

Relative velocity : $\overrightarrow{v_r} = \frac{d\overrightarrow{O\prime M}}{dt} = v\;\vec{i'}$

Training velocity: $\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt}\,| + \left(\vec{\omega}\wedge\overrightarrow{O'M}\right)$,

$$\left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = \begin{vmatrix} \vec{i'} & \vec{j'} & \vec{k'} \\ 0 & 0 & \omega \\ vt & 0 & 0 \end{vmatrix} = v\omega t\,\vec{j'}$$

$\Rightarrow \vec{v_e} = at\vec{j} + v\omega t\,\vec{j'}$

$\Rightarrow \vec{v_e} = at[\sin(\omega t)\vec{\imath'} + cos(\omega t)\,\vec{j'}] + v\omega t\,\vec{j'}$

$\Rightarrow \boldsymbol{\vec{v_e}} = [\boldsymbol{at\,sin(\omega t)}]\boldsymbol{\vec{\imath'}} + [\boldsymbol{atcos(\omega t) + v\omega t}\,]\,\boldsymbol{\vec{j'}}$

Absolute velocity: $\vec{v_a} = \vec{v_e} + \vec{v_r}$

$\Rightarrow\ \boldsymbol{\vec{v_a}} = [\boldsymbol{at\,sin(\omega t) + v}]\boldsymbol{\vec{i'}} + [\boldsymbol{atcos(\omega t) + v\omega t}\,]\,\boldsymbol{\vec{j'}}$

Relative acceleration:

$\vec{a_r} = \frac{d\vec{v_r}}{dt}| = 0$

Training acceleration: $\vec{a_e} = \frac{d^2\overrightarrow{OO\prime}}{dt^2} + (\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}) + \vec{\omega}\wedge(\vec{\omega}\wedge\overrightarrow{O'M})$

$$\Rightarrow \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = \begin{vmatrix} \vec{\imath'} & \vec{j'} & \vec{k'} \\ 0 & 0 & \omega \\ 0 & v\omega t & 0 \end{vmatrix} = -\,v\omega^2 t\,\vec{\imath'}$$

$\Rightarrow \vec{a_e} = at\left[\sin(\omega t)\,\vec{\imath'} + cos(\omega t)\vec{j'}\right] -\ v\omega^2 t\,\vec{\imath'}$

$\Rightarrow \boldsymbol{\vec{a_e}} = [\boldsymbol{a\,sin(\omega t) -\ v\omega^2 t}\,]\boldsymbol{\vec{\imath'}} + [\boldsymbol{acos(\omega t)}]\,\boldsymbol{\vec{j'}}$

Coriolis acceleration : $\vec{a_c} = 2\vec{\omega}\wedge\vec{v_r} = \begin{vmatrix} \vec{i'} & \vec{j'} & \vec{k'} \\ 0 & 0 & 2\omega \\ v & 0 & 0 \end{vmatrix} =$

$\Rightarrow \boldsymbol{\vec{a_c} = 2\omega\vec{k'}\wedge\vec{v_r} = 2\,v\,\omega\,\vec{j'}}$

Absolute acceleration : $\vec{a_a} = \vec{a_e} + \vec{a_r} + \vec{a_c}$

$\Rightarrow \boldsymbol{\vec{a_a}} = [\boldsymbol{a\,sin(\omega t) -\ v\omega^2 t}\,]\boldsymbol{\vec{\imath'}} + [\boldsymbol{acos(\omega t) + 2\,v\,\omega}\,]\,\boldsymbol{\vec{j'}}$

Exercise n°04:

$\boldsymbol{\overrightarrow{OO'} = at\vec{i}}$

The angular velocity is constant around the axis Oz $\vec{\omega} = \omega\vec{k} = \omega\vec{u_z}$

$\boldsymbol{\overrightarrow{O'M} = bt^2\vec{j'}}$

Determine in the fixed frame of basic $(\vec{\imath},\vec{j},\vec{k})$

Relative velocity : $\vec{v_r} = \frac{d\overrightarrow{O\prime M}}{dt} = 2bt\vec{j'} = 2bt[-sin(\omega t)\vec{i} + \cos(\omega t)\vec{j}]$

$\Rightarrow \overrightarrow{v_r}=-2btsin(\omega t)\vec{\imath}+2bt\cos(\omega t)\vec{\jmath}=\mathbf{2bt(\cos(\omega t)\vec{\jmath}-sin(\omega t)\vec{\imath})}$

Training velocity: $\overrightarrow{v_e}=\frac{d\overrightarrow{OO\prime}}{dt}\mid+\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)=a\vec{\imath}-b\omega t^2\vec{i'}$

$$\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)=\begin{vmatrix}\vec{i'} & \vec{j'} & \vec{k'}\\ 0 & 0 & \omega\\ 0 & bt^2 & 0\end{vmatrix}=-b\omega t^2\vec{i'}$$

$\Rightarrow \overrightarrow{v_e}=a\vec{\imath}-b\omega t^2\vec{i'}=a\vec{\imath}-b\omega t^2[cos(\omega t)\vec{\imath}+\sin(\omega t)\vec{\jmath}]$

$\Rightarrow \boldsymbol{\overrightarrow{v_e}}=[\boldsymbol{a-b\omega t^2cos(\omega t)}]\boldsymbol{\vec{\imath}}-\boldsymbol{b\omega t^2\sin(\omega t)\vec{\jmath}}$

Absolute velocity: $\overrightarrow{v_a}=\overrightarrow{v_e}+\overrightarrow{v_r}$

$\Rightarrow\ \overrightarrow{v_a}=-2btsin(\omega t)\vec{i}+2bt\cos(\omega t)\vec{j}+[a-b\omega t^2cos(\omega t)]\vec{i}-b\omega t^2\sin(\omega t)\,\vec{j}$

$\Rightarrow\ \boldsymbol{\overrightarrow{v_a}}=\boldsymbol{-2btsin(\omega t)+\ a-b\omega t^2cos(\omega t)]\vec{i}}+[\boldsymbol{2bt\cos(\omega t)-b\omega t^2\sin(\omega t)}]\,\boldsymbol{\vec{j}}$

Relative acceleration:

$\overrightarrow{a_r}=\frac{d\overrightarrow{v_r}}{dt}\mid=2b[-sin(\omega t)\vec{i}+\cos(\omega t)\vec{\jmath}]-2bt\omega[\cos(\omega t)\vec{\imath}+\sin(\omega t)\,\vec{\jmath}]$

$\overrightarrow{a_r}=\frac{d\overrightarrow{v_r}}{dt}\mid=-2b[\,(t\omega cos(\omega t)+sin(\omega t))\vec{i}+t\omega(\sin(\omega t)-\cos(\omega t))\vec{\jmath}]$

$\Rightarrow\ \boldsymbol{\overrightarrow{a_r}}=\boldsymbol{-2b}[\,(\boldsymbol{t\omega cos(\omega t)+sin(\omega t)})\boldsymbol{\vec{\imath}}+\boldsymbol{t\omega(\sin(\omega t)-\cos(\omega t))\vec{\jmath}}]$

Training acceleration: $\overrightarrow{a_e}=\frac{d^2\overrightarrow{OO\prime}}{dt^2}+(\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M})+\vec{\omega}\wedge(\,\vec{\omega}\wedge\overrightarrow{O'M})$

$$\Rightarrow\vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)=\begin{vmatrix}\vec{\imath'} & \vec{j'} & \vec{k'}\\ 0 & 0 & \omega\\ 0 & -b\omega t^2 & 0\end{vmatrix}=-0-\ \omega^2bt^2\vec{j'}$$

$\Rightarrow\overrightarrow{a_e}=-\omega^2bt^2[\,-sin(\omega t)\vec{\imath}+\ \cos(\omega t)\vec{\jmath}]$

$\Rightarrow\boldsymbol{\overrightarrow{a_e}}\ =\boldsymbol{\omega^2bt^2\ sin(\omega t)\vec{\imath}-\omega^2bt^2\cos(\omega t)\vec{\jmath}}$

Coriolis acceleration : $\overrightarrow{a_c}=2\vec{\omega}\wedge\overrightarrow{v_r}=\begin{vmatrix}\vec{i'} & \vec{j'} & \vec{k'}\\ 0 & 0 & 2\omega\\ -2btsin(\omega t) & 2bt\cos(\omega t) & 0\end{vmatrix}=$

$$\Rightarrow\overrightarrow{a_c}=2\omega\overrightarrow{u_z}\wedge(-2btsin(\omega t)\vec{\imath}+2bt\cos(\omega t)\vec{\jmath})=-4\omega bt\vec{\imath'}$$

$\Rightarrow\overrightarrow{a_c}=-4\omega bt[cos(\omega t)\vec{\imath}+\sin(\omega t)\vec{\jmath}]$

$\Rightarrow\boldsymbol{\overrightarrow{a_c}}=\ \boldsymbol{-4\omega btcos(\omega t)\vec{\imath}-4\omega bt\sin(\omega t)\,\vec{j}}$

Absolute acceleration : $\overrightarrow{a_a}=\overrightarrow{a_e}+\ \overrightarrow{a_r}+\overrightarrow{a_c}$

$$\Rightarrow\overrightarrow{a_a}=\omega^2bt^2\ sin(\omega t)\vec{\imath}-\omega^2bt^2\cos(\omega t)\vec{\jmath}-2bt\omega cos(\omega t)\vec{\imath}-2bsin(\omega t)\vec{\imath}$$
$$-2bt\omega\sin(\omega t)\,\vec{\jmath}-2bcos(\omega t)\vec{\jmath}-4\omega btcos(\omega t)\vec{\imath}-4\omega bt\sin(\omega t)\,\vec{\jmath}$$

$$\Rightarrow \overrightarrow{a_a} = [\omega^2 bt^2 \sin(\omega t) - 6bt\omega\cos(\omega t) - 2b\sin(\omega t)]\vec{i} - [2b\cos(\omega t) + \omega^2 bt^2 \cos(\omega t) + 6bt\omega \sin(\omega t)]\vec{j}$$

Exercise n°05:

$\overrightarrow{OO'} = \vec{0}$

$\overrightarrow{O'M} = v.t.\overrightarrow{u_x}$

Determine in the fixed frame of basic $(\overrightarrow{u_x}, \overrightarrow{u_y}, \overrightarrow{u_z})$

Relative velocity : $\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}| = v.\overrightarrow{u_x}$

Training velocity: $\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt}| + (\vec{\omega}\wedge\overrightarrow{O'M}) = 0 + v\omega t = v\omega t\overrightarrow{u_y}$

Absolute velocity: $\overrightarrow{v_a} = \overrightarrow{v_e} + \overrightarrow{v_r} = v.\overrightarrow{u_x} + v\omega t\,\overrightarrow{u_y}$

Relative acceleration: $\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}| = \vec{0}$

Training acceleration: $\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + (\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}) + \vec{\omega}\wedge(\vec{\omega}\wedge\overrightarrow{O'M})$

$\Rightarrow \overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \vec{\omega}\wedge(\vec{\omega}\wedge\overrightarrow{O'M}) = 0 - \omega^2 v.t\overrightarrow{u_x} = -\omega^2 v.t\,\overrightarrow{u_x}$

Coriolis acceleration : $\overrightarrow{a_c} = 2\vec{\omega}\wedge\overrightarrow{v_r} = 2\omega v\overrightarrow{u_y}$

Absolute acceleration: $\overrightarrow{a_a} = \overrightarrow{a_e} + \overrightarrow{a_r} + \overrightarrow{a_c} = -\omega^2 v.t\overrightarrow{u_x} + 2\omega v\overrightarrow{u_y}$

Exercise n°06:

$\overrightarrow{OM} = \overrightarrow{O'M} = r\vec{i'} = r_0\sin(\omega t)\vec{i'}$

The angular velocity is constant around the axis Oz $\vec{\omega} = \omega\vec{k} = \omega\overrightarrow{u_z}$

Determine in the mobile frame of basic $(\vec{i'}, \vec{j'}, \vec{k'})$

Relative velocity : $\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt} = r_0\omega\cos(\omega t)\vec{i'}$

Training velocity: $\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt}| + (\vec{\omega}\wedge\overrightarrow{O'M}) = (\vec{\omega}\wedge\overrightarrow{O'M}), \frac{d\overrightarrow{OO'}}{dt}| = 0$

$$(\vec{\omega}\wedge\overrightarrow{O'M}) = \begin{vmatrix} \vec{i'} & \vec{j'} & \vec{k'} \\ 0 & 0 & \omega \\ r_0\sin(\omega t) & 0 & 0 \end{vmatrix} = r_0\omega\sin(\omega t)\vec{j'}$$

$\Rightarrow \overrightarrow{v_e} = r_0\omega\sin(\omega t)\vec{j'}$

Absolute velocity: $\overrightarrow{v_a} = \overrightarrow{v_e} + \overrightarrow{v_r}$

$\Rightarrow \overrightarrow{v_a} = r_0\omega\sin(\omega t)\vec{j'} + r_0\omega\cos(\omega t)\vec{i'}$

Relative acceleration:

$\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}| = -r_0\omega^2\sin(\omega t)\vec{i'}$

Training acceleration: $\vec{a_e} = \frac{d^2\overrightarrow{OO\prime}}{dt^2} + (\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}) + \vec{\omega}\wedge(\vec{\omega}\wedge\overrightarrow{O'M})$

$$\Rightarrow \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = \begin{vmatrix} \vec{i'} & \vec{j'} & \vec{k'} \\ 0 & 0 & \omega \\ 0 & r_0\omega sin(\omega t) & 0 \end{vmatrix} = -r_0\omega^2 sin(\omega t)\vec{i'}$$

$$\Rightarrow \boldsymbol{\vec{a_e} = -r_0\omega^2 sin(\omega t)\vec{i'}}$$

Coriolis acceleration : $\vec{a_c} = 2\vec{\omega}\wedge\vec{v_r} = \begin{vmatrix} \vec{i'} & \vec{j'} & \vec{k'} \\ 0 & 0 & 2\omega \\ r_0\omega cos(\omega t) & 0 & 0 \end{vmatrix} =$

$$\Rightarrow \boldsymbol{\vec{a_c} = 2r_0\omega^2 cos(\omega t)\vec{j'}}$$

Absolute acceleration : $\vec{a_a} = \vec{a_e} + \vec{a_r} + \vec{a_c}$

$$\Rightarrow \vec{a_a} = -r_0\omega^2 sin(\omega t)\vec{i'} - r_0\omega^2 sin(\omega t)\vec{i'} + 2r_0\omega^2 cos(\omega t)\vec{j'}$$

$$\Rightarrow \boldsymbol{\vec{a_a} = -2r_0\omega^2 sin(\omega t)\vec{i'} + 2r_0\omega^2 cos(\omega t)\vec{j'}}$$

Exercise n°07:

$\overrightarrow{OM} = \overrightarrow{O'M} = r\vec{i'} = r = r_0(cos\omega t + sin\,\omega t)\vec{i'}$

The angular velocity is constant around the axis Oz $\vec{\omega} = \omega\vec{k} = \omega\vec{u_z}$

Determine in the mobile frame of basic $(\vec{i'}, \vec{j'}, \vec{k'})$

Relative velocity : $\vec{v_r} = \frac{d\overrightarrow{O\prime M}}{dt} = r_0\omega(-sin\omega t + cos\omega t)\vec{i'}$

Training velocity: $\vec{v_e} = \frac{d\overrightarrow{OO'}}{dt}\,| + (\vec{\omega}\wedge\overrightarrow{O'M}) = (\vec{\omega}\wedge\overrightarrow{O'M}), \frac{d\overrightarrow{OO'}}{dt}\,| = 0$

$$(\vec{\omega}\wedge\overrightarrow{O'M}) = \begin{vmatrix} \vec{i'} & \vec{j'} & \vec{k'} \\ 0 & 0 & \omega \\ r_0(cos\omega t + sin\,\omega t) & 0 & 0 \end{vmatrix} = r_0\omega(cos\omega t + sin\,\omega t)\vec{j'}$$

$$\Rightarrow \boldsymbol{\vec{v_e} = r_0\omega(cos\omega t + sin\,\omega t)\vec{j'}}$$

Absolute velocity: $\vec{v_a} = \vec{v_e} + \vec{v_r}$

$$\Rightarrow \vec{v_a} = r_0\omega\left[(-sin\omega t + cos\omega t)\vec{i'} + (cos\omega t + sin\,\omega t)\vec{j'}\right]$$

$$|\vec{v_a}| = r_0\omega\sqrt{(-sin\omega t + cos\omega t)^2 + (cos\omega t + sin\,\omega t)^2}$$

$\Rightarrow |\vec{v_a}| = r_0\omega\sqrt{2}$, $|\vec{v_a}|$ is constant

Relative acceleration:

$$\boldsymbol{\vec{a_r} = \frac{d\vec{v_r}}{dt}\,| = r_0\omega^2(-cos\omega t - sin\omega t)\vec{i'}}$$

Training acceleration: $\vec{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + (\frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M}) + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{O'M})$

$$\Rightarrow \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{O'M}\right) = \begin{vmatrix} \vec{i'} & \vec{j'} & \vec{k'} \\ 0 & 0 & \omega \\ 0 & r_0(\cos\omega t + \sin\omega t) & 0 \end{vmatrix} = -r_0\omega^2(\cos\omega t + \sin\omega t)\vec{i'}$$

$$\Rightarrow \boldsymbol{\vec{a_e} = -r_0\omega^2(\cos\omega t + \sin\omega t)\vec{i'}}$$

Coriolis acceleration : $\vec{a_c} = 2\vec{\omega} \wedge \vec{v_r} = \begin{vmatrix} \vec{i'} & \vec{j'} & \vec{k'} \\ 0 & 0 & 2\omega \\ r_0\omega(-\sin\omega t + \cos\omega t) & 0 & 0 \end{vmatrix} =$

$$\Rightarrow \boldsymbol{\vec{a_c} = 2r_0\omega^2(-\sin\omega t + \cos\omega t)\vec{j'}}$$

Absolute acceleration : $\vec{a_a} = \vec{a_e} + \vec{a_r} + \vec{a_c}$

$$\Rightarrow \vec{a_a} = r_0\omega^2(-\cos\omega t - \sin\omega t)\vec{i'} - r_0\omega^2(\cos\omega t + \sin\omega t)\vec{i'}$$
$$+ 2r_0\omega^2(-\sin\omega t + \cos\omega t)\vec{j'}$$

$$\Rightarrow \boldsymbol{\vec{a_a} = -2r_0\omega^2(\cos\omega t + \sin\omega t)\vec{i'} \quad + 2r_0\omega^2(-\sin\omega t + \cos\omega t)\vec{j'}}$$

$$|\boldsymbol{\vec{a_a}}| = \boldsymbol{2r_0\omega^2}\sqrt{-(\cos\omega t + \sin\omega t)^2 + (-\sin\omega t + \cos\omega t)^2}$$

$\Rightarrow |\boldsymbol{\vec{a_a}}| = \boldsymbol{2r_0\omega^2}\sqrt{2}$, $|\boldsymbol{\vec{a_a}}|$ is constant

Exercise n°08:

The point O' moves on the axis (Ox) with a speed v, then $\vec{v_{O'}} = \frac{d\overrightarrow{OO'}}{dt}\vec{i} = v\vec{i}$

$$\frac{dOO'}{dt} = v \Rightarrow OO' = vt \Rightarrow \overrightarrow{OO'} = vt\,\vec{i}$$

The angular velocity is constant around the axis Oz $\vec{\omega} = \omega\vec{k} = \omega\vec{u_z}$

$$\overline{O'M}\begin{cases} X = (t+1) \\ Y = t^2 \end{cases} \Rightarrow \overrightarrow{O'M} = (t+1)\,\vec{u_x} + t^2\vec{u_y}$$

Determine in the moving frame of basic $(\vec{u_x}, \vec{u_y}, \vec{u_z})$

Relative velocity : $\vec{v_r} = \frac{d\overrightarrow{O'M}}{dt} = \boldsymbol{\vec{u_x} + 2t\vec{u_y}}$

Training velocity: $\vec{v_e} = \frac{d\overrightarrow{OO'}}{dt}\,| + \left(\vec{\omega} \wedge \overrightarrow{O'M}\right) = v\,\vec{i} - \omega t^2\vec{u_x} + \omega(t+1)\vec{u_y}$

$$\left(\vec{\omega} \wedge \overrightarrow{O'M}\right) = \begin{vmatrix} \vec{u_x} & \vec{u_y} & \vec{u_z} \\ 0 & 0 & \omega \\ (t+1) & t^2 & 0 \end{vmatrix} = -\omega t^2\vec{u_x} + \omega(t+1)\vec{u_y}$$

$$\Rightarrow \vec{v_e} = v\,\vec{i} - \omega t^2\vec{u_x} + \omega(t+1)\vec{u_y}$$

We must write $\vec{v_e}$ in the same coordinate system, for this we will write $\vec{\boldsymbol{i}}$ according to $\vec{u_x}$ and $\vec{u_y}$

We have: $\begin{cases} \vec{u_x} = cos\omega t\, \vec{\imath} + sin\,\omega t\, \vec{\jmath} \\ \vec{u_y} = -sin\omega t\, \vec{\imath} + cos\omega t\, \vec{\jmath} \end{cases} \Rightarrow \vec{\imath} = cos\omega t\, \vec{u_x} - sin\,\omega t\, \vec{u_y}$

SO $\vec{v_e} = -\omega t^2 \vec{u_x} + \omega(t+1)\vec{u_y} + v\left(cos\omega t\, \vec{u_x} - sin\,\omega t\, \vec{u_y}\right)$

$\Rightarrow \boldsymbol{\vec{v_e} = (v\, cos\omega t - \omega t^2)\vec{u_x} + (\omega(t+1) - v\, sin\,\omega t)\vec{u_y}}$

Absolute velocity: $\vec{v_a} = \vec{v_e} + \vec{v_r}$

$\Rightarrow \quad \vec{v_a} = \vec{u_x} + 2t\vec{u_y} + (v\, cos\omega t - \omega t^2)\vec{u_x} + (\omega(t+1) - v\, sin\,\omega t)\vec{u_y}$

$\Rightarrow \quad \boldsymbol{\vec{v_a} = (1 + v\, cos\omega t - \omega t^2)\vec{u_x} + (2t + \omega(t+1) - v\, sin\,\omega t)\vec{u_y}}$

Relative acceleration: $\vec{a_r} = \frac{d\vec{v_r}}{dt}\Big| = 2\vec{u_y}$

Training acceleration: $\vec{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + (\frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M}) + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{O'M})$

$\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}\ ,\ \left(\frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M}\right) = \vec{0}$ because ω is constant

$\Rightarrow \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{O'M}\right) = \begin{vmatrix} \vec{u_x} & \vec{u_y} & \vec{u_z} \\ 0 & 0 & \omega \\ -\omega t^2 & \omega(t+1) & 0 \end{vmatrix} = -\omega^2(t+1)\,\vec{u_x} - \omega^2 t^2\,\vec{u_y}$

$\Rightarrow \boldsymbol{\vec{a_e} = -\omega^2(t+1)\,\vec{u_x} - \omega^2 t^2\,\vec{u_y}}$

Coriolis acceleration : $\vec{a_c} = 2\vec{\omega} \wedge \vec{v_r} = \begin{vmatrix} \vec{u_x} & \vec{u_y} & \vec{u_z} \\ 0 & 0 & 2\omega \\ 1 & 2t & 0 \end{vmatrix} = -4\omega t\,\vec{u_x} + 2\omega\vec{u_y}$

$\Rightarrow \boldsymbol{\vec{a_c} = 2\omega\vec{u_z} \wedge (\vec{u_x} + 2t\vec{u_y}) = -4\omega t\vec{u_x} + 2\omega\vec{u_y}}$

Absolute acceleration : $\vec{a_a} = \vec{a_e} + \vec{a_r} + \vec{a_c}$

$\Rightarrow \vec{a_a} = 2\vec{u_y} + -\omega^2(t+1)\,\vec{u_x} - \omega^2 t^2\,\vec{u_y} - 4\omega t\,\vec{u_x} + 2\omega\vec{u_y}$

$\Rightarrow \boldsymbol{\vec{a_a} = \left(-\omega^2(t+1) - 4\omega t\right)\vec{u_x} + (2 + 2\omega - \omega^2 t^2)\vec{u_y}}$

Exercise n°09:

The point O' moves on the axis (Ox) with a speed v, then $\vec{v_{O'}} = \frac{dOO'}{dt}\vec{\imath} = v\vec{\imath}$

$\frac{dOO'}{dt} = v \Rightarrow\ OO' = vt\ \Rightarrow\ \overrightarrow{OO'} = vt\,\vec{\imath}$

The angular velocity is constant around the axis Oz $\vec{\omega} = \omega\vec{k} = \omega\vec{u_z}$

$\overrightarrow{O'M} = t^2\,\vec{u_x}$

Determine in the moving frame of basic $(\vec{u_x}, \vec{u_y}, \vec{u_z})$

Relative velocity : $\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt} = \boldsymbol{2t\,\overrightarrow{u_x}}$

Training velocity: $\overrightarrow{v_e} = \frac{d\overrightarrow{OO'}}{dt}\,| + \left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = v\,\vec{\imath} + \ \omega t^2\overrightarrow{u_y}$

$$\left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = \begin{vmatrix} \overrightarrow{u_x} & \overrightarrow{u_y} & \overrightarrow{u_z} \\ 0 & 0 & \omega \\ t^2 & 0 & 0 \end{vmatrix} = \omega t^2\overrightarrow{u_y}$$

$\Rightarrow \overrightarrow{v_e} = v\,\vec{\imath} + \ \omega t^2\overrightarrow{u_y}$

We must write $\overrightarrow{v_e}$ in the same coordinate system, for this we will write $\vec{\boldsymbol{\imath}}$ according to $\overrightarrow{u_x}$ and $\overrightarrow{u_y}$

We have: $\begin{cases} \overrightarrow{u_x} = \cos\omega t\,\vec{\imath} + \sin\omega t\,\vec{\jmath} \\ \overrightarrow{u_y} = -\sin\omega t\,\vec{\imath} + \cos\omega t\,\vec{\jmath} \end{cases} \Rightarrow \ \vec{\imath} = \cos\omega t\,\overrightarrow{u_x} - \sin\omega t\,\overrightarrow{u_y}$

SO $\overrightarrow{v_e} = \omega t^2\overrightarrow{u_y} \ + v\left(\cos\omega t\,\overrightarrow{u_x} - \sin\omega t\,\overrightarrow{u_y}\right)$

$\Rightarrow \boldsymbol{\overrightarrow{v_e} = (v\cos\omega t)\overrightarrow{u_x} + (\omega t^2 - v\sin\omega t)\overrightarrow{u_y}}$

Absolute velocity: $\overrightarrow{v_a} = \overrightarrow{v_e} + \overrightarrow{v_r}$

$\Rightarrow \ \overrightarrow{v_a} = 2t\,\overrightarrow{u_x} + (v\cos\omega t)\overrightarrow{u_x} + (\omega t^2 - v\sin\omega t)\overrightarrow{u_y}$

$\Rightarrow \ \boldsymbol{\overrightarrow{v_a} = (2t + v\cos\omega t)\overrightarrow{u_x} + (\omega t^2 - v\sin\omega t)\overrightarrow{u_y}}$

Relative acceleration: $\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}\,|= 2\,\overrightarrow{u_x}$

Training acceleration: $\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \left(\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}\right) + \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)$

$\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}\ ,\left(\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}\right) = \vec{0}$ because ω is constant

$$\Rightarrow \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = \begin{vmatrix} \overrightarrow{u_x} & \overrightarrow{u_y} & \overrightarrow{u_z} \\ 0 & 0 & \omega \\ 0 & \omega t^2 & 0 \end{vmatrix} = -\,\omega^2 t^2\,\overrightarrow{u_x}$$

$\Rightarrow \boldsymbol{\overrightarrow{a_e} = -\,\omega^2 t^2\,\overrightarrow{u_x}}$

Coriolis acceleration : $\overrightarrow{a_c} = 2\vec{\omega}\wedge\overrightarrow{v_r} = \begin{vmatrix} \overrightarrow{u_x} & \overrightarrow{u_y} & \overrightarrow{u_z} \\ 0 & 0 & 2\omega \\ 2t & 0 & 0 \end{vmatrix} = 4\omega t\,\overrightarrow{u_y}$

$\Rightarrow \boldsymbol{\overrightarrow{a_c} = 2\omega\overrightarrow{u_z}\wedge(2t\overrightarrow{u_x}) = 4\omega t\,\overrightarrow{u_y}}$

Absolute acceleration : $\overrightarrow{a_a} = \overrightarrow{a_e} + \ \overrightarrow{a_r} + \overrightarrow{a_c}$

$\Rightarrow \overrightarrow{a_a} = 2\,\overrightarrow{u_x} - \ \omega^2 t^2\,\overrightarrow{u_x} + 4\omega t\,\overrightarrow{u_y}$

$\Rightarrow \boldsymbol{\overrightarrow{a_a} = (2 - \omega^2 t^2)\overrightarrow{u_x} + 4\omega t\,\overrightarrow{u_y}}$

Exercise n°10:

At $t=0, y=0, v=v_0$

$\overrightarrow{OM}=\overrightarrow{OO'}+\overrightarrow{O'M}$ with $\overrightarrow{OO'}=r\,(cos\omega t\,\vec{\imath}+sin\,\omega t\vec{\jmath})$

M moves on the axis (O'Y) parallel to (oy) with constant acceleration a

$\overrightarrow{OM}=y\overrightarrow{u_y}\,, a=\frac{dv}{dt}\Rightarrow dv=adt$

After integration $v=at+v_0$

$\frac{dy}{dt}=at+v_0\Rightarrow dy=at\,dt+v_0dt$, so $y=\frac{1}{2}at^2+v_0t$

$\overrightarrow{O'M}=\left(\frac{1}{2}at^2+v_0t\right)\overrightarrow{u_y}$

Since O'Y//Oy therefore $\overrightarrow{u_y}=\vec{\jmath}\Rightarrow\left(\frac{1}{2}at^2+v_0t\right)\overrightarrow{u_y}=\left(\frac{1}{2}at^2+v_0t\right)\vec{\jmath}$

Then $\overrightarrow{OM}=\overrightarrow{OO'}+\overrightarrow{O'M}=r\left(cos\omega t\,\vec{\imath}+sin\,\omega t\vec{\jmath}\right)+\left(\frac{1}{2}at^2+v_0t\right)\vec{\jmath}$

$\Rightarrow\boldsymbol{\overrightarrow{OM}=rcos\omega t\,\vec{\imath}+\left(rsin\,\omega t+\frac{1}{2}at^2+v_0t\right)\vec{\jmath}}$

Relative velocity : $\overrightarrow{v_r}=\frac{d\overrightarrow{O'M}}{dt}=\boldsymbol{(at+v_0)\vec{\jmath}}$

Training velocity: $\overrightarrow{v_e}=\frac{d\overrightarrow{OO'}}{dt}\,|+(\vec{\omega}\wedge\overrightarrow{O'M})$

$(\vec{\omega}\wedge\overrightarrow{O'M})=0$, because the unit vectors of the two coordinate systems are parallel, so there is no rotational motion.

There is a translation motion

$$\overrightarrow{v_e}=\frac{d\overrightarrow{OO'}}{dt}\,|=-r\omega\,sin\omega t\vec{\imath}+r\omega\,cos\omega t\vec{\jmath}$$

Absolute velocity: $\overrightarrow{v_a}=\overrightarrow{v_e}+\overrightarrow{v_r}$

$\Rightarrow\ \overrightarrow{v_a}=(at+v_0t)\vec{\jmath}-r\omega\,sin\omega t\vec{\imath}+r\omega\,cos\omega t\vec{\jmath}$

$\Rightarrow\ \boldsymbol{\overrightarrow{v_a}=-r\omega\,sin\omega t\vec{\imath}+(r\omega\,cos\omega t+at+v_0t)\vec{\jmath}}$

Relative acceleration:

$$\boldsymbol{\overrightarrow{a_r}=\frac{d\overrightarrow{v_r}}{dt}|=a\,\vec{\jmath}}$$

Training acceleration: $\overrightarrow{a_e}=\frac{d^2\overrightarrow{OO'}}{dt^2}+(\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M})+\vec{\omega}\wedge(\vec{\omega}\wedge\overrightarrow{O'M})$

$\left(\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}\right)=\vec{0}$ and $\vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)=\vec{0}$

Because there is a translation motion between the frames

$\Rightarrow\overrightarrow{a_e}=\frac{d^2\overrightarrow{OO'}}{dt^2}=-r\omega^2\,cos\omega t\vec{\imath}-r\omega^2 sin\omega t\,\vec{\jmath}$

$\Rightarrow \overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} = -r\omega^2 \left(cos\omega t\vec{i} + sin\omega t\,\vec{j}\right)$

Coriolis acceleration : $\overrightarrow{a_c} = 2\vec{\omega}\wedge\overrightarrow{v_r} = \vec{0}$

Absolute acceleration : $\overrightarrow{a_a} = \overrightarrow{a_e} + \overrightarrow{a_r} + \overrightarrow{a_c}$

$\Rightarrow \overrightarrow{a_a} = a\,\vec{j} - r\omega^2\, cos\omega t\vec{i} - r\omega^2 sin\omega t\,\vec{j}$

$\Rightarrow \boldsymbol{\overrightarrow{a_a} = -r\omega^2\, cos\omega t\vec{i} + (a - r\omega^2 sin\omega t)\,\vec{j}}$

Chapter5

DYNAMICS OF PARTICLE

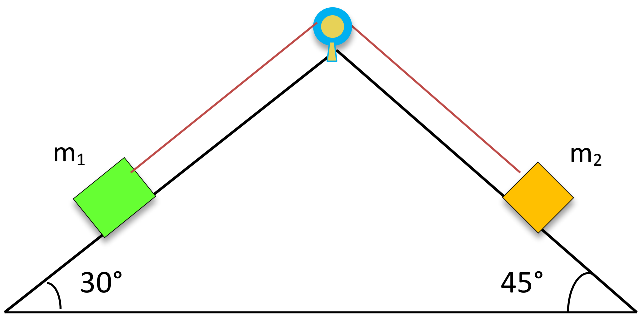
Learning Goals: After going through this chapter, students will be able to

- Learn and apply Newton's three laws.
- Understand the characteristics of certain forces (weight, normal force, friction force and restoring force).
- to understand the concepts of inertia, mass and force.
- Know how to take stock of the forces applied to a previously defined system.
- to solve statics problems using the first Newton's law;
- to solve dynamic problems using the second Newton's law. (Apply the fundamental principle of dynamics).
- Introduce the notion of angular momentum.
- Learn and apply the theorems of quantity of Motion and angular momentum.

Exercise n°01:

Two blocks of masses m_1 and m_2 are connected by a string of negligible mass which passes over a pulley connected with plane as shown in the figure below.. We neglect the mass and friction forces of the pulley.

1. Represent all the forces acting on the system.
2. Knowing that m_2=2.0 kg, find the value of m_1 in equilibrium.
3. We give m_1=2.5 kg and m_2=2.0 kg, plot the forces applied to the system. Using the second Newton's law, calculate the acceleration of motion.



Exercise n°02:

We consider an object of mass m=5Kg on an inclined plane making an angle α =30° with the horizontal. From point O, we launch the object upwards with an initial speed v_0= 2m/s Knowing that the coefficient of dynamic friction $\mu_d = 0{,}2$ and applying the fundamental principle of dynamics, calculate:

1) The acceleration of the object.
2) The distance traveled by this object before stopping at point A.
3) The minimum value of the coefficient of static friction μ_s so that the object does not slide down once stopped.

Exercise n°03:

We launch a block (M) of mass m, from the top of inclined plane AB = 1m with α = 45° with respect to the horizontal, and with an initial speed $v_A = 1\ m/s$

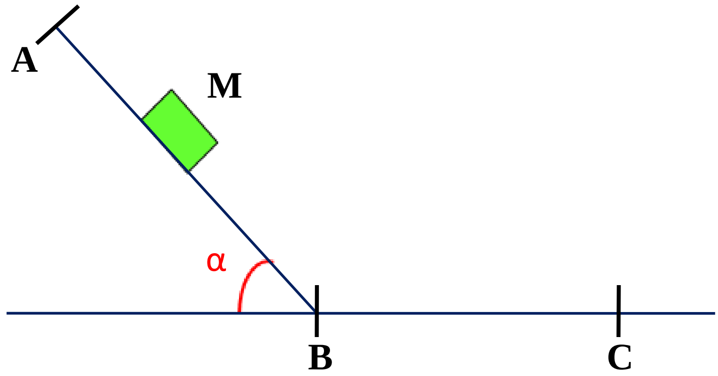
1. Knowing that the coefficient of friction $\mu = 0{,}5$ on AB,

calculate the acceleration of the motion on AB and the speed of (M) when it reaches point B.

2. We consider that the friction forces are negligible on the horizontal plane BC.

What is the nature of motion on the horizontal plane BC? Justify.

- At what distance from point B will block (M) stop?



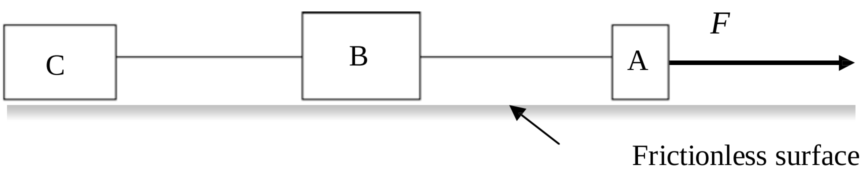
Exercise n°04:

Let's consider the system shown in Figure below. The magnitude of the applied force is F = 105N, and the masses are m_A = 10 kg, m_B = 18 kg and m_C = 14 kg. The masses of the connecting ropes between the blocks can be neglected.

a. Find the acceleration of the system.

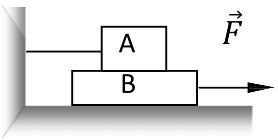
b. Find and represent the forces applied on each block.

c. Assuming that the maximum tension that can be applied on the rope before it breaks is 85N, find the maximum number of blocks that could be attached to the system provided (the force F is still applied).



Exercise n°05:

A 5.00 kg block A is placed on top of a 10.0 kg one B. A horizontal force of 45.0 N is applied to the block B, and A is tied to the wall (Figure above). The coefficient of kinetic friction between all surfaces is $\mu_k = 0.2$.



a. Represent the forces applied on each block and identify the action-reaction forces between the blocks.

b. Determine the tension in the string and the magnitude of the acceleration of the block B.

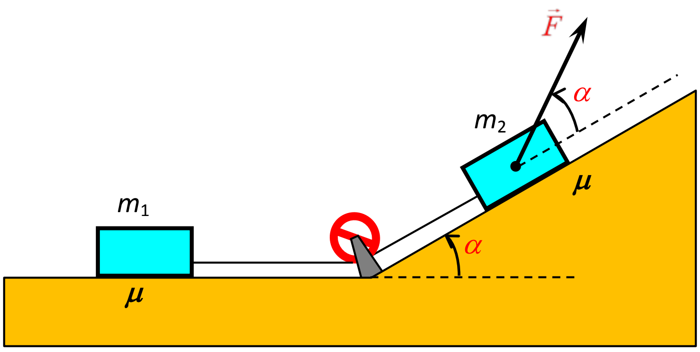
Exercise n°06:

Two masses m_1 and m_2 are connected by an inextensible rope of negligible mass which passes through a pulley of negligible mass. The mass m_2 moves on an inclined plane making an angle α with the horizontal and having a coefficient of friction μ and it is subjected to a force $\vec{F}$ making an angle α with respect to the surface of the inclined plane.

The mass m_1 moves on a horizontal plane having the same coefficient of friction μ (see figure blow).

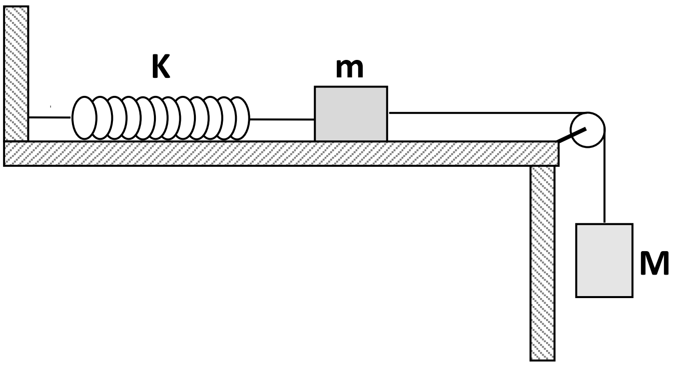
1. Write the Fundamental Principle of Dynamics and represent the forces for each mass.
2. Calculate the magnitude of the contact force C_2 applied to the mass m_2.
3. What is the magnitude of the acceleration a of the two masses?
4. Calculate the modulus of the tension of the string T.

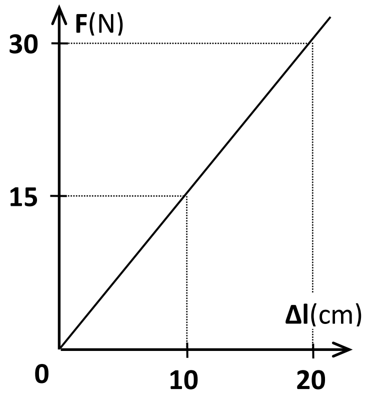
NC: $\alpha = 30°$; $m_1 = 2\ kg$; $m_2 = 4\ kg$; $\mu = 0{,}1$; $F = 40\ N$.



Exercise n°07:

A body of mass M is connected to a body of mass m = 1kg via an inextensible wire of negligible mass. A spring K of negligible mass, whose calibration curve is given below, is attached to the mass m and to the wall.





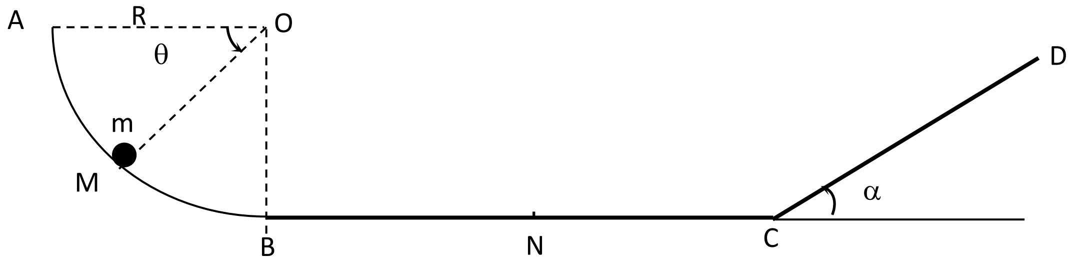
1) In the case where we neglect the friction of the mass **m** on the horizontal plane, literally calculate the acceleration taken by the system as well as the tension of the wire.

2) The friction is no longer negligible and the spring is not tensioned, what is the maximum value of the mass M to be suspended so that the system remains at rest? The value of the coefficient of static friction is μ_s= **0.8**.

3) The mass **m** has moved 10 cm, calculate at this position the acceleration of the system and the tension of the wire knowing that the coefficient of dynamic friction is μ_d= **0.25**

Exercise n°08:

A material point of mass m slides along the path ABCD shown in the figure below. The circular part AB with center O and radius R is completely smooth. The track BCD consists of a horizontal part BC characterized by a dynamic coefficient μ_{d1}, and another part CD, inclined at an angle α, characterized by a dynamic coefficient μ_{d2}.



1. a) Qualitatively, depict the forces acting on the mass m at the point M defined by the angle θ (see figure)

b) Show that the velocity of m at point M is given by: $V = \sqrt{2Rg(\sin\theta) + V_A^2}$, where V_A is the velocity of m at point A and g is the acceleration of gravity.

c) Derive the expression for the speed at point B.

2. a) Qualitatively, depict the forces acting on the mass m at point N on part BC.

b) Find the expression for the acceleration of mass m at point N.

c) We set BC=R, deduce the expression for the speed at point C of the path.

3.We give: $g=10m/s^2$,$\alpha=20°$, $\mu_{d2}=0.5$ and CD=R.

a) Calculate the acceleration of m on the inclined part CD.

b) If the initial speed of m is zero ($V_A=0$), what must be the value ofμ_{d1}so that the mass stops at point D?

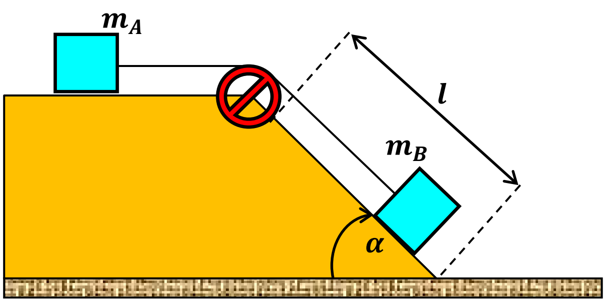
c)I n this case ($V_A=0m/s$), can the mass m go back down after stopping, on CD? Justify your answer.

Exercise n°09:

Consider the system represented by the figure below. The two inclined planes make an angle $\boldsymbol{\alpha}$ to the horizontal. The spring linked to the mass m_1 has a stiffness constant k , its mass being negligible. The masses of the pulleys and the wire are negligible, and the wire is inextensible. The mass values are given by: $m_2 = 2.m$; $m_3 = m_1 = m$.We release the system without initial speed, after a certain time we notice that the elongation of the spring is constant, noted $\Delta \boldsymbol{l}$, therefore the accelerations of the three masses are equal in magnitude $a_1 = a_2 = a_3 = a$

1. In the case where the plane is smooth (no friction). Show that the system moves in the positive direction indicated in the figure. Calculate its acceleration.
2. What is the elongation of the spring Δl?
3. If the friction coefficient of the plane is non-null ($\mu \neq 0$). What is the expression of acceleration a for the three masses?
4. In this case, calculate the elongation Δl of the spring as a function of $k,\ \mu,\ m,\ \alpha$ and g.

NC: $k = 10\ N/m$ **,** $= 0{,}1$, $m = 100\ g$, $\alpha = 60°$, $g = 10\ m/s^2$



Exercise n°10:

Consider the system represented in the figure below. It is composed of two point masses m_A and $m_B$connected by an inextensible wire of negligible mass passing through the throat of a pulley of negligible mass.

The mass $m_A = 6\ kg$ is placed on a horizontal plane having a static friction coefficient $\mu_s = 0{,}5$ and a dynamic friction coefficient μ_d.The mass is placed on a perfectly smooth inclined plane, of length $l = 50\ cm$ and making an angle $\alpha = 60°$ with the horizontal. ($g = 10\ m/s^2$)

1. **Static case:** Calculate the minimum value $m_{B\min}$ of the mass m_B at which the system is in equilibrium, and above which it starts to move.
2. **Dynamic case:** For $m_B = 4\ kg$ the system to start moving from rest, we then may identify two phases;

Phase I: The two masses are in motion and travel the same distance l until the mass m_B reaches the bottom of the inclined plane.

Phase II: It is assumed that the mass m_B stops when it reaches the bottom of the inclined plane and only the mass m_A continues its motion until it stops after having traveled a distance $d = 7\ cm$.

a. Write the fundamental principle of dynamics and represent the forces acting on the two masses during phase I.

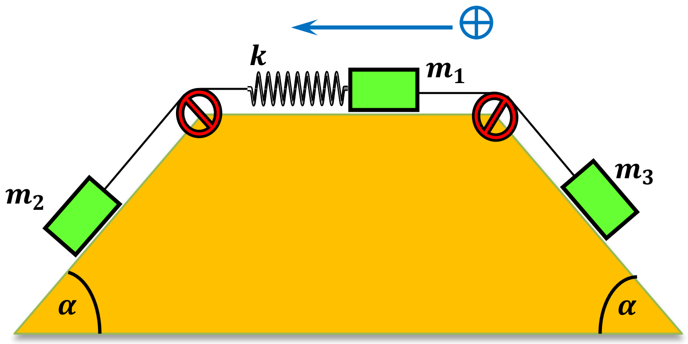
b. Find the expression of the acceleration a_I for the system during phase I. What is the nature of the motion?

c. Write the fundamental principle of dynamics and represent the forces acting on the mass m_A during phase II.

d. Find the expression of the acceleration a_{II} for the mass m_A during phase II. What is the nature of the motion?

e. Find the expression for the dynamic friction coefficient of the horizontal plane as a function of m_A, m_B, l, d and α.

f. Numerical Calculations: Questions 1. and 2.e.



Exercise n°11:

A body of mass m likened to a material point is found at the point at the top of a plane inclined at an angle α to the horizontal and having a coefficient of friction μ (constant). The height of the point A relative to ground level is noted h.

1. Calculate the limit value $\mu_{\lim}$ for which the mass m is in equilibrium (static case).
2. For $\mu < \mu_{\lim}$ calculate the acceleration a of the mass m.
3. What is the nature of the motion?
4. If the mass$m$ is released from the point A without initial velocity. Calculate the velocity V_B at ground level (point$B$).
5. What is the timet_B needed for the mass to reach at point B?
6. The mass m continues its motion on a horizontal plane having the same coefficient of friction μ. What is the distance d traveled by the mass before stopping at point C?

   Numerical Calculations: (for all questions) $= 30°$; $\mu = 0{,}2$;$h = 2\ m$ and $g = 10\ m/s^2$.

Noticed: Give all results in terms of α, μ, h and g (acceleration of gravity).

Exercise n°12:

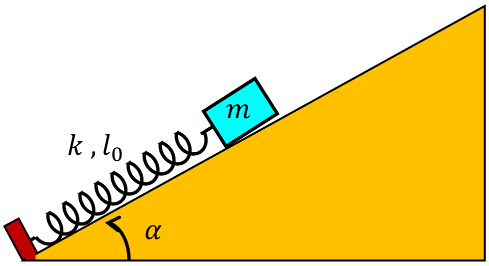
Given a spring with spring constant (k) and length (l), we fix a mass (m) at its end. What is the new length of the spring?

I-The mass is pulled down a distance d and then released without initial velocity.

1. By writing the FPD give the differential equation of motion (we recall that $\vec{F} = -k.\vec{x}$ Where $\vec{x}$ is the position vector relative to the equilibrium position)
2. If the solution of this equation is $x(t) = A.\sin(\omega.t) + B.\cos(\omega.t)$
   Determine A, B and ω.

II-Study the mass-spring system when it is horizontal.

III- Numerical Calculations: $k = 10\ N/m$; $l = 0{,}2\ m$; $m = 50\ g$; $d = 3\ cm$ .The origin of time is considered to be the moment when mass is released. (Air resistance is neglected)



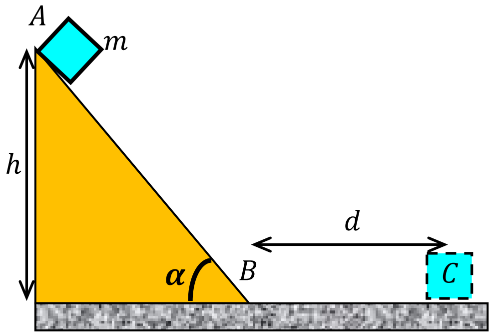
Exercise n°13:

Consider the system represented by the figure opposite. It is composed of a point mass$m$ connected to a spring of constant stiffness. k and empty length l_0. The mass-spring assembly is placed on a plane inclined at an angle α to the horizontal and offering no friction ($\mu = 0$).

1. Find the length l of the spring when the system is in equilibrium.

At instant $t_0 = 0$ considered as the origin of time, we give the equilibrium mass m an initial speed V_0 parallel to the inclined plane

2. Applying the Fundamental Principle of Dynamics, find the differential equation of motion for m at instant t.
3. If the solution to this equation is of the form $x(t) = A.\sin(\omega.t + \varphi)$
   a. Find ω, A and φ.
   b. Deduce the expression of the speed $V(t)$ and acceleration $a(t)$ for the mass m.



Exercise n°14:

A simple pendulum consists of a point mass m fixed at the end of a rod of negligible mass of length L, the other end of which is attached to the axis of rotation.

I-Using the Fundamental Principle of Dynamics:

1. Give the differential equation of motion for the case where the mass oscillates around its equilibrium position after being given an initial velocity V_0.
2. If the solution of this equation for fairly small angles is of the type

$$\theta(t) = \theta_0 . \sin(\omega . t + \varphi)$$

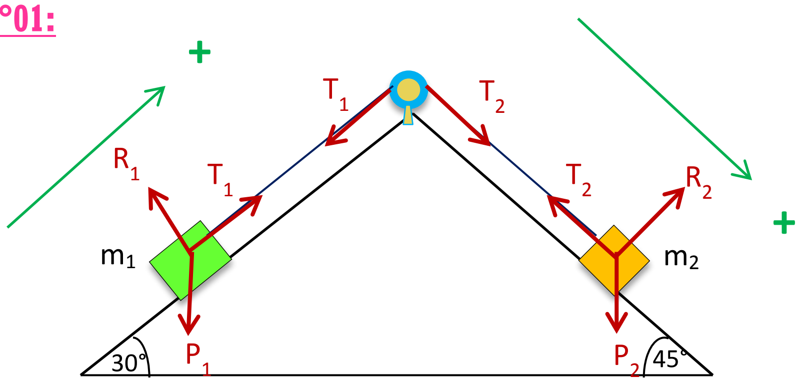
   a) Determine the value of the pulse ω and the period T.
   b) What does the value θ_0 represent? Calculate θ_0.
   c) What is the expression for angular velocity and acceleration?
   d) For what times is the speed (acceleration) maximum in magnitude?
   e) For what times is the speed (acceleration) zero?

II-Using the angular momentum theorem find the differential equation of motion.

III-Numerical Calculations $L = 19\,m$, $V(t = 0) = V_0 = 0{,}5\ m/s$ and we neglect air resistance.

SOLUTIONS TO EXERCISES

Exercise n°01:



- **The mass m_1:**

$\sum \vec{F} = \vec{0} \Rightarrow \vec{P_1} + \vec{T_1} + \vec{R_1} = \vec{0}$ …….(1)

After the projection on the axis of motion we obtain:

$\Rightarrow -P_{1x} + T_1 = \vec{0}$

(1) $\Rightarrow$ $\mathbf{m_1 . g . sin(30°)} = \boldsymbol{T_1}$**………….(2)**

- **The mass m_2:**

$\sum \vec{F} = \vec{0} \Rightarrow \vec{P_2} + \vec{T_2} + \vec{R_2} = \vec{0}$ …….(3)

After the projection on the axis of motion we obtain:

(3) $\Rightarrow P_{2x} - T_2 = 0$

$\Rightarrow m_2 . g . \sin(45°) = T_2$

$\Rightarrow \boldsymbol{T_2} = \mathbf{m_2 . g . sin(45°)}$**……………(4)**

From equation (2) and (4) we obtain:

$m_1 . g . \sin(30°) = m_2 . g . \sin(45°) \Rightarrow m_1 = \frac{m2.g.\sin(45°)}{g.\sin(30°)} \Rightarrow m_1 = \frac{m2.\sin(45°)}{\sin(30°)}$

N.C: $m_1 = \frac{2.\frac{\sqrt{2}}{2}}{\frac{1}{2}}$ **= 2.82 kg ⇒ $\mathbf{m_1}$= 0.35 m/s^2**

2. If **m_1=2.5 kg et m_2=2 kg ,** system in motion $\Rightarrow \sum \vec{F} = \mathbf{m_1} \vec{\boldsymbol{a}}$

- **The mass m_1:**

$\sum \vec{F} = \vec{0} \Rightarrow \vec{P_1} + \vec{T_1} + \vec{R_1} = m_1 \vec{a}$ …….(1)

After the projection on the axis of motion we obtain:

$\Rightarrow -P_{1x} + T_1 = m_1 a$

(1) $\Rightarrow$ $\mathbf{m_1 \boldsymbol{a} + m_1 . g . sin(30°)} = \boldsymbol{T_1}$**……………….(2)**

- **The mass m_2:**

$\sum \vec{F} = \vec{0} \Rightarrow \vec{P_2} + \vec{T_2} + \vec{R_2} = m_2 \vec{a}$ …….(1)

After the projection on the axis of motion we obtain:

$\Rightarrow +\boldsymbol{P_{2x}} - \boldsymbol{T_2} = \mathbf{m_2}\boldsymbol{a}$……………………..(3)

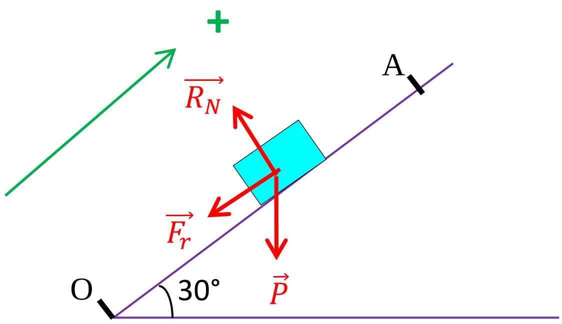
$\Rightarrow \mathbf{m_2.g.sin(45°)} - \mathbf{m_2}\boldsymbol{a} = \boldsymbol{T_2}$……………(4)

From equation (2) and (4) we obtain:

$m_1 a + m_1.g.\sin(30°) = m_2.g.\sin(45°) - m_2 a \Rightarrow \boldsymbol{a} = \frac{\mathbf{m2.g.sin(45°) - m1.g.sin(30°)}}{\mathbf{m_1+m_2}}$

N.C: $\boldsymbol{a} = \frac{2.10.\frac{\sqrt{2}}{2} - 2.5.10.\frac{1}{2}}{2.5+2} \Rightarrow \boldsymbol{a} = \mathbf{0.35\ m/s^2}$

Exercise n°02:



1. $\boldsymbol{a}$ = ?

Motion with friction a = ?

The system moving on the inclined plane with friction $\Rightarrow \sum \vec{F} = m\vec{a}$

$\sum \vec{F} = m\vec{a} \Rightarrow \vec{P} + \overrightarrow{R_n} + \overrightarrow{F_r} = m\vec{a}$ …….(1)

After the projection on the axis of motion we obtain:

$$(1) \Rightarrow \begin{cases} -P_x - F_r = ma \ldots\ldots\ldots\ldots\ldots (2) \\ R_n + P_y = 0 \ldots\ldots\ldots\ldots\ldots (3) \end{cases}$$

$$(2) \Rightarrow a = \frac{-P_x - F_r}{m} = \frac{-m.g.sin30 - F_r}{m}$$

$(3) \Rightarrow R_n = m.g.\cos\alpha$……………(4)

and we have $F_r = \mu.R_n = \mu.m.g.\cos\alpha$……(5)

$F_r = \mu.m.g.\cos\alpha$ = 0,2×5×10×0,86 =8,6 N

N.C: $a = \frac{-5x.10x.sin30 - 8,6}{5} \Rightarrow \boldsymbol{a} = \mathbf{-\ 6,72\ m/s^2}$

2. The distance covered x_{OA} , knowing that $v_O = 2\ m/s$

We have :

$v_A{}^2 - v_O{}^2 = 2.x_{OA}.a \Rightarrow x_{OA} = \frac{v_A{}^2 - v_O{}^2}{2.a}$ When $v_A = 0\ m/s$

$\Rightarrow x_{AB} = \frac{0^2 - 2^2}{2x(-6,72)}.$

$\Rightarrow \boldsymbol{x_{AB} = 0,29\ m}$

3. The minimum value of the coefficient of static friction μ_s so that the object does not slide down once stopped.

$\sum \vec{F} = \vec{0} \Rightarrow \vec{P} + \vec{R_n} + \vec{F_r} = \vec{0}$ …….(6)

After the projection on the axis of motion we obtain:

$\Rightarrow \begin{cases} P_x - F_r = 0 \ldots\ldots(7) \\ R_n + P_y = 0 \ldots\ldots(8) \end{cases}$

(7) $\Rightarrow F_r = m.g.sin30$……..(9)

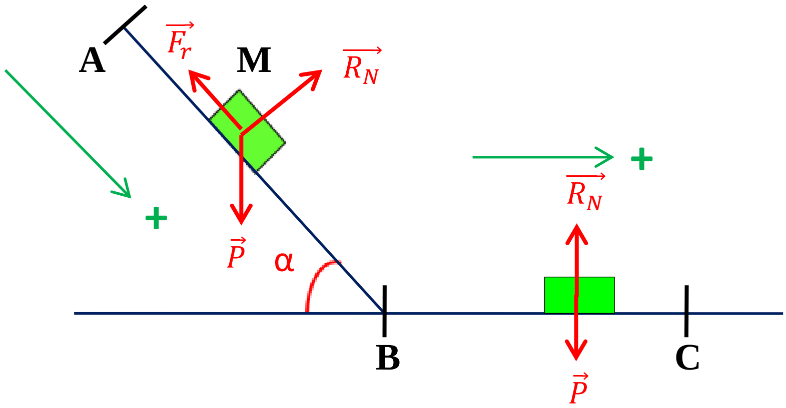
(8) $\Rightarrow R_n = \mathrm{m.g.\cos\alpha}$…………..(10)

μ_s =?

We have $\mu_s = tg\,\varphi = \frac{F}{R_n} = \frac{F}{P_y} = \frac{m.g.sin30}{m.g.cos30}$

NC: $\mu_s = \frac{0,5}{0,86} \Rightarrow \boldsymbol{\mu_s = 0,58}$

Exercise n°03:



1- The system moving on AB with friction $\Rightarrow \sum \vec{F} = m\vec{a}$

$\sum \vec{F} = m\vec{a} \Rightarrow \vec{P} + \vec{R} = m\vec{a}$ …….(1)

After the projection on the axis of motion we obtain:

(1) $\Rightarrow \begin{cases} P_x - F_r = ma \ldots\ldots(2) \\ R_n + P_y = 0 \ldots\ldots(3) \end{cases}$

(3) $\Rightarrow R_n = \mathrm{m.g.\cos\alpha}$…………..(4)

And we have $F_r = \mu.R_n = \mu.\mathrm{m.g.\cos\alpha}$…… (5)

(2) $\Rightarrow a = \frac{P_x - F_r}{m} = \frac{m.g.sin\alpha - \mu.\mathrm{m.g.\cos\alpha}}{m} = g(sin\alpha - \mu.\cos\alpha)$

N.C : a =9.81(707-0.5×0.707) ⇒ $\boldsymbol{a}$**=3.46 m/s^2**

v_B =?

We have:

$v_B{}^2 - v_A{}^2 = 2.AB.a \;\; \Rightarrow v_B{}^2 = 2.AB.a + v_A{}^2$ When $v_A = 1\,m/s$

$v_B = \sqrt{v_A{}^2 + 2.AB.a} = \sqrt{1^2 + 2 \times 1 \times 3{,}46}.$

$\Rightarrow \boldsymbol{v_B = 2{,}81\ m/s}$

The system moving on BC without friction $\Rightarrow \sum \vec{F} = m\overrightarrow{a'}$

$\sum \vec{F} = m\overrightarrow{a'} \;\Rightarrow \vec{P} + \overrightarrow{R_n} = m\overrightarrow{a'}$ …….(6)

After the projection on the axis of motion we obtain:

$$(6) \Rightarrow \begin{cases} 0 = ma' \dots\dots\dots\dots\dots\dots.(7) \\ R_n + P = 0 \dots\dots\dots..\dots\dots\dots(8) \end{cases}$$

$(\mathbf{7}) \Rightarrow a' = \mathbf{0}$ Therefore the motion is **Uniform rectilinear (MRU)**

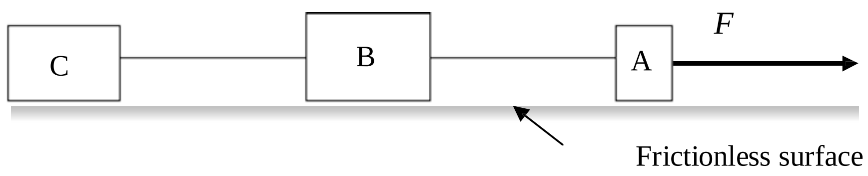
$v_c = ?$

we have :

$v_c{}^2 - v_B{}^2 = 2.\ \text{BC}.\ a' \Rightarrow \text{BC} = \frac{v_c{}^2 - v_B{}^2}{2.a'}$ When $v_C = 0$ and $a' = 0$

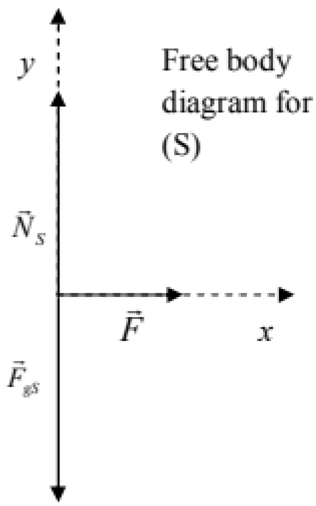
NC: BC=∞ the body M will never stop because the motion of the block is **Uniform rectilinear (MRU)** with a constant speed $\boldsymbol{v} = v_B$

Exercise n°04:



The acceleration of the system:

We consider the system (S) made of the three blocks and the ropes connecting them, the external forces applied on this system are: the total force of the gravity: $\vec{F}_{gS}$, the total normal force: $\vec{N}_S$ and the force $\vec{F}$.



2nd Newton's law:

$$\sum \vec{F}_{ext/S} = m\vec{a} \Rightarrow : \vec{F}_{gS} + \vec{F} + \vec{N}_S = m\vec{a}$$

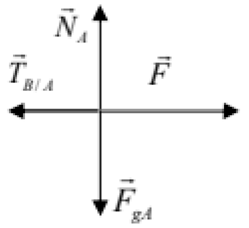
Resolving vectors on the x and y axis we get:

$$\begin{cases} (ox): F = ma_X = ma = (m_A + m_B + m_C)a \\ (oy): N_S - F_{gS} = ma_y = 0 \end{cases}$$

From these equations we get:

$$a = \frac{F}{m_A + m_B + m_C} = \frac{105}{10 + 18 + 14} = 2.5\text{ms}^{-2}$$

The forces applied on each block:



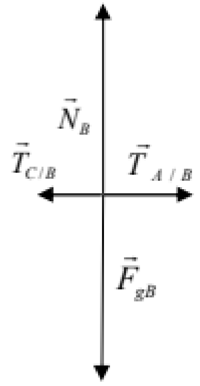
On the block A:

$\sum \vec{F}_{ext/S} = m_A\vec{a} \Rightarrow \vec{F} + \vec{F}_{gA} + \vec{T}_{B/A} + \vec{N}_A = m_A\vec{a}$

Resolving vectors on the x and y axis we get:

$$\begin{cases}(ox):\ F - T_{B/A} = m_A a_X = m_A a \Rightarrow T_{B/A} = F - m_A a \\ (oy): N_A - F_{gA} = m_A a_y = 0 \Rightarrow N_A = F_{gA} = m_A g\end{cases}$$

$$T_{B/A} = F - m_A a = F - \frac{m_A}{m_A + m_B + m_C}F = \frac{m_B + m_C}{m_A + m_B + m_C}F$$

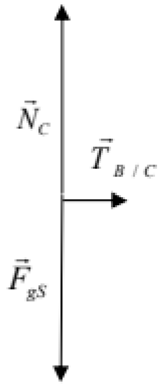


On the block B:

$\sum \vec{F}_{ext/S} = m_B\vec{a} \Rightarrow \vec{T}_{c/B} + \vec{F}_{gB} + \vec{T}_{A/B} + \vec{N}_B = m_B\vec{a}$

$$\Rightarrow \begin{cases}(ox):\ T_{A/B} - T_{c/B} = m_B a_x = m_B a \Rightarrow T_{c/B} = T_{A/B} - m_B a \\ \vec{T}_{C/B} \\ (oy):\ N_B - F_{gB} = m_B a_y = 0 \Rightarrow N_B = F_{gB} = m_B g\end{cases}$$

$$T_{c/B} = T_{A/B} - m_B a = \frac{m_c}{m_A + m_B + m_C}F = T_{B/C}$$



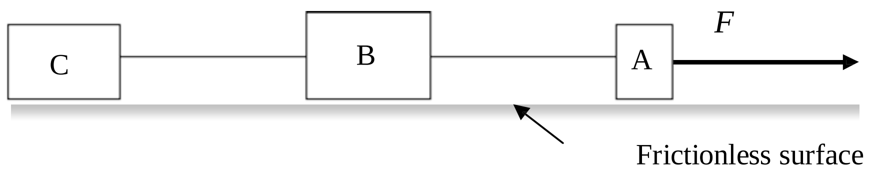
On the block C:

$$\sum \vec{F}_{ext/S} = m_C\vec{a} \Rightarrow \vec{T}_{B/C} + \vec{F}_{gC} + \vec{N}_C = m_C\vec{a}$$

Following the same method we find the values of all forces.

To find the maximum numbers of blocks of equal masses: let n the number ofthe blocks and m the mass of each block.

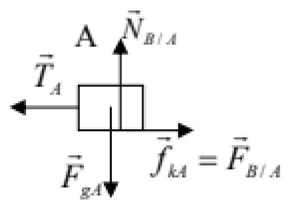
First we need to find the acceleration of a such system, we find: $= \frac{F}{nm}$, we can show that the maximum tension is applied to the first block and is given by:



$T_1 = F - ma = F - \frac{m}{nm}F = \frac{(n-1)m}{nm}F = \frac{n-1}{n}F$ is the strongest tension applied to the rope

We have: $T_1 \le T_{\max} \Rightarrow \frac{n-1}{n}F \le T_{\max} \Rightarrow n \le \frac{F}{F - T_{\max}} = \frac{105}{105-85} = 5.25$ thus the number of blocks that can be attached is: n = 5.

Exercise n°05:



1.The tension of the cord:

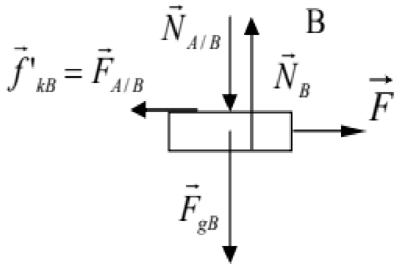
$$\sum \vec{F}_{ext/S} = \vec{0} \Rightarrow \vec{T}_A + \vec{F}_{gA} + \vec{N}_{B/A} + \vec{f}_{kA} = \vec{0}$$

Resolving on two perpendicular axis we get:

$$\begin{cases} (ox): -T_A + f_u = 0 \\ (oy): N_{B/A} - m_A g = 0 \end{cases} \Rightarrow \begin{cases} -T_A + \mu_k m_A g = 0 \\ \Rightarrow T_A = \mu_k m_A g \\ N_{B/A} = m_A g \end{cases}$$

Numerical Value:

$T_A = \mu_k m_A g = 0.200 \times 5.00 \times 9.80 \Rightarrow \boldsymbol{T_A = 9.80\mathrm{N}}.$



2.The acceleration of the bloc B:

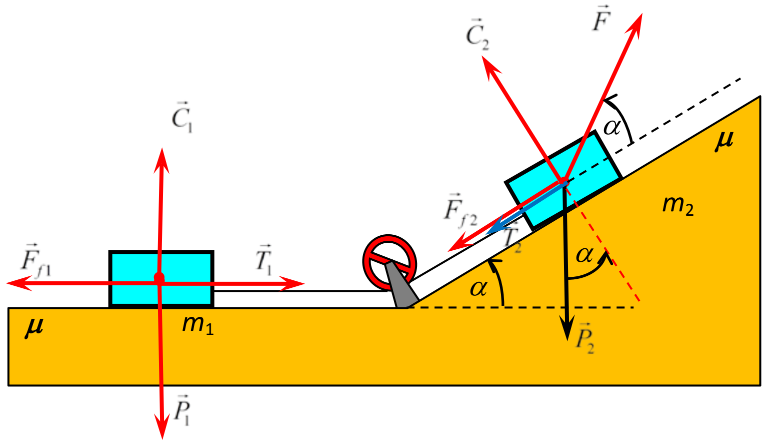
$$\sum \vec{F}_{ext/S} = m_B \vec{a} \Rightarrow \vec{F} + \vec{F}_{gB} + \vec{N}_{B/A} + \vec{N}_B + \vec{f}_{kB} = m_B \vec{a}$$

Resolving on two perpendicular axis we get:

$$\begin{cases} (ox): F - f_{kB} = m_B a \\ (oy): N_B - N_{A/B} - m_B g = 0 \end{cases} \Rightarrow a = \frac{F - f_{kB}}{m_B} \text{ With } f_{kB} = f_{kA} = \mu_k m_A g \text{ so:}$$

$$a = \frac{F - \mu_k m_A g}{m_B} = \frac{45.0 - 0.200 \times 5.00 \times 9.80}{10.0} \Rightarrow \boldsymbol{a = 3.52\mathrm{ms}^{-2}}$$

Exercise n°06:



1. Fundamental Principle of Dynamics.

Mass m_1 : $\vec{P}_1 + \vec{T}_1 + \vec{C}_1 + \vec{F}_{f1} = m_1 . \vec{a}_1$

Mass m_2 : $\vec{P}_2 + \vec{T}_2 + \vec{C}_2 + \vec{F}_{f2} + \vec{F} = m_2 . \vec{a}_2$

2.

By projecting onto the axis O_2Y_2 : $-m_2 g.\cos\alpha + C_2 + F.\sin\alpha = 0$

So $\boxed{C_2 = m_2 g.\cos\alpha - F.\sin\alpha}$

3.

By projecting onto the axis O_1X_1 : $-F_{f1} + T_1 = m_1 . a_1$ ....(1)

By projecting onto the axis O_1Y_1 : $-m_1 g + C_1 = 0$ ....(2)

By projecting onto the axis O_2X_2 : $-m_2 g.\sin\alpha - T_2 - F_{f2} + F.\cos\alpha = m_2 . a_2$ .....(3)

By projecting onto the axis O_2Y_2 : $-m_2 g.\cos\alpha + C_2 + F.\sin\alpha = 0$ ....(4)

Since ($a_1 = a_2 = a$) et ($T_1 = T_2 = T$) (wire, pulley and spring of negligible masses)

By adding (1) et de (3) we find:

$$-m_2 g.\sin\alpha - F_{f1} - F_{f2} + F.\cos\alpha = (m_1 + m_2).a$$

(2) $\Rightarrow$ $F_{f1} = \mu.C_1 = \mu.m_1 g$

(4) $\Rightarrow$ $F_{f2} = \mu.C_2 = \mu.m_2 g.\cos\alpha - \mu.F.\sin\alpha$

Replacing:

$$-m_2 g.\sin\alpha - \mu.m_1 g - \mu.m_2 g.\cos\alpha + \mu.F.\sin\alpha + F.\cos\alpha = (m_1 + m_2).a$$

And the acceleration is given by:

$$\boxed{a = \frac{(\mu.F - m_2 g).\sin\alpha + (F - \mu.m_2 g).\cos\alpha - \mu.m_1 g}{(m_1 + m_2)}}$$

4. According to equation (1) : $T_1 = m_1.a + F_{f1} = m_1.a + \mu.m_1 g$ et ($T_1 = T_2 = T$)

So $T = \frac{m_1}{(m_1 + m_2)}[(\mu.F - m_2 g).\sin\alpha + (F - \mu.m_2 g).\cos\alpha - \mu.m_1 g] + \mu.m_1 g$

Ou $\boxed{T = \frac{m_1}{(m_1 + m_2)}[(\mu.F - m_2 g).\sin\alpha + (F - \mu.m_2 g).\cos\alpha + \mu.m_2 g]}$

5. Numerical calculation : We have $\cos 30° = \frac{\sqrt{3}}{2}$ et $\sin 30° = \frac{1}{2}$

$\boxed{C_2 = 14{,}4\ \text{N}}$; $\boxed{a = 2\ \text{m/s}^2}$; $\boxed{T = 6\ \text{N}}$.

Exercise n°07:

1- If we neglect the friction of mass m on the horizontal plane, we literally calculate the acceleration taken by the system and the tension of the wire, using the fundamental relation of dynamics applied to two bodies of mass m and M.

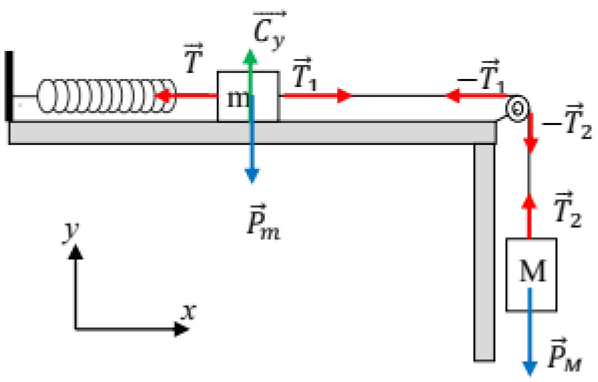
The system is in motion, hence:

For mass m:

$$\overrightarrow{P_m} + \vec{T} + \overrightarrow{C_y} + \overrightarrow{T_1} = m\,\overrightarrow{a_m}$$

For mass M:

$$\overrightarrow{P_M} + \overrightarrow{T_2} = m\,\overrightarrow{a_M}$$



Since the mass of the wire and the mass of the pulley are negligible, we have:

$\|\overrightarrow{T_1}\| = \|\overrightarrow{T_2}\|$

We also have a track with negligible friction, so: $\overrightarrow{C_x} = \vec{0}$ and $\vec{C} = \overrightarrow{C_y}$

The wire is inextensible, hence: $a_m = a_M = a$

The elastic force is given by: $\vec{T} = -k \cdot \Delta l\, \vec{\imath}$

Projecting on the Ox and Oy axes, we find:

$$\begin{cases} -k\Delta l + \|\overrightarrow{T_1}\| = ma & (1) \\ Mg - \|\overrightarrow{T_1}\| = Ma & (2) \\ mg - \|C_y \rightarrow\| = 0 & (3) \end{cases}$$

Summing (1) and (2), we find:

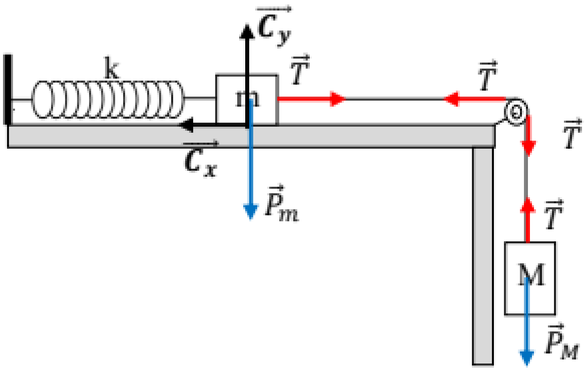
$$a = \frac{Mg - k\Delta l}{M + m}$$

By replacing the expression of a in equation (1) or (2), we find:

$$\boldsymbol{\|\overrightarrow{T_1}\| = M(g - a) = \frac{M(gm + k\Delta l)}{(M + m)}}$$

2-As friction is no longer negligible and the spring is not tensioned, the maximum value of the mass M that must be suspended for the system to remain at rest is obtained from the static state.

The fundamental relation of dynamics in the static state is given by:



$\overrightarrow{P_m} + \vec{C} + \overrightarrow{T_1} = \vec{0}$

$\overrightarrow{P_M} + \overrightarrow{T_2} = \vec{0}$

Knowing that $\|\overrightarrow{T_1}\| = \|\overrightarrow{T_2}\|$ and by projection we obtain:

$$\begin{cases} -C_X + \|\overrightarrow{T_1}\| = 0 & \rightarrow \|\overrightarrow{T_1}\| = C_X \\ Mg - \|\overrightarrow{T_1}\| = 0 & \rightarrow \|\overrightarrow{T_1}\| = Mg \\ mg - C_y = 0 & \rightarrow mg = C_y \end{cases}$$

Hence:

$$\begin{cases} C_X = Mg \\ C_y = mg \end{cases} \rightarrow \mu_s = \frac{C_X}{C_y} = \frac{M}{m} \rightarrow \boldsymbol{M = m\,\mu_s}$$

Knowing that $\mu_s = 0.8$ and $m = 1$kg, we obtain :

3. The mass m has moved 10 cm. To calculate the acceleration of the system and the tension of the wire at this position, given that the coefficient of dynamic friction is

$\mu_d = 0{,}25$, we use the dynamic state.

Using the fundamental relation of dynamics, we have :

$\overrightarrow{P_m} + \vec{T} + \vec{C} + \overrightarrow{T_1} = m\vec{a}$

$\overrightarrow{P_M} + \overrightarrow{T_2} = M\vec{a}$

By projection, we obtain :

$$\begin{cases} -\|\vec{T}\| - \|\overrightarrow{C_x}\| + \|\overrightarrow{T_1}\| = ma \;\rightarrow\; \|\overrightarrow{T_1}\| = ma + \|\vec{T}\| + \|\overrightarrow{C_x}\| \\ \text{Mg} - \|\overrightarrow{T_1}\| = Ma \;\rightarrow\; \|\overrightarrow{T_1}\| = \text{Mg} - Ma \;\; (*) \\ mg - \|\overrightarrow{C_y}\| = 0 \;\rightarrow\; \|\overrightarrow{C_y}\| = mg \end{cases}$$

Equating between the first and second equations gives :

$$ma + \|\vec{T}\| + \|\overrightarrow{C_x}\| = \text{Mg} - Ma \;\rightarrow\; \|\overrightarrow{C_x}\| = Mg - (M+m)a - \|\vec{T}\|$$

Knowing that $\|\vec{T}\| = k\Delta l$, we have:

$$\mu_{cl} = \frac{\|\overrightarrow{C_x}\|}{\|\overrightarrow{C_y}\|} = \frac{Mg - (M+m)a - k\Delta l}{mg}$$

Hence:

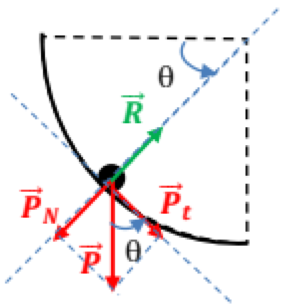
$$a = \frac{(M - m\mu_d)g - k\Delta l}{(M+m)} \;\rightarrow\; \boldsymbol{a = 0{,}72\ m/s^2}$$

By replacing the acceleration in equation (*), we find:

$$\boldsymbol{\|\overrightarrow{T_1}\| = M(g-a) = M\left(\frac{gm\,(1-\mu_d)g - k\Delta l}{(M+m)}\right) \;\rightarrow\; \|\overrightarrow{T_1}\| = 18{,}175\ N}$$

Exercise n°08:

1. a) Qualitative representation of the forces applied to the mass **m** on point **M** defined by the angle $\boldsymbol{\theta}$.



b) We show that the speed of **m** on point **M** is given by: $V = \sqrt{2Rg(sin\theta) + V_A^2}$, Or V_A is the speed of m at point A and g is the acceleration of gravity.

According to the fundamental relation of dynamics, we have:

$\sum \vec{F} = m\vec{a} \rightarrow \vec{P} + \vec{R} = m\vec{a}$

By projection on the axis perpendicular and tangential to the movement, we have:

$$\begin{cases} P_t = ma_t \rightarrow mg\cos\theta = ma_t \\ R - P_N = ma_N \rightarrow R - mg\sin\theta = ma_N \end{cases}$$

We have:

$$a_t = \frac{dV}{dL} \rightarrow dV = a_\iota dt \rightarrow \boldsymbol{dV = g\cos\theta\ dt}$$

We also have:

$$\begin{cases} \omega = \dfrac{V}{R} \\ \omega = \dfrac{d\theta}{dt} \end{cases} \rightarrow \quad \boldsymbol{dt = \frac{R}{V}d\theta}$$

We insert dt in dV , we find:

$$VdV = g\cos\theta R d\theta \rightarrow \int VdV = \int gR\cos\theta d\theta$$

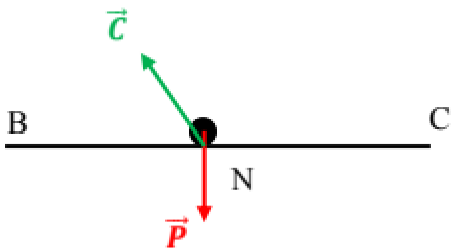
$$\frac{1}{2}(V^2 - V_A^2) = gR\ \mathrm{sins}\ \theta \rightarrow \boldsymbol{V = \sqrt{2gR(sin\theta) + V_A^2}}$$

c) Derive the expression for the speed at point B.

On point B we have $\theta = \frac{\pi}{2}$, hence:

$$V_B = \sqrt{2gR\sin\frac{\pi}{2} + V_A^2} \rightarrow \boldsymbol{V_B = \sqrt{2gR + V_A^2}}$$

2. a) Qualitative representation of the forces applied for m on point M on the BC part.



b) To find the expression for the acceleration of the mass m on point M, we have the fundamental relation of the dynamics given by:

$$\vec{P} + \vec{C} = m\vec{a}$$

By projection on the axis parallel and perpendicular to the movement, we find:

$$C_1 - P = 0 - C_{//} = ma \rightarrow \begin{cases} C_{//} = -ma \\ c_\perp = P \end{cases}$$

We also have:

$$\mu_{ai} = \frac{C_{//}}{c_\perp} = \frac{-ma}{mg} = \frac{-a}{g} \rightarrow \boldsymbol{a = -g\mu_{d1}}$$

c) We put BC = R , deduce the expression for the speed C on the path.

We have: $a = \frac{dV}{dL} \rightarrow dV = adt \rightarrow dV = -g\mu_{d1}dr$

Knowing that:

$$dx = Vdt \rightarrow dt = \frac{dx}{V}$$

We replace dt in the expression of dV, and BC = R , we find:

$$VdV = -g\mu_{d1}dx \rightarrow \frac{1}{2}(V_c^2 - V_B^2) = -g\mu_{d1}(BC)$$

$$V_c^2 = -2g\mu_{d1}R + V_B^2$$

We have :

$$V_B = \sqrt{2gR + V_A^2} \rightarrow \boldsymbol{V_C = \sqrt{2gR(1-\mu_{d1}) + V_A^2}}$$

3. We give: $g = 10\text{m/s}^2$, $\alpha = 20^o$, $\mu_{d2} = 0{,}5$ and $CD = R$.

a) To calculate the acceleration of m on the inclined part CD, we have:

$$\sum \vec{F} = m\vec{a} \ \rightarrow \ \vec{P} + \vec{C} = m\vec{a}$$

By projection on the axis parallel and perpendicular to the movement, we find:

$$-C_{\parallel} - mg\sin\alpha = ma \rightarrow \ C_{\parallel} = -mg\sin\alpha - ma$$

$$mg\cos\alpha - c_{\perp} = 0 \rightarrow c_{\perp} = mg\cos\alpha$$

With:

$$\mu_{d2} = \frac{C_{//}}{C_1} \rightarrow \mu_{d2} = \frac{-mg\sin\alpha - ma}{mg\cos\alpha}$$

$$\boldsymbol{a = -g(\mu_{cl2}\cos\alpha + \ \sin\alpha)\ ,\ a = -9.81\text{m}/s^2}$$

c) To find the value of μ_{d1} so that the masss'stop at the point D with initial speed $V_A = 0$, we use the previous question.

The acceleration is a negative constant, so the mobile makes a uniformly decelerated rectilinear motion, so we can write:

$$V_D^2 - V_C^2 = 2Ra$$

According to the expression of V_c found previously, we find:

$$V_D^2 = 2gR(1-\mu_{d1}) + V_A^2 = 2Ra \quad \rightarrow \quad V_D = \sqrt{2gR(1-\mu_{d1}) + 2Ra + V_A^2}$$

The masss'stop at the point D SO $V_D = 0$: , hence :

$$2gR(1-\mu_{d1}) + 2Ra + V_A^2 = 0$$

$$V_A = 0 \rightarrow \mu_{d1} = \frac{a+g}{g} \rightarrow \ \boldsymbol{\mu_{d1} = 0{,}19}$$

c) When the mass **m** reached the point D and stops, is-whether it goes down or not, the answer is obtained from the coefficient of friction.

We know that the value of the static friction coefficient is greater than the value of the dynamic coefficient:

$\mu_{s2} \geq \mu_{d2}$

Let us take as a minimum value of the coefficient of static friction:

$\mu_{s2} = \mu_{d2} = 0.5$

This corresponds to a minimum inclination of the CD part such that:

$\tan \alpha_{min} = 0.5 \rightarrow \boldsymbol{\alpha_{min} = 26.56^o}$

The inclination of the CD part being equal to 20^o, the mass m can't come down.

<u>2nd method</u>

If the mobile does not descend then the mobile is in a static state:

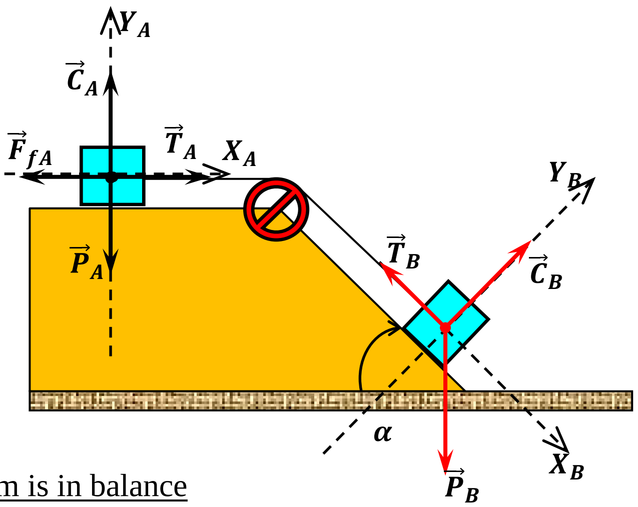
$$\vec{P} + \vec{C} = 0 \rightarrow \begin{cases} -C_{//} + mg \sin\alpha = 0 \rightarrow C_{\parallel} = mg \sin\alpha \\ mg \cos\alpha - c_{\perp} = 0 \rightarrow C_1 = mg\cos\alpha \end{cases}$$

We have:

$$\mu_{s2} = \frac{C_{//}}{C_1} = \frac{\sin\alpha}{\cos\alpha} \rightarrow \mu_{s2} = \tan 20 = 0.36$$

We have $\mu_{d2} = 0.5$, so $\mu_{s2} < \mu_{d2}$, which is not correct. So for the mobile to descend, an angle greater than 20^o. Mass cannot descend at an angle of 20^o.

<u>Exercise n°09:</u>



3. **<u>Static case:</u>** <u>the system is in balance</u>

Fundamental Principle of Dynamics: $\sum \vec{f} = m.\vec{a} = \vec{0}$

$$\begin{cases} (m_A) \quad \vec{P}_A + \vec{C}_A + \vec{T}_A + \vec{F}_{fA} = m_A.\vec{a}_A = \vec{0} \\ (m_B) \quad \vec{P}_B + \vec{C}_B + \vec{T}_B = m_B.\vec{a}_B = \vec{0} \end{cases}$$

Projection on:

$$\begin{matrix} O_AX_A \\ O_AY_A \end{matrix} \quad \begin{cases} T_A - F_{fA} = 0 \\ -P_A + C_A = 0 \end{cases} \quad \text{with} \quad F_{fA} = \mu_s . C_A$$

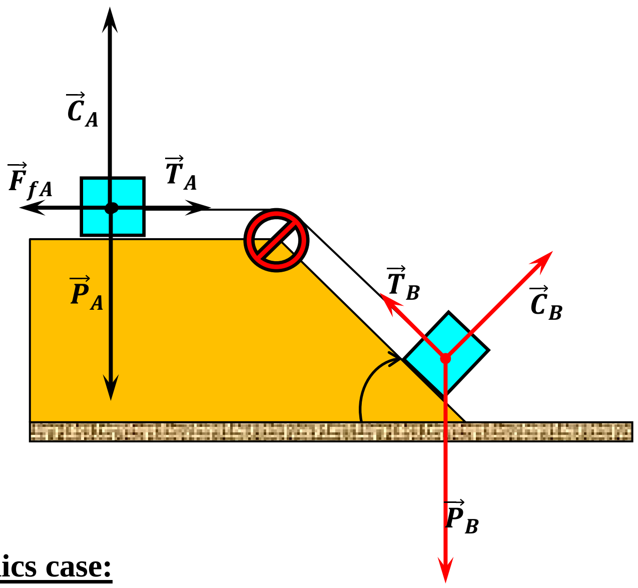
$$\begin{matrix} O_BX_B \\ O_BY_B \end{matrix} \quad \begin{cases} -T_B + P_B . \sin\alpha = 0 \\ -P_B . \cos\alpha + C_B = 0 \end{cases}$$

The wire and the pulley having negligible masses:$T_A = T_B = T$

By adding the first and third equations and replacing F_{fA}, it turns out that:

$$P_B . \sin\alpha = \mu_s . P_A$$

$$\Rightarrow m_{B\min} . \sin\alpha = \mu_s . m_A \qquad \text{and} \qquad \boxed{m_{B\min} = \frac{\mu_s . m_A}{\sin\alpha}}$$



4. **<u>Dynamics case:</u>**

a. Fundamental principle of dynamics.$\sum \vec{f} = m.\vec{a}$

$$\begin{cases} (m_A) \quad \vec{P}_A + \vec{C}_A + \vec{T}_A + \vec{F}_{fA} = m_A . \vec{a}_A \\ (m_B) \quad \vec{P}_B + \vec{C}_B + \vec{T}_B = m_B . \vec{a}_B \end{cases}$$

Screening on:

$$\begin{matrix} O_AX_A \\ O_AY_A \end{matrix} \quad \begin{cases} T_A - F_{fA} = m_A . a_A \\ -P_A + C_A = 0 \end{cases} \quad \text{avec} \quad F_{fA} = \mu_d . C_A$$

$$\begin{matrix} O_BX_B \\ O_BY_B \end{matrix} \quad \begin{cases} -T_B + P_B . \sin\alpha = m_B . a_B \\ -P_B . \cos\alpha + C_B = 0 \end{cases}$$

The wire and the pulley having negligible masses:

$T_A = T_B = T$

The wire being inextensible: (is the acceleration of the first phase)$a_A = a_B = a_I a_I$

Forces magnitude:

$C_A = P_A = m_A . g = 60\ N \ ; \ P_B = m_B . g = 40\ N \ ; \ C_B = m_B . g . \cos\alpha = 20\ N$

b. Expression of acceleration.a_I

By adding the first and third equations and replacingF_{fA}, it turns out that:

$$P_B . \sin\alpha - \mu_d . P_A = (m_A + m_B) . a_I$$

SO

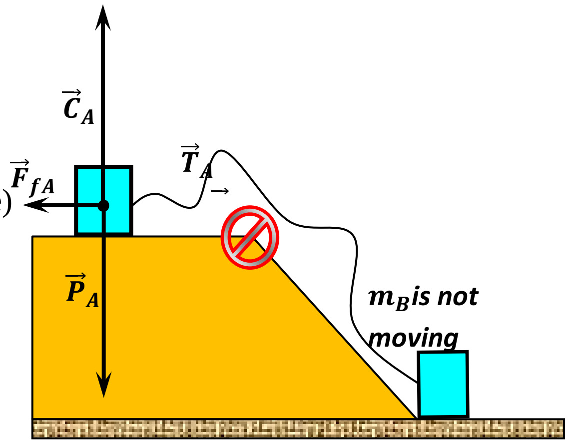
$$a_I = g\frac{m_B.\sin\alpha - \mu_d.m_A}{m_A + m_B}$$

$a_I$is constant, hence the motion is **uniformly varied rectilinear** both masses.

c. *Phase II*: only the mass is in motion.m_A

$\vec{P}_A + \vec{C}_A + \vec{F}_{fA} = m_A.\vec{a}_A$

Screening on:



$$\begin{matrix} O_AX_A \\ O_AY_A \end{matrix} \quad \begin{cases} -F_{fA} = m_A.a_A \\ -P_A + C_A = 0 \end{cases} \quad \text{avec} \quad F_{fA} = \mu_d.C_A$$

$a_A = a_{II}$ ($a_{II}$is the acceleration of the second phase)

Forces magnitude:$C_A = P_A = m_A.g = 60\ N$

d. Using the two projection equations:

$$-F_{fA} = -\mu_d.m_A.g = m_A.a_A$$

And

$$a_{II} = -\mu_d.g$$

$a_{II}$is constant, hence the motion is **uniformly varied rectilinear** for the mass m_A.

e. Dynamic friction coefficient.

Phase I: uniformly varied rectilinear movement

$$V_{If}^2 - V_{0I}^2 = 2a_I.(x_{If} - x_{0I}) \qquad \text{avec} \quad V_{0I} = 0 \quad \text{et} \quad x_{If} - x_{0I} = l$$

Phase II: uniformly varied rectilinear motion

$$V_{IIf}^2 - V_{0II}^2 = 2a_{II}.(x_{IIf} - x_{0II}) \qquad \text{avec} \quad V_{0II} = V_{If}\ ,\ V_{IIf} = 0 \quad \text{et} \quad x_{IIf} - x_{0II} = d$$

Hence the two equations

$$V_{If}^2 = 2a_I.l \qquad \text{et} \qquad -V_{If}^2 = 2a_{II}.d$$

By replacing the accelerations

$$2\left(g\frac{m_B.\sin\alpha - \mu_d.m_A}{m_A + m_B}\right).l = -2(-\mu_d.g).d$$

Eventually

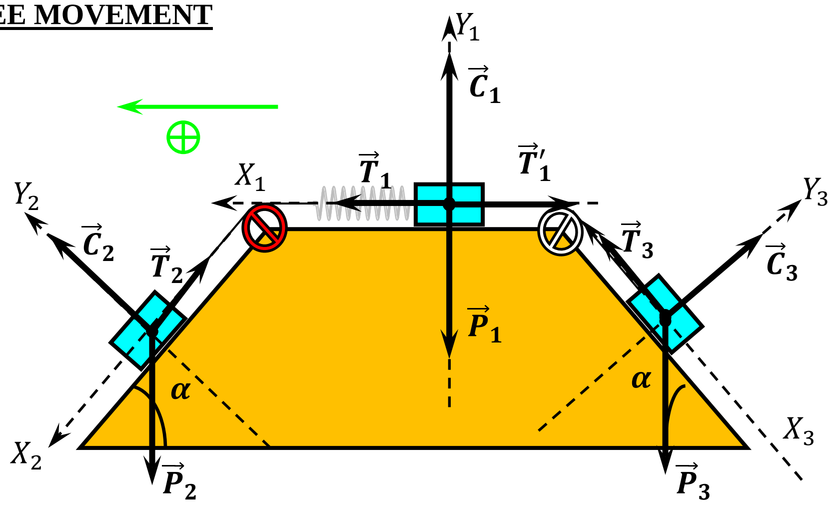
$$\mu_d = \frac{m_B.l.\sin\alpha}{m_A.l + (m_A + m_B).d}$$

f. Numerical calculation:

$$m_{B\min} = 3{,}464\ kg \quad \text{and} \quad \mu_d = 0{,}468$$

Exercise n°10:

FRICTION-FREE MOVEMENT



1. Fundamental Principle of Dynamics: $\sum \vec{f} = m.\vec{a}$

$$(m_1) \qquad \vec{P}_1 + \vec{C}_1 + \vec{T}_1 + \vec{T}'_1 = m_1.\vec{a}_1$$
$$(m_2) \qquad \vec{P}_2 + \vec{C}_2 + \vec{T}_2 = m_2.\vec{a}_2$$
$$(m_3) \qquad \vec{P}_3 + \vec{C}_3 + \vec{T}_3 = m_3.\vec{a}_3$$

Screening on:

$$\begin{matrix} O_1X_1 \\ O_1Y_1 \end{matrix} \begin{cases} T_1 - T'_1 = m_1.a_1 \\ -P_1 + C_1 = 0 \end{cases} \quad ; \quad \begin{matrix} O_2X_2 \\ O_2Y_2 \end{matrix} \begin{cases} P_2.\sin\alpha - T_2 = m_2.a_2 \\ -P_2.\cos\alpha + C_2 = 0 \end{cases} \quad ;$$

$$\begin{matrix} O_3X_3 \\ O_3Y_3 \end{matrix} \begin{cases} T_3 - P_3.\sin\alpha = m_3.a_3 \\ -P_3.\cos\alpha + C_3 = 0 \end{cases}$$

The wires, pulleys and spring having negligible masses: $T_1 = T_2 = T \quad et \quad T'_1 = T_3 = T'$

The elongation of the spring being constant and the wires inextensible: $a_1 = a_2 = a_3 = a$

$$T - T' = m_1.a \quad \ldots\ldots\ldots\ldots\ldots \ (1) \qquad C_1 = m_1 g \quad \ldots\ldots\ldots\ldots \ (2)$$
$$m_2 g.\sin\alpha - T = m_2.a \quad \ldots\ldots\ldots\ldots \ (3) \qquad C_2 = m_2 g.\cos\alpha \quad \ldots\ldots\ldots\ldots \ (4)$$
$$-m_3 g.\sin\alpha + T' = m_3.a \quad \ldots\ldots\ldots \ (5) \qquad C_3 = m_3 g.\cos\alpha \quad \ldots\ldots\ldots\ldots \ (6)$$

By adding up: we have: $(1) + (3) + (5)$

$$(m_2 - m_3)g.\sin\alpha = (m_1 + m_2 + m_3).a$$

$$\Rightarrow a = \frac{m_2 - m_3}{m_1 + m_2 + m_3} g.\sin\alpha$$

For $m_2 = 2.m \ ; \ m_3 = m_1 = m$

$$\boxed{\Rightarrow a = \frac{1}{4} g.\sin\alpha}$$

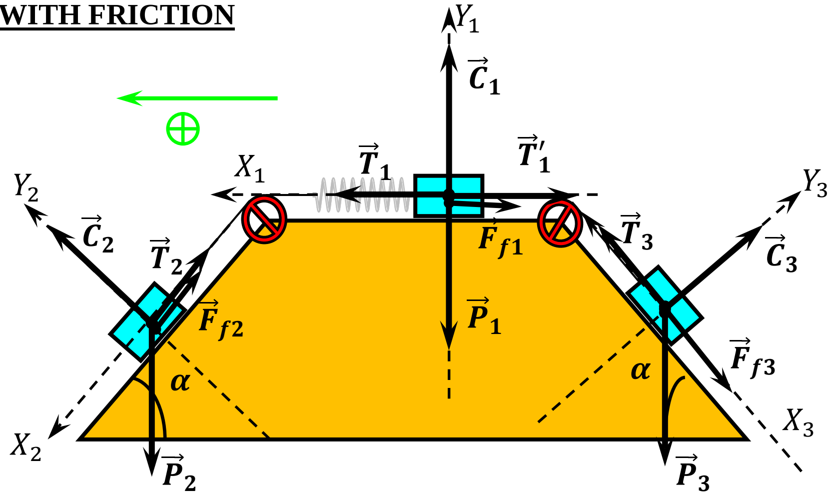
$a > 0$ The system moves towards the positive direction shown in the figure.

2. From the equation (3):

$$T = k.\Delta l = m_2 g.\sin\alpha - m_2.a$$

$$\Rightarrow \boxed{\Delta l = \frac{3m}{2k} g.\sin\alpha}$$

MOVEMENT WITH FRICTION



3. PFD: $\sum \vec{f} = m.\vec{a}$

(m_1) $\quad \vec{P}_1 + \vec{C}_1 + \vec{T}_1 + \vec{T}'_1 + \vec{F}_{f1} = m_1.\vec{a}_1$

(m_2) $\quad \vec{P}_2 + \vec{C}_2 + \vec{T}_2 + \vec{F}_{f2} = m_2.\vec{a}_2$

(m_3) $\quad \vec{P}_3 + \vec{C}_3 + \vec{T}_3 + \vec{F}_{f3} = m_3.\vec{a}_3$

Screening on:

$$\begin{matrix} O_1X_1 \\ O_1Y_1 \end{matrix} \begin{cases} T_1 - T'_1 - F_{f1} = m_1.a_1 \\ -P_1 + C_1 = 0 \end{cases} \quad ; \quad \begin{matrix} O_2X_2 \\ O_2Y_2 \end{matrix} \begin{cases} P_2.\sin\alpha - T_2 - F_{f2} = m_2.a_2 \\ -P_2.\cos\alpha + C_2 = 0 \end{cases}$$

$$\begin{matrix} O_3X_3 \\ O_3Y_3 \end{matrix} \begin{cases} T_3 - P_3.\sin\alpha - F_{f3} = m_3.a_3 \\ -P_3.\cos\alpha + C_3 = 0 \end{cases}$$

With : $\boxed{F_f = \mu.C}$

$T - T' - \mu.C_1 = m_1.a$ .... (1) $\qquad C_1 = m_1 g$ .... (2)

$m_2 g.\sin\alpha - T - \mu.C_2 = m_2.a$ .... (3) $\qquad C_2 = m_2 g.\cos\alpha$ .... (4)

$-m_3 g.\sin\alpha + T' - \mu.C_3 = m_3.a$ .... (5) $\qquad C_3 = m_3 g.\cos\alpha$ .... (6)

By adding up: we have: $(1) + (3) + (5)$

$$(m_2 - m_3)g.\sin\alpha - \mu.(C_1 + C_2 + C_3) = (m_1 + m_2 + m_3).a$$

And replacing $(2), (4)$ and (6) with $m_2 = 2.m\ ;\ m_3 = m_1 = m$

$$mg.\sin\alpha - \mu.(3mg.\cos\alpha + mg) = 4m.a$$

$$\Rightarrow \boxed{a = \frac{1}{4} g(\sin\alpha - 3\mu\cos\alpha - \mu)}$$

4. From the equation: (3)

$$T = k.\Delta l = m_2 g.\sin\alpha - \mu.m_2 g.\cos\alpha - m_2.a$$

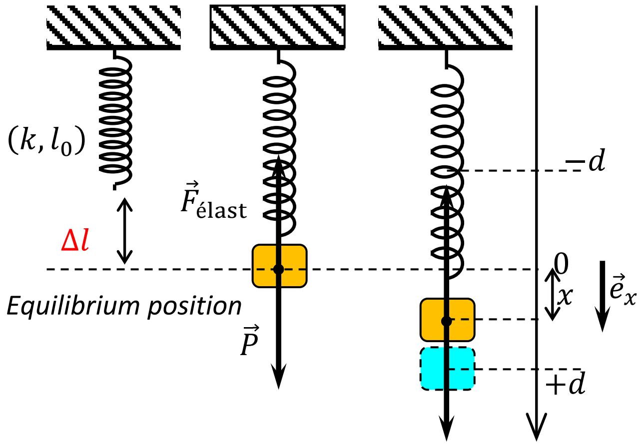
$$\Rightarrow k.\Delta l = \frac{mg}{2}(3.\sin\alpha - \mu.\cos\alpha + \mu) \quad \Rightarrow \quad \boxed{\Delta l = \frac{mg}{2k}(3.\sin\alpha - \mu.\cos\alpha + \mu)}$$

Numerical calculation:

Frictionless: $\boxed{a = 2{,}165\ m/s^2}$ and $\boxed{\Delta l = 0{,}1299\ m = 12{,}99\ cm}$

With friction: $\boxed{a = 1{,}54\ \ m/s^2}$ and $\boxed{\Delta l = 0{,}1324\ m = 13{,}24\ cm}$

Exercise n°11:



Calculation of elongation:Δl

By Applying the PFD to Equilibrium

$$\sum \vec{f} = \vec{P} + \vec{F}_{\text{élast}} = m.\vec{a} = \vec{0}$$

With $\vec{F}_{\text{élast}} = -k.\Delta\vec{l}$

and its magnitude $F_{\text{élast}} = k.\Delta l$

By projecting onto the axis:(OX)

$$m.g - k.\Delta l = 0$$

$$\Rightarrow \boxed{\Delta l = \frac{m.g}{k}}$$

1. Fundamental Principle of Dynamics:$\sum \vec{f} = \vec{P} + \vec{F}_{\text{élast}} = m.\vec{a}$

By projecting onto the axis:(OX)

$$m.g - k.(\Delta l + x) = m.a$$

As

$$m.g - k.\Delta l = 0$$

SO

$$\boxed{a = -\frac{k}{m}x}$$

It is a 2nd degree differential equation.

2. Solution of the form:$x(t) = A.\sin(\omega.t) + B.\cos(\omega.t)$

By deriving:

$$\begin{cases} x(t) = A.\sin(\omega.t) + B.\cos(\omega.t) \\ V = x^{\bullet} = \omega A.\cos(\omega.t) - \omega B.\sin(\omega.t) \\ a = x^{\bullet\bullet} = -\omega^2 A.\sin(\omega.t) - \omega^2 B.\cos(\omega.t) = -\omega^2.x \end{cases}$$

Hence by identifying we have:

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

Initial conditions: $t = 0$, $x = d$ and $V_0 = 0$ $\begin{cases} d = B \\ 0 = \omega A \end{cases}$

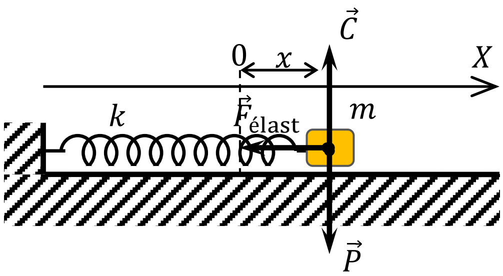
By replacing in $x(t)$ we have:

$$\boxed{x(t) = d.\cos(\omega.t)}$$

3. Speed and acceleration:

$$\begin{cases} V = x^{\bullet} = -\omega d.\sin(\omega.t) \\ a = x^{\bullet\bullet} = -\omega^2 d.\cos(\omega.t) \end{cases}$$

	$\omega.t = n.\pi \;\; \Rightarrow \;\; t = 2n\left(\frac{\pi}{2\omega}\right)$	$\omega.t = (2n+1)\frac{\pi}{2} \;\; \Rightarrow \;\; t = (2n+1)\left(\frac{\pi}{2\omega}\right)$
$x(t)$	$\lvert x(t)\rvert = x_{\text{Max}} = d$	$x(t) = 0$
$V(t)$	$V(t) = 0$	$\lvert V(t)\rvert = V_{\text{Max}} = \omega.d$
$a(t)$	$\lvert a(t)\rvert = a_{\text{Max}} = \omega^2 d$	$a(t) = 0$



Horizontal mass-spring system:

Fundamental Principle of Dynamics:

$$\sum \vec{f} = \vec{P} + \vec{F}_{\text{élast}} = m.\vec{a}$$

By projecting on to the axis(OX):

$-k.x = m.a$ and $\boxed{a = -\frac{k}{m}x}$

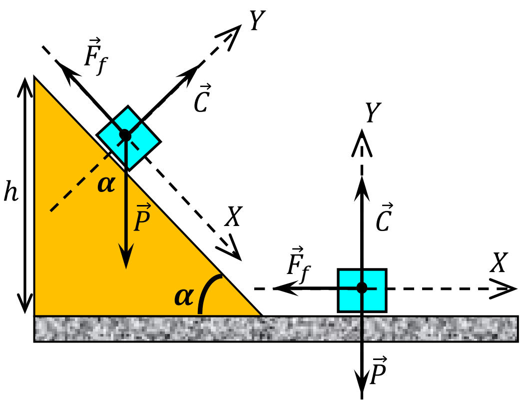
We find the same differential equation of the 2nd degree. The solution is of the same form as what preceded.

Numerical calculation: $k = 10 \; N/m \, ; \; l = 0{,}2 \, m \, ; \;\; m = 50 \, g \, ; \; d = 3 \, cm$

$\Delta l = 0{,}04905 \, m$; $\omega = 14{,}142 \; rad/s$; $B = d = 0{,}03 \, m$

$V_{\text{Max}} = 0{,}424 \; m/s$; $a_{\text{Max}} = 6 \; m/s^2$

Exercise n°12:



1. Fundamental Principle of Dynamics: Static Case.

System in equilibrium:$\sum \vec{f} = \vec{P} + \vec{C} + \vec{F}_f = \vec{0}$

Projection

on $\begin{matrix} OX \\ OY \end{matrix}$ $\begin{cases} +P.\sin\alpha - F_f = 0 & \ldots\ldots\ldots(1) \\ -P.\cos\alpha + C = 0 & \ldots\ldots\ldots(2) \end{cases}$

In the equation (1) we have: , therefore $F_f = \mu_{\text{lim}}.C$

$$\begin{cases} mg.\sin\alpha = \mu_{\text{lim}}.C \\ mg.\cos\alpha = C \end{cases}$$

By dividing: $\boxed{\mu_{\text{lim}} = \tan\alpha}$

2. Dynamic case:

$\sum \vec{f} = \vec{P} + \vec{C} + \vec{F}_f = m.\vec{a}$ Projection on $\begin{matrix} OX \\ OY \end{matrix}$ $\begin{cases} +P.\sin\alpha - F_f = m.a & \ldots\ldots\ldots(3) \\ -P.\cos\alpha + C = 0 & \ldots\ldots\ldots(4) \end{cases}$

From the equilibrium equation(4) we have: $mg.\cos\alpha = C$
SO

$$\boxed{a = g.(\sin\alpha - \mu.\cos\alpha)}$$

3. $a =$ Constante then the rectilinear movement is uniformly accelerated.

4. Distance calculation:$AB \sin\alpha = \frac{h}{AB}$ $\Rightarrow$ $AB = \frac{h}{\sin\alpha}$

Uniformly accelerated rectilinear motion

$$V_B^2 - V_A^2 = 2a.AB$$

$$V_A = 0 \quad \Rightarrow \quad V_B^2 = 2g.(\sin\alpha - \mu.\cos\alpha)\frac{h}{\sin\alpha}$$

And

$$\boxed{V_B = \sqrt{2gh.(1 - \mu/\tan\alpha)}}$$

5. Uniformly accelerated rectilinear motion

$$V(t) = a.t + V_0 \quad \text{avec} \quad V_0 = V_A = 0$$

From where

$$t_B = \frac{V_B}{a} = \frac{\sqrt{2a.AB}}{a} \quad \Rightarrow \quad \boxed{t_B = \sqrt{\frac{2h}{g.\sin\alpha.(\sin\alpha - \mu.\cos\alpha)}}}$$

6. Fundamental Principle of Dynamics:

$$\sum \vec{f} = \vec{P} + \vec{C} + \vec{F_f} = m.\vec{a}'$$

$$\text{Projection on}\begin{matrix} OX \\ OY \end{matrix} \quad \begin{cases} -F_f = -\mu.C = m.a' \\ -P + C = 0 \end{cases} \quad \Rightarrow \quad a' = -\mu.g$$

Uniformly accelerated rectilinear motion

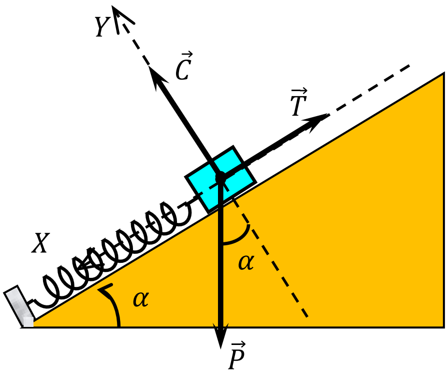
$$V_C^2 - V_B^2 = 2a'.d$$

$$V_C = 0 \quad \Rightarrow \quad d = \frac{-V_B^2}{-2\mu.g} \quad \text{et} \quad \boxed{d = h\left(\frac{1}{\mu} - \frac{1}{\tan\alpha}\right)}$$

7. <u>Numerical calculation</u>:

$\boxed{\mu_{\lim} = 1/\sqrt{3} = 0{,}577}$; $\boxed{a = 3{,}268 \ \ m/s^2}$; $\boxed{V_B = 5{,}113 \ \ m/s}$; $\boxed{t_B = 1{,}565 \ \ s}$; $\boxed{d = 6{,}536 \ \ m}$

<u>Exercise n°13:</u>



1. Fundamental Principle of Dynamics.

System in equilibrium:$\sum \vec{f} = \vec{P} + \vec{C} + \vec{T} = \vec{0}$

Projection

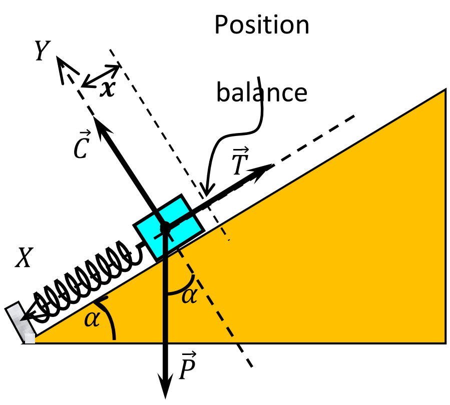
$$\text{on}\begin{matrix} OX \\ OY \end{matrix} \quad \begin{cases} +P.\sin\alpha - T = 0 & \dots\dots\dots(1) \\ -P.\cos\alpha + C = 0 & \dots\dots\dots(2) \end{cases}$$

From the equation (1) we have

:$T = k.\Delta l = mg.\sin\alpha$, and $\Delta l = \frac{mg.\sin\alpha}{k}$

With $\Delta l = l_0 - l$ spring compression,

$$\Rightarrow \boxed{l = l_0 - \frac{mg.\sin\alpha}{k}}$$



2. Dynamic case:

$$\sum \vec{f} = \vec{P} + \vec{C} + \vec{T} = m.\vec{a}$$

Projection

$$\text{on}\begin{matrix} OX \\ OY \end{matrix} \quad \begin{cases} +P.\sin\alpha - T = m.a \\ -P.\cos\alpha + C = 0 \end{cases} \quad \ldots\ldots\ldots.(3)$$

With $T = k.(\Delta l + x)$:

From the equilibrium equation(1) we have: $mg.\sin\alpha - k.\Delta l = 0$

$$\Rightarrow +mg.\sin\alpha - k.(\Delta l + x) = m.a \qquad \Rightarrow \qquad -k.x = m.a$$

And the differential equation is given by

$$\boxed{a = x^{\bullet\bullet} = -\frac{k}{m}x}$$

3. The solution to the 2nd degree differential equation is of the form:

$$\begin{cases} x(t) = A.\sin(\omega.t + \varphi) \\ V(t) = x^{\bullet} = A\omega.\cos(\omega.t + \varphi) \\ a(t) = x^{\bullet\bullet} = -A\omega^2.\sin(\omega.t + \varphi) \end{cases} \quad \ldots\ldots\ldots \quad (1)$$

$$\boxed{a = x^{\bullet\bullet} = -\omega^2.x}$$

Comparing with the differential equation, we find:

$$\omega^2 = k/m \qquad \Rightarrow \qquad \boxed{\omega = \sqrt{k/m}}$$

We calculate A and φ using the initial conditions:

At; $t = 0\,s x = 0$(equilibrium) $\Rightarrow \quad A.\sin(\varphi) = 0 \quad \Rightarrow \quad \sin(\varphi) = 0$

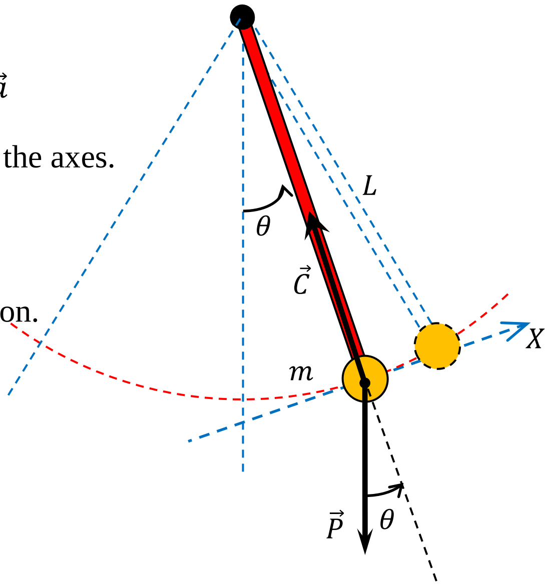
$$\boxed{\varphi = 0} \quad \text{ou} \quad \boxed{\varphi = \pi}$$

At; $t = 0s\; V = V_0 \Rightarrow \quad A\omega.\cos(\varphi) = V_0$

$$\boxed{\begin{cases} \varphi = 0 \quad \Rightarrow \quad A = V_0/\omega = V_0\sqrt{m/k} \\ \varphi = \pi \quad \Rightarrow \quad A = -V_0/\omega = -V_0\sqrt{m/k} \end{cases}}$$

By replacing in the equation(1):
$$\boxed{\begin{cases} x(t) = V_0\sqrt{m/k}.\sin\left(\sqrt{k/m}.t\right) \\ V(t) = V_0.\cos\left(\sqrt{k/m}.t\right) \\ a(t) = -V_0\sqrt{k/m}.\sin\left(\sqrt{k/m}.t\right) \end{cases}}$$

Exercise n°14:



Fundamental Principle of dynamics:

$$\sum \vec{f} = \vec{P} + \vec{C} = m.\vec{a}$$

By projecting normal and tangential accelerations along the axes.

$$\begin{matrix} \text{axe} \quad (OX) \\ \text{axe} \quad (OY) \end{matrix} \begin{cases} -mg.\sin\theta = m.a_T = m.L.\theta^{\bullet\bullet} \\ -mg.\cos\theta + C = m.a_N = m.L.\theta^{\bullet 2} \end{cases}$$

The first equation gives the differential equation of motion.

$$\theta^{\bullet\bullet} = -\frac{g}{L}\sin\theta$$

In the case of small angles($\theta \ll$)we have andsin $\theta \approx \theta$

$$\boxed{\theta^{\bullet\bullet} = -\frac{g}{L}\theta}$$

1. The solution to the previous differential equation is of the form:

$$\begin{cases} \theta(t) = \theta_0.\sin(\omega.t + \varphi) \\ \theta^{\bullet}(t) = \omega.\theta_0.\cos(\omega.t + \varphi) \\ \theta^{\bullet\bullet}(t) = -\omega^2.\theta_0.\sin(\omega.t + \varphi) \end{cases} \qquad \text{avec} \qquad \boxed{\omega = \sqrt{\frac{g}{L}}}$$

Period

$$\boxed{\omega = \sqrt{\frac{g}{L}}} \qquad \Rightarrow \qquad \boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}}$$

Initial conditions($t = 0 \quad , \quad \theta(0) = 0$) equilibrium position.($t = 0 \quad , \quad \theta^{\bullet}(0) = V_0/L$)

Replacing :

$$0 = \theta_0.\sin(\varphi) \quad \Rightarrow \quad \varphi = 0 \quad \text{ou} \quad \varphi = \pi$$

$$\frac{V_0}{L} = \omega.\theta_0.\cos(\varphi) = \sqrt{\frac{g}{L}}.\theta_0.\cos(\varphi)$$

$$\Rightarrow \quad \boxed{\begin{cases} \varphi = 0 \quad \Rightarrow \quad \theta_0 = V_0/\omega L = V_0/\sqrt{gL} \\ \varphi = \pi \quad \Rightarrow \quad \theta_0 = -V_0/\omega L = -V_0/\sqrt{gL} \end{cases}}$$

Both scenarios giving the same solution

$$\boxed{\theta(t) = \frac{V_0}{\sqrt{gL}}.\sin\left(\sqrt{\frac{g}{L}}.t\right)}$$

$$\boxed{\theta^{\bullet}(t) = \frac{V_0}{L}.\cos\left(\sqrt{\frac{g}{L}}.t\right)} \qquad \text{and} \qquad \boxed{\theta^{\bullet\bullet}(t) = -\sqrt{\frac{g}{L}}\frac{V_0}{L}.\sin\left(\sqrt{\frac{g}{L}}.t\right)}$$

	$\omega.t = (2n+1)\frac{\pi}{2} \quad \Rightarrow \quad \boxed{t = (2n+1)\left(\frac{\pi}{2\omega}\right)}$	$\omega.t = n.\pi \quad \Rightarrow \quad \boxed{t = 2n\left(\frac{\pi}{2\omega}\right)}$
$\theta^{\bullet}(t)$	$\theta^{\bullet}(t) = 0$	$\lvert\theta^{\bullet}(t)\rvert = \theta^{\bullet}_{\max} = V_0/L$
$\theta^{\bullet\bullet}(t)$	$\lvert\theta^{\bullet\bullet}(t)\rvert = \theta^{\bullet\bullet}_{\max} = V_0\sqrt{g}/L\sqrt{L}$	$\theta^{\bullet\bullet}(t) = 0$

2. Angular momentum theorem:

$$\vec{M}(\vec{C}) + \vec{M}(\vec{P}) = m.L^2\theta^{\bullet\bullet}.\vec{e}_z$$

As we have: $\vec{C} \parallel \vec{r} \quad \Rightarrow \quad \vec{M}(\vec{C}) = \vec{r} \times \vec{C} = \vec{0}$

$$\Rightarrow \vec{r} \times \vec{P} = -mg.L.\sin\theta\,.\vec{e}_z = m.L^2\theta^{\bullet\bullet}.\vec{e}_z$$

We find the same differential equation

$$\theta^{\bullet\bullet} = -\frac{g}{L}\sin\theta \qquad \text{and for } (\theta \ll) \text{ on a} \qquad \boxed{\theta^{\bullet\bullet} = -\frac{g}{L}\theta}$$

8. **<u>Numerical calculation</u>**: $L = 19\ m\ ,\ V_0 = 0{,}5\ m/s$

$$\boxed{\omega = 0{,}719\ \ rad/s} \quad ; \quad \boxed{T = 8{,}744\ \ s} \ ; \ \boxed{\theta_0 = 0{,}0366\ rad \approx 2°}$$

$$\boxed{\theta^{\bullet}_{\max} = 0{,}026\ \ rad/s} \quad ; \quad \boxed{\theta^{\bullet\bullet}_{\max} = 0{,}019\ \ rad/s^2}$$

Chapter6

WORK AND ENERGY

Learning Goals: After going through this chapter, students will be able to

- Introduce the concepts of work, power and energy.
- Determine kinetic energy, potential energy and mechanical energy.
- Apply the kinetic energy theorem.
- Understand and apply the principle of conservation of mechanical energy.
- Define the condition of stable equilibrium of a physical system.

Exercise n°01:

A particle of mass m moves under the action of the force: $\vec{F} = x^2\vec{\imath} + 2xy\vec{\jmath}$ from the origin $0(0,0)$ to the point $B(2,3)$ passing through the point $A(2,0)$.

1) Give the expression for elementary work dw.

2) Calculate the work of the force $\vec{F}$ when the particle moves from point 0 to point A.

3) Calculate the work of the force $\vec{F}$ when the particle moves from A point to point B.

4) Calculate the work of the force $\vec{F}$ when the particle moves directly from point 0 to B.

5) The force $\vec{F}$ is conservative? Justify.

Exercise n°02:

A body is subjected to a force: $\vec{F} = (y^2 - x^2)\vec{\imath} + 3xy\vec{\jmath}$. Find the work of the force if the body moves from the point $A(0,0)$ to the point $(2,4)$ following the following trajectories:

a) On the axis$[Ox)$ from $A(0,0)$ to $C(2,0)$ and then parallel to$[Oy)$ from C to B.

b) On the axis$[Oy)$ from $A(0,0)$ to $D(0,4)$ and then parallel to$[Ox)$ from D to B.

c) On the straight line$[AB]$.

d) On the trajectory $y = x^2$

Exercise n°03:

Let a material point M be subject to a force field: $\vec{F} = (x - ay)\vec{\imath} + (3y - 2x)\vec{\jmath}$.Calculate the work of the force $\vec{F}$ for moving M from point $O(0,0)$ to point $A(2,4)$ passing through point $C(0,4)$.

2) Find the value of a such that $\vec{F}$ is conservative; deduce potential energy E_P resulting from this force field.

3) Determine the work of $\vec{F}$ for the displacement of M along a circular path of radius R and center $0(0,0)$.

Exercise n°04:

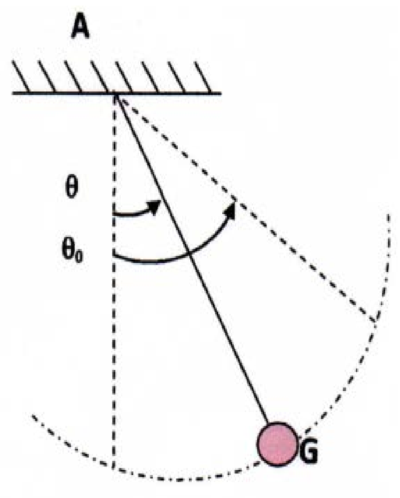
A body of mass m moves along an inclined plane making an angle α with the horizontal. To move from point A to point B we apply a force $\vec{F}$ such that the velocity of the body remains constant during the movement (friction is negligible).

1. Calculate the work of force $\vec{F}$, reaction $\vec{R}$ and weight $\vec{F}_G$
2. Find the variation in kinetic energy between A and B.

Exercise n°05:

Consider a simple pendulum AG (the mass m fixed at the end of an inextensible wire), suspended at point A (Fig.). We define its oscillations by $\theta(t)$. Find the differential equation of motion (without solving it) using:

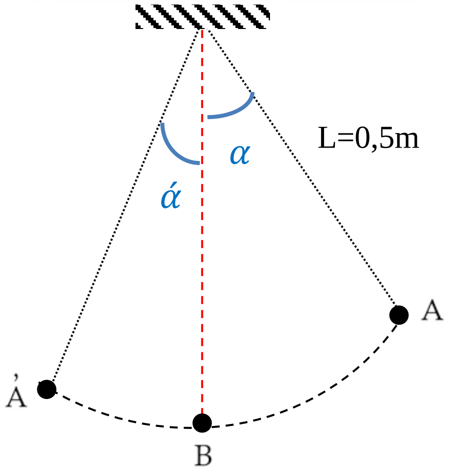
1. The fundamental principle of dynamics (FPD).
2. The angular momentum theorem.
3. The kinetic energy theorem.
4. The total mechanical energy theorem.



Exercise n°06:

A pendulum OB formed from a solid rod of negligible mass and length L carries at its end B a mass m=50g. The pendulum is launched from A (the rod makes an angle α=60° with the vertical) without initial velocity. Find the kinetic energy and velocity of the body when

1. The mass passes through the vertical at a point B.
2. The mass passes through A′ (the rod makes an angle α′ = 30° with the vertical on the opposite side to the starting one)



Exercise n°07:

A body of mass m = 2 kg is allowed to slide without initial velocity from point A on an inclined plane AB of slope α = 30°, of length AB = 20 m, and offering no friction.

1. Calculate the velocity of the body at point B.

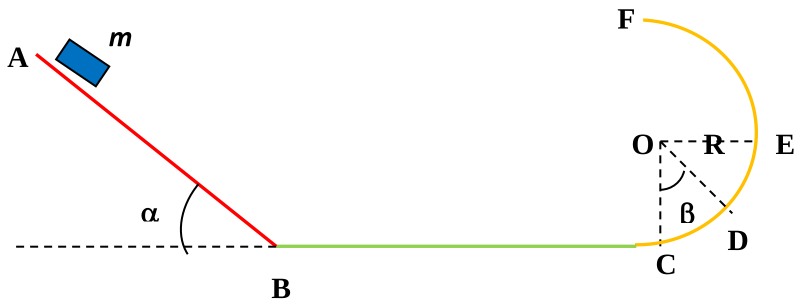
The body continues its motion on the horizontal plane BC, arriving at point C with a velocity VC = 10 m/s. knowing that the length of part BC = 5 m.

2. Calculate the value of the friction force – considered constant – applied to the body.

From point CEF the body follows a circular track of radius R, arriving at point E mid-height with a speed VD = 8 m/s, neglecting the friction of this section.

3. Calculate the radius of the circular part CEF.

4. Calculate the speed of the body at point D such that OD makes an angle β = 60° with the normal.
5. Calculate the force exerted by the track on the body at point D.
6. Calculate the speed of the body and the force exerted by the track on the body at the highest point F.



Exercise n°08:

A runway lying in a vertical plane consists of a circular section ABD, with center O and radius $R = 1m$, and a horizontal section DEF, shown in the figure below. A spring is attached to a wall at point F and E is its free end, resting on the horizontal part with stiffness constant $k = 100N/m$. Let's consider a body of mass m=1kg, which has to move on the track. B is a point located in the middle of the circular section ($\alpha = 45°$).

1) Friction is negligible over the entire track, so the mass m is released without any initial velocity initial speed from point A.

a) Calculate the velocity of mass m at point B.

b) Find the contact force C exerted by the track on m at point B.

c) Calculate the acceleration of m at point B.

d) Calculate the maximum compression of the spring when mass m presses it.

2) Friction is negligible on the ABD track, while the DEF section is characterized by static friction μ_s=0.2 and sliding friction μ_g=0.1. The mass m is released without initial velocity from point A.

- Calculate the maximum compression of the spring when the mass m hits it with DE=1m.

Exercise n°09:

A Block of mass $m = 2kg$ is dropped without initial velocity from the top of a curved track in the shape of a quarter circle AB of radius $R = 1m$. The block then slides along an inclined plane BC which makes an angle $\alpha = 45°$ with the horizontal (Figure blow). Friction along AB is negligible, while friction along BC is characterized by the dynamic friction coefficient $\mu_d = 0.5$. We take $g = 10m/s^2$.

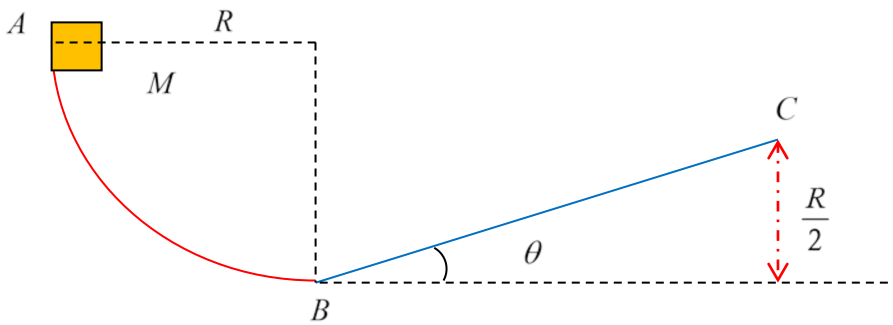
1) Calculate the variation in kinetic energy E_c between points A and B, and deduce the velocity V_B at point B.

2) Show the forces acting on mass m at point A.

3) As mass m continues to move along the inclined plane, represent the forces acting on it along BC, and then determine its acceleration a.

4) Calculate the work W of the friction force along BC and deduce the speed V_C of mass m at point C.

5) How long will it take the block m to cover the distance BC?



Exercise n°10:

A skier of mass m=80 kg glides along the beginning of a track made up of three parts AB, BC and CD.

- Part AB is a circular arc of radius $r = 5$m and center 0' such that $\widehat{AO'B} = \alpha = 60^o$
- BC is a horizontal straight section of length r.
- CD is a vertical quarter-circumference of radius r and center 0.

The entire trajectory is in the same vertical plane. The skier starts from A with no initial speed. To simplify calculations, his motion will be assimilated to that of a material point throughout the problem.

1. On a first attempt, the ABC track is icy. Friction is low enough to be neglected. Under these conditions, calculate the speeds v_B and v_C which the skier passes through B and C.

2. In another test, the ABC track is covered with snow. For simplicity's sake, we'll assume that the resultant of the friction forces, constantly tangent to the trajectory, maintains a constant magnitude f along the entire path ABC.

2.1. Express v_B as a function of m, r, f and g.

2.2 Express v_C as a function of m, r, f and v_B

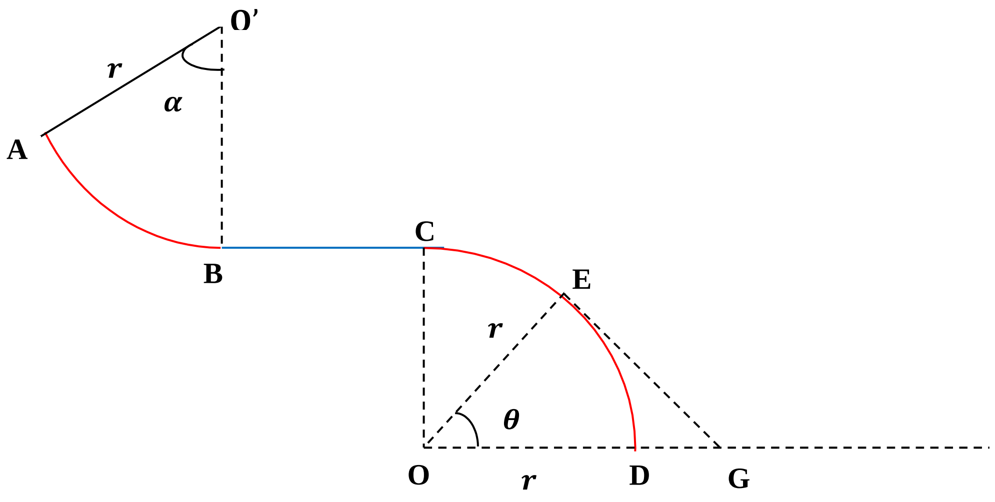
2.3 Calculate the intensity of the friction force if the skier arrives at C with zero speed.

3. The skier arrives at C with zero speed; he approaches the CD section, which is icy; friction will therefore be neglected.

3.1. The skier passes a point E on the track CD, defined by (OD,OE) $= \theta$; OD being carried by the horizontal. Express his speed v_B as a function of g, r and θ.

3.2. The skier leaves the track at E with speed $v_\mathrm{E} = 5{,}77\mathrm{m.s}^{1}$, calculate the value of angle θ

4. What is the skier's speed when they arrive at the landing slope, at a point G?



SOLUTIONS TO EXERCISES

Exercise n°01:

1) The expression of elementary work dw:

$\vec{F} = x^2\vec{\imath} + 2xy\vec{\jmath}$

$\overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath}$

$dw = \vec{F}\overrightarrow{dl} = x^2dx + 2xydy$

2) The work $W_O^A(\vec{F})$

On the way OA, we have $y = 0 = cte \Rightarrow dy = 0$

$dW_O^A(\vec{F}) = x^2dx \Rightarrow W_O^A(\vec{F}) = \int_O^A d\,W_O^A(\vec{F}) = \int_0^2 x^2\,dx = \frac{1}{3}x^3|_0^2 = \mathbf{\frac{8}{3}}$ **Joules**

3) The work $W_A^B(\vec{F})$

On the way AB, we have $x = 2 = cte \Rightarrow dx = 0$

$dW_A^B(\vec{F}) = 4ydy \Rightarrow W_A^B(\vec{F}) = \int_A^B d\,W_A^B(\vec{F}) = \int_2^3 4\,ydy = 2y^2|_2^3 = \mathbf{10}$ **Joules**

4) The work $W_O^B(\vec{F})$

The straight line connecting points 0 and B is given by: $y = ax + b$ $(b = 0)$ because the straight line passes through O $\Rightarrow y = ax$

Point $B(2,3)$ belongs to the straight line, so: $3 = 2a \Rightarrow a = \frac{3}{2}$

The equation of the line connecting 0 and B is given by: $y = \frac{3}{2}x \Rightarrow dy = \frac{3}{2}dx$

$$dW_O^B(\vec{F}) = x^2dx + 2x\left(\frac{3}{2}x\right)\left(\frac{3}{2}dx\right) = \frac{11}{2}x^2dx$$

$W_O^B\left(\vec{F}\right) = \int_0^B d\,W_O^B\left(\vec{F}\right) = \int_0^2 \frac{11}{2}x^2dx = \frac{11}{6}x^3|_0^2 = \mathbf{\frac{44}{3}}$ **Joules**

5)

$W_O^A(\vec{F}) + W_A^B(\vec{F}) = \frac{8}{3}$ Joules $+10$ Joules $= \mathbf{\frac{38}{3}}$ **Joules**

$W_O^B(\vec{F}) = \mathbf{\frac{44}{3}}$ **Joules**

Clearly, the force $\vec{F}$ is not conservative, since its work depends on the path followed.

Exercise n°02:

1) The expression of elementary work dw:

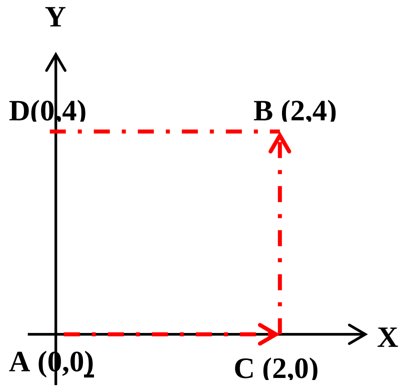
$\vec{F} = (y^2 - x^2)\vec{\imath} + 3xy\vec{\jmath}$

$\overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath}$

$dw = \vec{F}\vec{dl} = (y^2 - x^2)dx + 3xydy$

a) On the axis$[Ox)$ from $A(0,0)$ to $C(2,0)$ and then parallel to$[Oy)$ from C to B.

The work $W_A^B(\vec{F})$



$W_A^B = \int_A^B (y^2 - x^2)dx + \int_A^B 3xydy$ ....(1)

$W_A^B = W_A^C + W_C^B$

$W_A^C \begin{cases} y = 0 \\ x = 2 \end{cases} \Rightarrow \{dy = 0$

By replacing in the equation (1) we obtain:

$$W_A^C = \int_0^2 -x^2 dx = \left[-\frac{1}{3}x^3\right]_0^2 \Longrightarrow \boldsymbol{W_A^C = -\frac{8}{3}\text{Joules}}$$

$W_A^C \begin{cases} x = 2 \\ y = 4 \end{cases} \Rightarrow \{dx = 0$

By replacing in the equation (1) we obtain:

$$W_C^B = \int_0^4 6ydy = 6\left[\frac{1}{2}y^2\right]_0^4 \Longrightarrow \boldsymbol{W_C^B = 48\ \text{Joule}}$$

$$W_A^B = W_A^C + W_C^B = -\frac{8}{3} + 48 \Longrightarrow \boldsymbol{W_A^B = 45,3\text{Joules}}$$

b) On the axis$[Oy)$ from $A(0,0)$ to $D(0,4)$ and then parallel to$[Ox)$ from D to B.

$W_A^B = W_{A-D} + W_{O-D}$

$W_A^D \begin{cases} x = 0 \\ y = 4 \end{cases} \Rightarrow \{dx = 0$

$\boldsymbol{W_A^D = 0\ \text{Joule}}$

$\boldsymbol{W_D^B} \begin{cases} x = 2 \\ y = 4 \end{cases} \Rightarrow \{dy = 0$

By replacing in the equation (1) we obtain:

$$W_D^B = \int_0^2 (16 - x^2)dx = 16x\left[-\frac{1}{3}x^3\right]_0^2 = 32 - \frac{8}{3} \Longrightarrow \boldsymbol{W_D^B = \frac{88}{3}\ \text{Joules}}$$

$$W_A^B = W_A^D + W_O^D = 0 + \frac{88}{3} \Longrightarrow \boldsymbol{W_A^B = 29,3\text{Joules}}$$

c) On the straight line$[AB]$.

The straight line connecting points A and B is given by: $y = ax + b$ $(b = 0)$ because the straight line passes through O $\Rightarrow y = ax$

Point $B(2,4)$ belongs to the straight line, so: $4 = 2a \Rightarrow a = 2 \Rightarrow y = 2x \Rightarrow dy = 2dx$

By replacing in the equation (1) we obtain:

$$W_A^B = \int_0^2 ((2x)^2 - x^2)dx + \int_0^2 3x\ 2\ dx\ 2x$$

$$W_A^B = \int_0^2 (4x^2 - x^2)dx + \int_0^2 12x^2\, dx$$

$$W_A^B = \int_0^2 15x^2\, dx = \frac{15}{3}x^3 \Rightarrow \boldsymbol{W_A^B = 40\ Joules}$$

d) On the trajectory $y = x^2$

$y = x^2 \Rightarrow dy = 2x\, dx$

By replacing in the equation (1) we obtain:

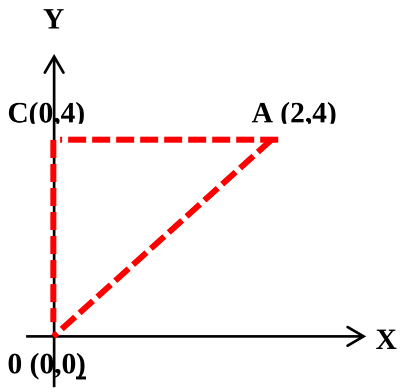
$$W_A^B = \int_0^2 ((x^2)^2 - x^2)dx + \int_0^2 3x^3\, 2x\, dx$$

$$W_A^B = \int_0^2 (x^4 + x^2 + 6x^6)dx$$

$$W_A^B = \int_0^2 (7x^4 - x^2)dx = \frac{7}{5}x^5 \left[-\frac{1}{3}x^3\right]_0^2 \Rightarrow \boldsymbol{W_A^B = 42,13\ Joules}$$

The force $\vec{F}$ is not conservative. A conservative force $\vec{F}$ is a force whose work is independent of the path followed by its point of application when it moves between two points A and B. Example: The weight of a body is a conservative force.

Exercise n°03:



1) The expression of elementary work dw:

$\vec{F} = (x - ay)\vec{\imath} + (3y - 2x)\vec{\jmath}$ …. (1)

$\vec{dl} = dx\vec{\imath} + dy\vec{\jmath}$

$dw = \vec{F}\vec{dl} = F_x dx + F_y dy = (x - ay)dx + (3y - 2x)dy$

$W_O^A = W_O^C + W_C^A$

$W_O^C \begin{cases} x = 0 \\ y = 4 \end{cases} \Rightarrow \{dx = 0$

By replacing in the equation (1) we obtain:

$$W_O^C = \int_0^4 3y dy = \left[\frac{3}{2}y^2\right]_0^4 \Longrightarrow \boldsymbol{W_O^C = 24\ Joule}$$

$W_C^A \begin{cases} x = 2 \\ y = 4 \end{cases} \Rightarrow \{dx = 0$

By replacing in the equation (1) we obtain:

$$W_C^A = \int_0^2 (x - 4a)dx = \left[\frac{x^2}{2} - 4ax\right]_0^2 = (2 - 8a) \Longrightarrow \boldsymbol{W_C^A = (2 - 8a)}$$

2) For the force to be conservative it must verify: $\overrightarrow{rot}(\vec{F}) = \vec{0}$

$$\overrightarrow{rot}(\vec{F}) = \vec{\nabla} \wedge \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x-ay) & (3y-2x) & B_z \end{vmatrix} = (-2+a)\vec{k} = \vec{0} \Longrightarrow \boldsymbol{a = 2}$$

$\vec{F} = (x-ay)\vec{\imath} + (3y-2x)\vec{\jmath}$ which is the first conservative force so it is the first force which derives from a potential E_P

$$\vec{F} = -\overrightarrow{grad}\, E_p(x,y) = -\frac{\partial\, E_p}{\partial x}\vec{\imath} - \frac{\partial\, E_p}{\partial y}\vec{\jmath}$$

$\frac{\partial\, E_p}{\partial x} = -(x-2y)$ ….. (2) and $\frac{\partial\, E_p}{\partial y} = -(3y-2x)$ …. (3)

From equation (2): $E_p = -\int (x-2y)dx = -\frac{x^2}{2} + 2yx + C(y)$

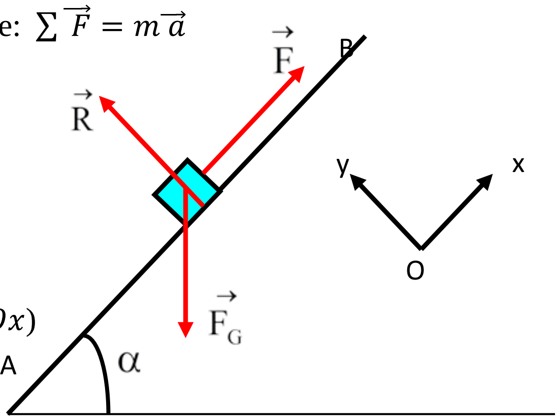
From equation (3): $\frac{\partial\, E_p}{\partial y} = 2x + \frac{\partial C(y)}{\partial y} = -(3y-2x) \Rightarrow C(y) = \frac{-3y^2}{2} + C$

$$\Rightarrow E_p(x,y) = -\frac{x^2}{2} + 2yx - \frac{3y^2}{2} + C$$

Exercise n°04:

1) Applying Newton's second law, we can write: $\sum \vec{F} = m\vec{a}$

$\sum \vec{F} = \vec{0} \Rightarrow \vec{P} + \vec{F} + \vec{R} = \vec{0}$ … … (1)



1- The work of $\vec{F}$: $W_A^B(\vec{F})$

$$W_A^B(\vec{F}) = \int_A^B \vec{F}\,\overrightarrow{dr}$$

When $\overrightarrow{dr} = dx\vec{\imath}$ (displacement along the ox axis Ox)

$$W_A^B(\vec{F}) = \int_A^B F.dx$$

$$(1) \Rightarrow \begin{cases} Ox/\ F - mg\sin\alpha = 0 \Rightarrow \\ Oy/\ R - mg\cos\alpha = 0 \end{cases} \quad F = mg\sin\alpha$$

$$W_A^B(\vec{F}) = \int_A^B mg\sin\alpha\,.dx = mg[\sin\alpha]_A^B = mg\,\boldsymbol{\sin\alpha\,.AB} = mg\boldsymbol{h}$$

$$\Rightarrow \boldsymbol{W_A^B(\vec{F}) = mgh}$$

2- The work of $\vec{F}_G$: $W_A^B(\vec{\boldsymbol{F}}_{\boldsymbol{G}})$

$W_A^B(\vec{F}_G) = \int_A^B \vec{F}_G\,.\overrightarrow{dr} = \int_A^B F_{Gx}.dx$ (displacement along the ox axis Ox)

$$W_A^B(\vec{F}_G) = \int_A^B -mg\sin\alpha\,.dx = -mg[\sin\alpha]_A^B = -mg\,\boldsymbol{\sin\alpha\,.AB} = -mg\boldsymbol{h}$$

$$\boldsymbol{W_A^B(\vec{F}_G) = -mgh}$$

3- The work of $\vec{R}$: $W_A^B(\vec{R})$

The friction forces are negligible, the reaction force $\vec{R}$ is perpendicular to the plane, and therefore to the displacement, and its work is zero.

$$W_A^B(\vec{R}) = \int_A^B \vec{R}.\overrightarrow{dr} \quad \rightarrow (\vec{R} \perp \overrightarrow{dr}) \Rightarrow \quad \boldsymbol{W_A^B(\vec{R}) = 0}$$

2) the variation in kinetic energy between A and B

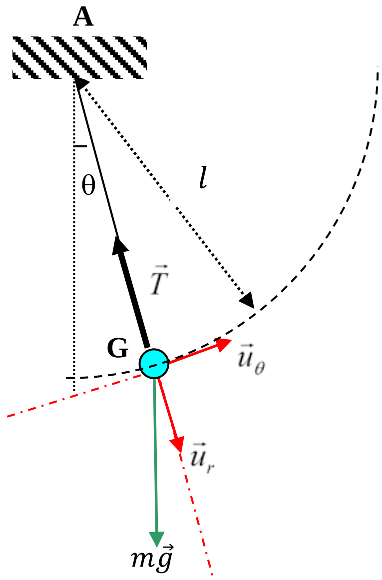
$W_A^B(\vec{F}_1) = E_{CB} - E_{CA}$, $\vec{F}_1$: the resultant force

$$E_{CB} - E_{CA} = W_A^B(\vec{F} + \vec{F}_G + \vec{R}) = W_A^B(\vec{F}) + W_A^B(\vec{F}_G) + W_A^B(\vec{R})$$

$$E_{CB} - E_{CA} = mgh - mgh + O$$

$\Rightarrow \boldsymbol{E_{CB} - E_{CA} = 0 \Rightarrow v_A = v_B}$ $\Rightarrow$ The motion is uniform and the trajectory is a straight line$\Rightarrow$ Uniformly rectilinear motion.

Exercise n°05:



1) Fundamental Principle of dynamics:

$$\sum \vec{F}_{ext} = m\vec{g} + \vec{T} = m.\vec{a}$$

By projecting normal and tangential accelerations along the axes.

$$\begin{matrix} (\vec{u}_r) \\ (\vec{u}_\theta) \end{matrix} \begin{cases} mg.\cos\theta - T = -m.l.\theta^{\bullet 2} \\ -mg.\sin\theta = m.l.\theta^{\bullet\bullet} \end{cases}$$

The 2nd equation gives the differential equation of motion.

$$\Rightarrow \qquad \theta^{\bullet\bullet} + \frac{g}{l}\sin\theta = 0$$

2) Angular momentum theorem:

$$\frac{d\vec{L}}{dt} = \sum \vec{M}/_A(\vec{F}_{ext}),\ \vec{L} = \overrightarrow{AG} \wedge m\vec{v},\ \overrightarrow{AG} = l\,\vec{u}_r,\ \vec{v} = l.\theta^{\bullet}.\vec{u}_\theta$$

$$\vec{L} = l\,\vec{u}_r \wedge m.l.\theta^{\bullet}\vec{u}_\theta = m.l.\theta^{\bullet}\,\vec{k} \Rightarrow$$

$$\frac{d\vec{L}}{dt} = m.l^2.\theta^{\bullet\bullet}\vec{k},\ \vec{M}/_A(m\vec{g} + \vec{T}) = \vec{M}/_A(m\vec{g}) + \underbrace{\vec{M}/_A(\vec{T})}_{0} = \vec{M}/_A(m\vec{g})$$

$$\vec{M}/_A(m\vec{g}) = \overrightarrow{AG} \wedge (mg.\cos\theta\,\vec{u}_r - mg.\sin\theta\,\vec{u}_\theta) = -mgl.\sin\theta\,\vec{k}$$

$$\Rightarrow m.l^2.\theta^{\bullet\bullet} = -mgl.\sin\theta$$

$$\Rightarrow \qquad \theta^{\bullet\bullet} + \frac{g}{l}\sin\theta = 0$$

3) Kinetic energy theorem:

$dw = \vec{F}d\vec{l} = (m\vec{g} + \vec{T})\vec{dl} = (-T\vec{u}_r + mg.\cos\theta\,\vec{u}_r - mg.\sin\theta\,\vec{u}_\theta).l\,d\theta\,\vec{u}_\theta$

With $\vec{dl} = l\,d\theta\,\vec{u}_\theta \Rightarrow dw = -mgl.\sin\theta\,d\theta$

$E_C = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\theta^{\bullet\,2} \Rightarrow dE_C = m.l^2.\theta^{\bullet\bullet}d\theta$

$mgl.\sin\theta\,d\theta = m.l^2.\theta^{\bullet\bullet}d\theta$

$$\Rightarrow \quad \theta^{\bullet\bullet} + \frac{g}{l}\sin\theta = 0$$

4) Total mechanical energy theorem:

$dw = -dE_P = -mgl.\sin\theta\,d\theta \Rightarrow E_P = -mgl.\cos\theta + cst$

$E_P = 0$ for $\theta = 0 \Rightarrow C = m.g.l \Rightarrow \boldsymbol{E_P = mgl(1-\cos\theta)}$

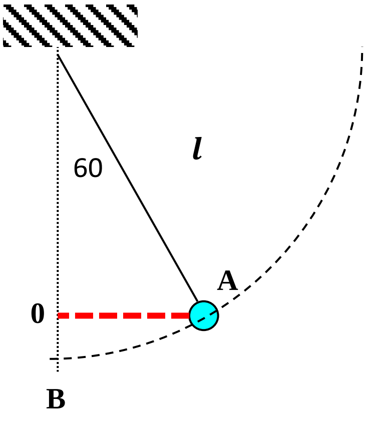
$E_T = E_C + E_P = cst \Rightarrow \frac{d(E_C+E_P)}{d} = 0 \quad ....(*)$

$E_C + E_P = \frac{1}{2}ml^2\theta^{\bullet\,2} + mgl(1-\cos\theta) = cst$

According to (*) : $ml^2\theta^{\bullet\,2}\theta^{\bullet\bullet} + mgl.\theta^{\bullet}.\sin\theta$

$\Rightarrow \quad \theta^{\bullet\bullet} + \frac{g}{l}\sin\theta = 0$

Exercise n°06:



$m = 50g\ ,\ L = 0.5m,\ g = 10m/s2, v_0 = 0m/s, \theta = 60°$

1) At $= 0s\ , \theta = 60°$

$W_A^B = E_{CB} - E_{CA}$

Let's take point A as reference O $\Rightarrow W_A^B = E_{CB} - E_{CA}$

$ET_O - ET_B \Rightarrow \frac{1}{2}mv_0^2 + E_{PO} = \frac{1}{2}mv_B^2 + E_{PB}$

$E_{PB} = 0 \Rightarrow E_{PO} = \frac{1}{2}mv_B^2$ with $E_{PO} = mgh$

$mgh = \frac{1}{2}mv_B^2 \Rightarrow v_B = \sqrt{2gh}$ with $h = l - l\cos\theta = l(1-\cos\theta)$

$\Rightarrow v_B = \sqrt{2gl(1-\cos\theta)}$ NC: $\boldsymbol{v_B = 2,24\text{m/s}}$

2) $E_{CB} = \frac{1}{2}mv_B^2 = mgh \Rightarrow \boldsymbol{E_{CB} = 0,125\ \text{joules}}$

Exercise n°07:

$m = 2$ kg ; α = 30° ; β = 60° ; AB = 20 m ; BC = 5 m ; $V_C = 10$ m/s ; $V_E = 8$ m/s

1 –

Conservation of total mechanical energy $E_C(A) + E_P(A) = E_C(B) + E_P(B)$

$\frac{1}{2}mV_A^2 + mg.h_A = \frac{1}{2}mV_B^2 + mg.h_B$ With $\begin{cases} h_A = AB.\sin\alpha \\ h_B = 0.m \end{cases}$ and $V_A = 0.m/s$

Finally $\boxed{V_B = \sqrt{2g.AB.\sin\alpha}}$ N.C. $V_B = 14{,}1$ *m/s*

2 –

The work of force is non-conservative $W(\vec{F}_f) = \Delta E_T = E_T(C) + E_T(B)$

$-F_f.BC = \frac{1}{2}mV_C^2 - \frac{1}{2}mV_B^2$ Finally $\boxed{F_f = \frac{m(V_B^2 - V_C^2)}{2.BC}}$ N.C. $F_f = 20$ *N*

3 –

Conservation of total mechanical energy $E_C(C) + E_P(C) = E_C(E) + E_P(E)$

$\frac{1}{2}mV_C^2 + mg.h_C = \frac{1}{2}mV_E^2 + mg.h_E$ With $\begin{cases} h_E = R \\ h_C = 0.m \end{cases}$

Finally $\boxed{R = \frac{(V_C^2 - V_E^2)}{2.g}}$ N.C. $R = 1{,}8$ *m*

4 –

Conservation of total mechanical energy $E_C(C) + E_P(C) = E_C(D) + E_P(D)$

$\frac{1}{2}mV_C^2 + mg.h_C = \frac{1}{2}mV_D^2 + mg.h_D$ With $\begin{cases} h_D = R(1-\cos\beta) \\ h_C = 0.m \end{cases}$

Finally $\boxed{V_D = \sqrt{V_C^2 - 2g.R(1-\cos\beta)}}$ N.C. $V_D = 9$ *m/s*

5 – Using the ***FPD*** $\Rightarrow$ $\vec{P} + \vec{C} = m.\vec{a}$ $(\vec{F} = \vec{C})$

Projected on the (OD) axis, the equation becomes: $C - mg.\cos(\beta) = m.a_N = m.\frac{V^2}{r}$ So

$\boxed{C_D = m.\frac{V_D^2}{r} + mg.\cos(\beta)}$ NC. $C_D = 101{,}11$ *N*

6 –

Conservation of total mechanical energy $E_C(C) + E_P(C) = E_C(F) + E_P(F)$

$\frac{1}{2}mV_C^2 + mg.h_C = \frac{1}{2}mV_F^2 + mg.h_F$ With $\begin{cases} h_F = 2R \\ h_C = 0.m \end{cases}$

Finally $\boxed{V_F = \sqrt{V_C^2 - 4gR}}$ N.C. $V_F = 5{,}29$ *m/s*

$C + mg = m.a_N = m.\frac{V^2}{r}$ So $\boxed{C_F = m.\frac{V_F^2}{r} - mg}$ N.C. $C_F = 11{,}11$ *N*

Exercise n°08:

1-Motion without friction.

a) Calculation of the speed at point B.

Total energy conserved: $E_{tot}(A) = E_{tot}(B) \Rightarrow E_P(A) + E_C(A) = E_P(B) + E_C(B)$

We consider the horizontal plane (ED) as a reference for the potential energy $E_P = 0$J.

$$\Rightarrow mgR = \frac{1}{2}mv_B^2 + mgR(1 - cos\alpha)$$

$$\Rightarrow v_B = \sqrt{2gRcos\alpha}$$

N. C : $R = 1m, \alpha = 45^o \rightarrow \boldsymbol{v_B = 3,72m/s}$

b) The contact force $\vec{C}$.

P.F.D $\Rightarrow \vec{P} + \vec{C} = m\vec{a}$

By projection on the normal axis:

$$C - mgcos\alpha = ma_n \Rightarrow C = mgcos\alpha + m\frac{v_B^2}{R}$$

N. C : $\boldsymbol{C = 20,76N}$

c) Calculation of the acceleration at point B.

We have: $a = \sqrt{a_t^2 + a_n^2}$

The normal acceleration is: $\boldsymbol{a_n = \frac{v_B^2}{R} = 13.84m/s^2}$

The tangential acceleration from the projection of the P.F.D on the tangent axis:

$mgsin\alpha = ma_t \Rightarrow \boldsymbol{a_t = gsin\,\alpha = 6,94m/s^2}$

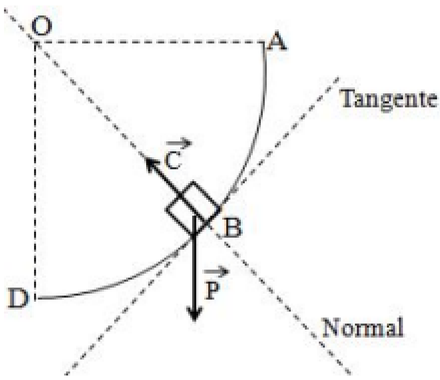
So : $a = \sqrt{(6{,}94)^2 + (13{,}84)^2} = \boldsymbol{15,51m/s^2}$

d) Calculation of spring compression.

$$E_{tot}(E) = E_{tot}(A) \Rightarrow E_P(E) + E_C(E) = E_P(A) + E_C(A)$$

$$\Rightarrow \frac{1}{2}K(\Delta x)^2 = mgR$$

$$\Rightarrow \Delta x = \sqrt{2mgRK} = \boldsymbol{44,29cm}$$



2) Calculation of spring compression with non-negligible friction on DEF. The variation in (total) mechanical energy is equal to the work of the non-conservative forces.

$$\Delta E_{tot} = \Sigma W(\vec{F})_{NC} \Rightarrow E_{tot}(E) - E_{tot}(D) = W(\vec{C}_x)$$

$\frac{1}{2}K(\Delta x)^2 - \frac{1}{2}mv_D^2 = -C_x . DE$

$\Delta x = \sqrt{2m\left(v_D^2 - \mu_g gDE\right)K}$

Determination of the velocity at point D, $\boldsymbol{v_D = \sqrt{2gR} = 4,43m/s}$

N. C : $\boldsymbol{\Delta X = 42,02cm}$

Exercise n°09:

1) The variation of the kinetic energy ΔE_C and v_B :

Let us consider the reference of the potential energy $E_P(B) = 0J$.

The motion is frictionless along AB therefore:

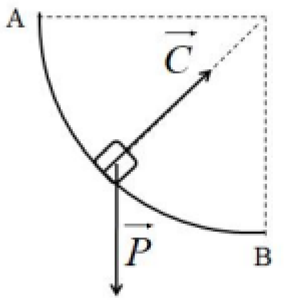
$$\Delta E_C = -\Delta E_P = -(0 - mgR)$$

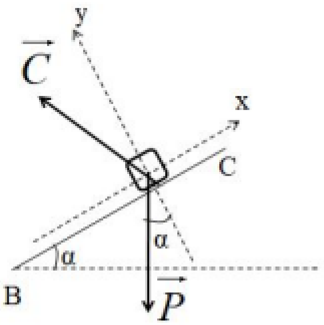
N.C : $\boldsymbol{\Delta E_C = 20J}$

The speed at point B : $\Delta E_C = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$

$$v_A = 0 \Rightarrow v_B = \sqrt{2\frac{\Delta E_C}{m}} \Rightarrow \boldsymbol{v_A = 4.47m/s}$$

2) Representation of the forces in sections AB and BC:





3) The acceleration of the mass on the inclined plane BC: The F.P.D applied to the mass gives: $\vec{P} + \vec{C} = m\vec{a}$

By projection on the axes:

$-P_x - C_x = ma \rightarrow -mgsin\alpha - C_x = ma$

$C_y - P_y = 0 \rightarrow C_y = mgcos\,\alpha$

We have : $C_x = \mu_d C_y = \mu_d(mgcos\alpha)$

$\boldsymbol{a = -(sin\alpha + \mu_d cos\alpha)g = -10.61m/s^2}$

4) The work $W(\vec{C})$ along BC and v_C is:

$W(\vec{C}) = -\int C_x\, dx$

$C_x = \mu_d(mgcos\alpha)$) and $dx = BC$

La distance $BC = \frac{\left(\frac{R}{2}\right)}{\sin\alpha} = \mathbf{0.71m}$

So : $W(\vec{C}) = -(0{,}5)(2)(10)\ (cos45)\ (0.71) = \mathbf{-5J}$

The speed at point C is calculated by: $\Delta\ E_{tot} = W(\vec{C})$

$\left(\frac{1}{2}mv_C^2 + \frac{mgR}{2}\right) - \left(\frac{1}{2}mv_B^2 + 0\right) = -5J \Rightarrow \boldsymbol{v_C} = \sqrt{\mathbf{5}} = \mathbf{2.24m/s}$ 5)

The time taken to cover the distance BC

$v_C = at + v_B \Rightarrow \boldsymbol{t} = \frac{\boldsymbol{v_C - v_B}}{\boldsymbol{a}} = \mathbf{0.21s}$

Exercise n°10:

1) Let's calculate the speeds $\boldsymbol{v_B}$ and $\boldsymbol{v_C}$ of the skier

Let's apply the kinetic energy theorem between A and B: $\mathrm{E_{CB}} - \mathrm{E_{CA}} = \mathrm{W}(\vec{P}) + \mathrm{W}(\vec{R})$

$\frac{1}{2}mv_B^2 - 0 = \mathrm{mgh} = \mathrm{mgR}(1 - \cos\alpha)\ \mathrm{v_B^2} = 2\mathrm{gR}(1 - \cos\alpha)\ v_\mathrm{B} = \sqrt{2\mathrm{gR}(1 - \cos\alpha)}$

NC:$v_\mathrm{B} = \sqrt{2 \times 9{,}8 \times 5 \times (1 - \cos 60)}, \cdot\ v_\mathrm{B} = 7\mathrm{m.s^{-1}}$

Let us apply the theorem of kinetic energy between B and C:

$\mathrm{E_{CC}} - \mathrm{E_{CB}} = \mathrm{W}(\vec{P}) + \mathrm{W}(\vec{R}) = 0 + 0$ therefore $v_\mathrm{C} = v_\mathrm{B} = 7\mathrm{m.s^{-1}}$

Let us express v_C and function of m, R, F, v_B

Let's apply the kinetic energy theorem between B and C:

$\mathrm{E_{cC}} - \mathrm{E_{cB}} = \mathrm{W}(\vec{P}) + \mathrm{W}(\vec{R}) + \mathrm{W}(\vec{F})$

$\frac{1}{2}mv_c^2 - \frac{1}{2}mv_B^2 = -F \times \mathrm{R} \Rightarrow \frac{1}{2}mv_c^2 = \frac{1}{2}mv_B^2 - F \times \mathrm{R}$ where $v_\mathrm{C} = \sqrt{v_B^2 - \frac{2FR}{m}}$

b) Let us express v_B as a function of m, R, F, g.

Let's apply the kinetic energy theorem between A and B:

$\mathrm{E_{CB}} - \mathrm{E_{CA}} = \mathrm{W}(\vec{P}) + \mathrm{W}(\vec{R}) + \mathrm{W}(\vec{F})$

$\frac{1}{2}mv_B^2 = m\mathrm{gR}(1 - \cos\alpha) - F \times \mathrm{R}\frac{\pi}{3}$;

$v_\mathrm{B} = \sqrt{2\mathrm{gR}(1 - \cos\alpha) - F \times \mathrm{R}\frac{2\pi}{3m}}$ or $\alpha = \frac{\pi}{3}$ so $v_\mathrm{B} = \sqrt{\mathrm{gR} - F \times \mathrm{R}\frac{2\pi}{3m}}$

The intensity of the friction force

$v_C = \sqrt{v_B^2 - \frac{2FR}{m}} = 0 \Rightarrow v_B^2 = \frac{2FR}{m} \Rightarrow gR - F \times R\frac{2\pi}{3m} = \frac{2FR}{m}$

$\Rightarrow g = F \times \frac{2\pi}{3m} + \frac{2F}{m}$

$F = \frac{mg}{2+\frac{2\pi}{3}}$ NC: $F = \frac{8010}{2+\frac{2\pi}{3}}$ $\Rightarrow$ F=195N

3) Let us express its speed v_E as a function of g, R and θ

Let's apply the kinetic energy theorem between C and E:

$E_{CE} - E_{CC} = W(\vec{P}) + W(\vec{R})$

$\frac{1}{2}mv_E^2 - 0 = mgh = mgR(1 - \sin\alpha) \Rightarrow v_E = \sqrt{2gR(1 - \sin\theta)} \Rightarrow v_E = 5{,}77 m/S$

The value of the angle θ

$\frac{v_E^2}{2gR} = 1 - \sin\theta \Rightarrow \sin\theta = 1 - \frac{v_E^2}{2gR}$ NC: $\sin\theta = 1 - \frac{5{,}77^2}{2x10x5} = 0{,}67 \Rightarrow \theta = 41{,}8°$

4) Speed in G

Let us apply the kinetic energy theorem between E and G:

$E_{CG} - E_{CE} = W(\vec{P})$

$\frac{1}{2}mv_G^2 - \frac{1}{2}mv_E^2 = mgR\sin\theta \Rightarrow v_G = \sqrt{v_E^2 + 2gR\sin\theta}$

NC: $v_G = \sqrt{(5{,}77)^2 + 2 \times 10 \times 5 \times 0{,}67} \Rightarrow v_G = 10 m/S$

Bibliography

1. LAMRIA.B, DEBIAN.M, GOURARI .A, et MAHAMDIA .A ; Physique I Mécanique du Point Matériel, Edité par la Faculté de Physique U.S.T.H.B. B.P. N 32 El-Alia Alger, 2011.
2. ALONSO.M. and FINN.E.J. and DAUNE.M., Physique générale : cours et exercices corrigés. Mécanique et thermodynamique.
3. HALLIDAY, D. and RESNICK, R. and WALKER, J, Fundamentals of Physics Extended, 09th Edition.
4. GIANCOLI.D.C., Physics Principles with Applications, Global Edition, 2016.
5. BENSON, HARRIS., University physics, 05th Edition. John Wiley & Sons, 2008.
6. FEYNMAN, R. and LEIGHTON, R. and SANDS, M., Exercices pour le cours de physique de Feynman : 900 exercices.
7. SAMUEL J. L., JEFF S, and BILL M, University Physics Volume 1, Openstax, 2016.
8. IRODOV I. E., PROBLEM IN GENERAL PHYSICS, 1981.
9. SERWAY.A & JEWETT.J.W., Physics for Scientists and Engineers 9th Edition.
10. TIPLER.P. A., MOSCA.G., Physics For Scientists and Engineers, 6th Ed., W. H. Freeman Company, 2008.
11. SINGH.M, VERMA L. P., SINGH.M, DEVLAL.K, BHATT.T, Mechanics (BSCPH 101), Department Of Physics School Of Sciences Uttarakhand Open University, 2017.
12. SPIEGEL.M.R , Schaum's Outline of Theory and Problems of Theoretical Mechanics SI(metric) edition, Including 720 solved problems.
13. GIBAUD.A, HENRY.M., Cours de physique - Mécanique du point - Cours et exercices corrigés; Dunod, 2007.
14. ROUSSEL.J., Cours de physique mécanique classique, 2023, https://femto-physique.fr/mecanique/index.php.
15. ROUSSEL.J., OUTILS & MÉTHODES, 87 exercices et problèmes corrigés, https://femto-physique.fr/
16. PÉREZ, J.P.,Mécanique, fondements et applications avec 320 exercices et problèmes résolus, 7e édition, Édition Dunod.
17. DUBARRY-BARBE.J.P, FREY.A., Mécanique - Exercices et problèmes corrigés, classes préparatoires, MPSI, PCSI, PTSI, Ellipses,2009
18. BOUDAHRI.F., Polycopié Travaux Dirigés Corrigés Mécanique du point matériel, 2022.
19. BENYAGOUB.H., Travaux Dirigés Corrigés Mécanique du point matériel, Université Ziane Achour – Djelfa, socle commun 1ère année SM, 2013.
20. KCHIDI.Z., Polycopié de travaux dirigés de physique (01) : Mécanique du point matériel, Octobre2017.