Lecture: Mathematics-2

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Preface

This manuscript is intended for first-year students in the common core curriculum preparing for a bachelor's degree in commercial, economic, and management sciences. It consists of three chapters covering the entire program of Mathematics 2 module.

The first chapter deals with solving first and second-order differential equations.

The second chapter explains fundamental and advanced concepts related to matrices. It also aims to provide readers with a thorough understanding of matrix operations and properties. It also introduces essential notions such as calculating the determinant and the inverse of a square matrix.

Finally, the last chapter describes Cramer's linear systems of equations and presents various solution methods. Additionally, non-Cramer systems are also addressed in this section.

We are aware that our work can always be improved, and we strongly encourage readers to provide us with their feedback or suggestions to enhance our document

Chapter 01 :

Différential Equations

Definition 01:

An Ordinary Differential Equation (ODE) of order n is an equation where the unknown is a function $y(t)$. It is of the form

$$F\left(t, y, y', y'', \dots \ , y^{(n)}\right) = 0 ,$$

With :

- F:is a continues function.
- y : The unknown function of the variable t (to be determined)
- t : is the real variable (in physics, representing time).
- n : It is the highest order of the derivative of y, which is also the order of the ODE.

Definition 02 : A first-order ordinary differential equation (ODE) is of the form:

$$y' = F(t, y) \qquad \dots .(1)$$

Remark : To find the solution of an ODE, one must search for a function y(t) that satisfies this ODE, according to each type. Integration is the process that allows us to do this.

1. Separable Variable Equation:

Definition 03: The ODE (1) is said to be in separated variables form if it is of the form:

$$f(y)dy = g(t)dt$$

Where f and g are two real (continuous) functions of the real variable.

Resolution method: To find the solution from (1), you need to follow the following steps:

1. **Use the relation** $\boldsymbol{y' = \frac{dy}{dt}}$.
2. **Separate the y terms from the t variable:** Put the y terms on one side and what depends on t on the other side.
3. **Integrate both sides of the equation with respect to both t and y.**

Example: Solve the following ODE:

$$\boldsymbol{y^2 - (1 + 3t)y' = 0} \qquad \dots \dots (\boldsymbol{E})$$

- **From the relation** $\boldsymbol{y'} = \frac{dy}{dt}$, we obtain :

$$y^2 - (1 + 3t)y' = 0 \iff y^2 - (1 + 3t)\frac{dy}{dt} = 0$$

- To begin, we separate the variables, we have

$$y^2 - (1+3t)y' = 0 \quad \Leftrightarrow y^2 = (1+3t)\frac{d\boldsymbol{y}}{dt}$$

$$\Leftrightarrow \frac{1}{(1+3t)} = \frac{1}{y^2}y'$$

$$\Leftrightarrow \frac{1.dt}{(1+3t)} = \frac{1}{y^2}dy$$

- By integrating each side of the equation:

$$\int \frac{1.dt}{(1+3t)} = \int \frac{1}{y^2}dy \quad \Leftrightarrow \frac{1}{3}\,ln|1+3t| + c = -\frac{1}{y}$$

- Finally, the solution is:

$$\boldsymbol{y} = -\frac{\mathbf{1}}{\frac{1}{3}ln|1+3t| + c}$$

2. Homogeneous Differential Equation:

Definition 04 : A homogeneous differential equation is written as:

$$y' = f\left(\frac{y}{t}\right) \ldots..(2)$$

Resolution method: Let's consider the differential equation $y' = f\left(\frac{y}{t}\right)$,

1. Introduce the change of variable : $\boldsymbol{u(t)} = \frac{\boldsymbol{y(t)}}{\boldsymbol{t}} \quad \Rightarrow \quad y(t) = t.u(t)$
2. Use the derivative: $\boldsymbol{y' = t.u'(t) + u(t)}$
3. Replace the values of u and y' in equation (2).
4. Use the method of separated variables to solve the resulting equation.
5. Find the final solution $y(t)$.

Example: Solve the following ODE:

$$ty' + y = t$$

Solution :

We have,

$$ty' + y = t$$

In order to express$(\boldsymbol{E})$ in the form $\boldsymbol{y' = f}\left(\frac{y}{t}\right)$, We need to divide everything by t.

$$ty' + y = t \quad \Leftrightarrow \quad \frac{t.y'}{t} + \frac{y}{t} = \frac{t}{t}$$

$$\Rightarrow \quad y' = 1 - \frac{y}{t} \ldots \; (\boldsymbol{E})$$

Let's introduce a variable transformation

$$u(t) = \frac{y(t)}{t} \quad \Rightarrow \quad y(t) = t.u(t)$$

The derivative is $y' = t.u'(t) + u(t)$

By substituting into (E)

$$y' = 1 - \frac{y}{t}$$

We get ,

$$t.u'(t) + u(t) = 1 - u$$

$$\Rightarrow \quad t.u'(t) = 1 - 2u$$

- **Using the method of separated variables, we solve the equation:**

$$t.u'(t) = \frac{1}{2} - 2u$$

Remark that $u'(t) = \frac{du}{dt}$, so ,

$$t.u'(t) = 1 - 2u \qquad \Leftrightarrow \; t.\frac{du}{dt}. = \quad 1 - 2u$$

$$\Leftrightarrow \frac{du}{1-2u}. = \frac{dt}{t}$$

We integrate both sides.

$$\int \frac{du}{1-2u}. = \int \frac{dt}{t}$$

$$\frac{-1}{2} ln|1-2u| = ln|t| + c_1$$

Then ,

$$ln|1-2u| = -2ln|t| + c \quad \Leftrightarrow \quad e^{ln|1-2u|} = e^{-2ln|t|+c}$$

$$\Leftrightarrow \quad |1-2u| = e^{-ln|t|^2}.e^c$$

$$\Leftrightarrow \quad 1 - 2u = \pm e^{ln\frac{1}{t^2}}.e^c$$

$$\Leftrightarrow u = \frac{-1}{2}\left(\pm\frac{1}{t^2}.e^c - 1\right)$$

Then the solution is

$$u(t) = \frac{1}{2}\left(\frac{k}{t^2} + 1\right) \qquad where \;\; k = \mp e^c$$

- **In order to find $y(t)$, we simply substitute the value of u into the solution, such that,**

$$u(t) = \frac{y(t)}{t} = \frac{1}{2}\left(\frac{k}{t^2} + 1\right)$$

Finally, the solution is :

$$y(t) = \frac{t}{2}\left(\frac{k}{t^2} + 1\right) = \frac{k}{2t} + \frac{t}{2}, \qquad k \in \mathbb{R}$$

I. 2nd Order Differential Equations

Definition : A second-order linear differential equation with constant coefficients has the form

$$a.y'' + by' + c.y = f(t) \qquad (E)$$

Where : a, b and c are a reals constants with $a \neq 0$, $\forall t \in \mathbb{R}$.

And $f(t)$ is the second member.

if $f(t) = 0$, then (E) becomes an equation without a second member (EWSM), called a ***linear homogeneous equation,*** denoted by(E_h) :

$$a.y'' + by' + c.y = 0 \qquad (E_h)$$

Resolution method:

The general solution y of (E) is the sum of the homogeneous solution y_h of (E_h) and a particular solution (y_p) of (E): such that

$$\boldsymbol{y = y_h + y_p}$$

1. *How to find y_h?*

Let the homogeneous equation be

$$a.y'' + by' + c.y = 0$$

a) Write the caracteristic equation: $a.r^2 + b.r + c = 0.$

b) Findthe root r according to the sign of Δ given in the following table where $\Delta = b^2 - 4ac$: $hier\ a = 1$

Sign of Δ	**The roots : r_i**	**The soltion y_h**
$\Delta > 0$	There is $tow\ roots$ $r_1 = \dfrac{-b - \sqrt{\Delta}}{2a}$ $r_2 = \dfrac{-b + \sqrt{\Delta}}{2a}$	$y_h = C_1 e^{r_1.t} + C_2 e^{r_2.t}$
$\Delta = 0$	$r_0 = \dfrac{-b}{2a}$	$y_h = (C_1 t + C_2) e^{r_0.t}$

2. *How to find the particular solution y_p ?*

We determine the particular solution y_p of (E1), according to the form of the second member $f(t)$, and using the identification method of coefficient, the following table shows how to choose the form of y_p

f(t) takes the form :	y_p
$\boldsymbol{f(t) = P(t)e^{\alpha t}}$ *with* $P(t)$ is a polynomial of degree n, where α is a real number, and m is not a root of $\boldsymbol{P}$	$\boldsymbol{y_p = Q(t)e^{\alpha t}}$ $Q(t)$ is a polynomial $deg(Q) = n$
$\boldsymbol{f(t) = P(t)e^{\alpha t}}$ *with* $P(t)$ is a polynomial of degree 2, α is a real number, and $\boldsymbol{m}$ **Is a simple root**	$\boldsymbol{y_p = Q(t).t.e^{\alpha t}}$
$\boldsymbol{f(t) = P(t)e^{\alpha t}}$ *with* $P(t)$ is a polynomial of degree 2, α is a real number, and $\boldsymbol{m}$ **Is a double root**	$\boldsymbol{y_p = Q(t).t^2.e^{\alpha t}}$

3. *Donne la solution finale* $\boldsymbol{y(t) = y_h + y_p}$.

Exemple : *Solve the equation (E) given by*

$$y'' + 2y' = 2e^{-2x} \qquad (E)$$

Solution :

- **Find the homogeneous solution of** $\boldsymbol{y'' + 2y' = 0}$

The characteristic equation associate to (E) is:

$$r^2 + 2r = 0$$

Wich has tow roots : $r_1 = 0$ *and* $r_2 = -2$

Then , the solution y_h is

$$y_h = k_1 + k_2 e^{-2x} \quad where\ k_1 and\ k_2 \in \mathbb{R}$$

- **Find the particular solution of (E)**

Hier $\alpha = -2$, then y_p tak the form

$$y_p = kxe^{-2x} \quad , \qquad k \in \mathbb{R}$$

Then the derivative of y_p gives

$$y_p' = ke^{-2x} - 2kxe^{-2x} \text{ et } y_p'' = -4ke^{-2x} + 4kxe^{-2x}$$

By substituting into (E)

$$y_p'' + 2y_p' = -4ke^{-2x} + 4kxe^{-2x} + 2ke^{-2x} - 4kxe^{-2x} = -2ke^{-2x} = 2e^{-2x}$$

Using the identification method, it follow :

$$-2k = 2 \Rightarrow k = -1$$

Then ,

$$y_p = -xe^{-2x}$$

Finally , the general solution of (E) is

$$y = y_h + y_p = k_1 + k_2 e^{-2x} - xe^{-2x}$$

$$where\ k_1 and\ k_2 \in \mathbb{R}$$

Chapitre 02 :

The Matrices

The matrices play a fundamental role in economics, as they are used to represent the relationships between different economic variables such as production or consumption. They are utilized in various fields including the analysis of inter-sectoral exchanges, inventory management, or solving economic equations. Matrices help to understand and analyze economic systems effectively.

I. Definitions :

- A matrix is a table with n rows and p columns. It represents data that consists of real numbers (called coefficients or terms).
  Matrices are often denoted by uppercase letters such as A, B, C, M, ..., and their coefficients are represented by lowercase letters.

- We denoted by a_{ij} the coefficient of the matrix A, situated on the *i*-th row of A and at the *j*-th colunn it represents the numbers that appear inside the matrix. Then, the A is Witten by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$$

- The dimension of the matrix (called also the size, the order) :
  To describe the size of this matrix is to state how many rows and columns it has. Rows are listed first, followed by columns. is denoted by size (.)

$$\text{size}(.) = umber\ of\ rows \times \quad number\ of columns$$

Exampls :

- The size of the matrix $A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -1 & 2 \end{pmatrix}$ $is\ \ 2 \times 3$

- Let $B = \begin{pmatrix} -1 & 0 & 2 \\ -2 & 0 & 0 \\ 1 & 1 & 3 \end{pmatrix}$ *, the order of B is* 3

- *Let* $C = \begin{pmatrix} -1 & 0 \\ -2 & 4 \\ 1 & 1 \end{pmatrix}$ *and* $\text{size}(C) = 3 \times 2$

Definition 02 :

- **A zero matrix** is a matrix where all its coefficients are zeros.
- **A Square matrix** Is a matrix where the number of rows is equal to the number of columns
- A **row matrix** is a matrix where the number of rows is equal to 1. It is also called a "row vector".
- A **column matrix** is a matrix where the number of columns is equal to

Exampls :

- $A = (4 \quad -2 \quad 0)$ *is* row matrix
- $B = \begin{pmatrix} -1 & 0 & 2 \\ -2 & 0 & 0 \\ 1 & 1 & 3 \end{pmatrix}$ *is a square matrix, the size is* 3×3,

  in order to simplify , we can say , *B is a square matrix order* 3d' ordre 3.
- $C = \begin{pmatrix} 5 \\ 6 \\ 3 \\ 4 \end{pmatrix}$ *is a column matrix.*

Definition 03 :

The main diagonal of a matrix refers to the diagonal that connects the top-left corner to the bottom-right corner. In other words, the main diagonal elements have the same row and column numbers are : a_{11} , a_{22}, a_{33}, … .

Exemple :

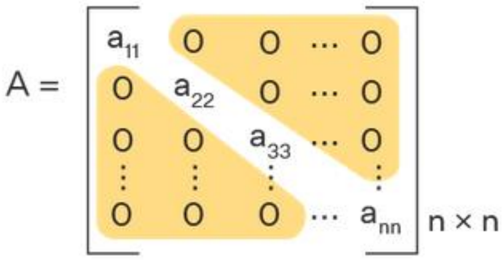
Let

$$A = \begin{pmatrix} 1 & -1 & -3 & -5 \\ 7 & 3 & -3 & 0 \\ 0 & 8 & 2 & 0 \\ -1 & 6 & 0 & -8 \end{pmatrix}$$

The diagonal éléments of A are : $a_{11} = 1$, $a_{22} = 3$, $a_{33} = 2$, and $a_{44} = -8$.

<u>Definition</u> 03 :

The diagonal matrix is a square matrix where all non-diagonal coefficients (those not on the diagonal) are zero.



<u>Example</u> :

Let

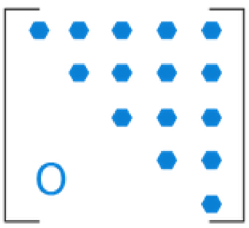
$$A=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 \end{pmatrix}, \qquad B=\begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

$$C=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad et \qquad M=\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

A, B and M are a diagonal matrices, but C is not.

<u>Definition</u> 04 :

An **upper triangular matrix** is a square matrix where all coefficients below the diagonal are zero



Upper Triangular Matrix

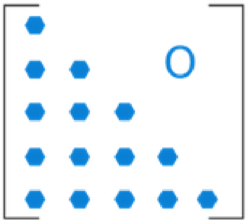
<u>Example</u>: Let the matrices:

$$A=\begin{pmatrix} -1 & 3 & 0 & -1 \\ 0 & 3 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 \end{pmatrix}, \qquad B=\begin{pmatrix} 3 & 1 & 3 \\ 0 & 0 & 2 \\ 9 & 0 & 5 \end{pmatrix},$$

We remark that A is an upper triangular matrix supérieure, however B is not,
Because there is a coefficient below the diagonal that is not zero, which is $a_{31} = 9 \neq 0$

Definition 05 :

A lower triangular matrix is a square matrix where all coefficients above the diagonal are zero.



Lower Triangular Matrix

Exemple :

Let the matrices

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 2 & 0 & 7 & 6 \end{pmatrix}, \quad and \quad C = \begin{pmatrix} 4 & 0 & 0 \\ 8 & 1 & 4 \\ 9 & -2 & 0 \end{pmatrix},$$

We remark that A is a lower triangular matrix; however, C is not, because there is a coefficient above the diagonal that is not zero, namely $a_{23} = 4 \neq 0$.

Définition 06 :

An **identity matrix** is a diagonal matrix where all the diagonal elements are equal to 1. It is denoted by I_n where n is the order of the matrix.

Example :

Let

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

We notice that:

- I_3 is an identity matrix of order 3(in $\mathbb{R}^3$
- I_2 is an identity matrix of order 2 (in $\mathbb{R}^2$).

II. Elementary Operations with Matrices:

We consider the positive natural numbers n_1, n_2 , p_1 et p_2.

Let A be a matrix of dimension: $n_1 \times p_1$ and B Another matrix of dimension: $n_2 \times p_2$.

1) Equality: $($= المساواة $)$

We say that two matrices are equal if these two conditions are satisfied:

- They have the same dimension
- Any term $a_{ij} = b_{ij}$ for each i, j.

Exercise : Let :

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -1 & 2 \end{pmatrix} \quad et \;\; B = \begin{pmatrix} a+2 & -2 & 0 \\ 1+d & 4-c & 2 \end{pmatrix}$$

Find the values of th reals : a, b and c such that the matrices A and B are equals

<u>Solution :</u>

For A and B to be equal, it is necessary that:

- $dim(A) = (2,3) = dim(B)$
- The termes satisfy

$$a + 2 = 1 \quad \Longrightarrow a = -1$$
$$4 - c = -1 \quad \Longrightarrow c = 5$$
$$1 + d = 3 \quad \Longrightarrow d = 2$$

2) Transposition

We call the transpose of A, the matrix denoted by ${}^tA$ $(ou\ bien\ A^T)$, obtained by writing the rows of A as columns.

Example :

The matrix transpose of $A = \begin{pmatrix} \mathbf{1} & \mathbf{4} & \mathbf{0} \\ 3 & -1 & 2 \end{pmatrix}$ is ${}^tA = \begin{pmatrix} \mathbf{1} & 3 \\ \mathbf{4} & -1 \\ \mathbf{0} & 2 \end{pmatrix}$

3) Addition and Subtraction (+, -):

In order to compute the sum $A + B$, the following condition must be satisfied:

$$size(A) = 2 \times 3 = size(B).$$

Additionally, $A + B$ is computed by adding the terms of A to the elements of B located at the same position.

Exercise :

Let be :

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 1 & 4 \\ -2 & 1 & -5 \end{pmatrix} \quad and \quad M = \begin{pmatrix} 3 & 0 \\ -3 & \frac{1}{2} \\ 2 & 2 \end{pmatrix},$$

Compute the matrices $A + B$ and , $A + M$

Solution :

Since $dim(A) = dim(B) = 2 \times 3$, then we can calculate $A + B$, indeed:

$$A + B = \begin{pmatrix} 1-3 & -2+1 & 0+4 \\ 3-2 & -1+1 & 2-5 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 4 \\ 1 & 0 & -3 \end{pmatrix}$$

For $A + M$, we cannot compute this sum because

$$dim(A) = 2 \times 3 \neq dim(B) = 3 \times 2$$

Therefore, the sum $A + M$ does not exist.

Remark :

In the case where we want to find the difference of the two matrices, A and B, we follow the same process, but we need to use element-wise subtraction.

4) Product ($\times$) الجداء :

Definition 07 (scalar multiplication) :

Let k be a real number (a constant). The product $k.A$ is the matrix obtained by multiplying each element of matrix A by k

Exemple : Let

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -1 & 2 \end{pmatrix},$$

The scalar multiplication of matrix is 3A and ,

$$3.A = \begin{pmatrix} 1\times 3 & -2\times 3 & 0\times 3 \\ 3\times 3 & -1\times 3 & 2\times 3 \end{pmatrix} = \begin{pmatrix} 3 & -6 & 0 \\ 9 & -3 & 6 \end{pmatrix},$$

Définition 08 : (Matrix Multiplication) :

Let m, n and p be integers. Let A be a matrix of size $m \times n$, and B be a matrix of size $(n \times p)$.

The matrix product $A \times B$ is a matrix of dimension (m, p) obtained by calculating the product of A by the columns of B.

$$AB = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m1} & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} =$$

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mj} & & c_{mp} \end{pmatrix} = C \ (dimension\ m \times p)$$

With

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{n1}b_{nj}$$

$$= \sum^{n} a_{ik}b_{kj} \ (i = 1,2, \ldots, m; j = 1,2, \ldots, p)$$

Example :

Let the matrices :

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 4 & -1 \\ -1 & 5 \end{pmatrix} \quad and \quad C = \begin{pmatrix} 1 & 4 \\ 1 & -5 \end{pmatrix}$$

We notice that:

- Since the number of columns of $A = 3$ = the number of rows of B, then we can compute the matrix product AB .
  Moreover, the size of the resulting matrix AB is 2×2, and the calculation gives:
  $$AB = \begin{pmatrix} -6 & 3 \\ 0 & 14 \end{pmatrix}$$
- We cannot compute the matrix product AC because the number of columns of A is 3, while the number of rows of C is 2, so:
  the number of columns of A is not equal to the number of rows of B.

$$\text{the number of columns of } A \neq \text{ the number of rows of } B$$

Therfore AC doesn't exist.

Remarks :

- The matrix product is not commutative. In other words, $AB \neq BA$.
- There is no specific method for computing the power of a matrix. A^n. It is computed using matrix multiplication as follows:

$$A^n = \underbrace{A.A.A \dots .A}_{n\,fois}$$

And this can only be done if matrix A is square.

Properties

- ${}^t(A.B) = {}^tB.\,{}^tA$
- $A.I_n = A$
- $k.A = A.k \quad with \quad k \in \mathbb{R}.$

III. Déterminant d'une matrice carrée :

Let's consider A a square matrix of order n

$$A = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix}$$

Definition :

The determinant of a square matrix A is a number (a value) denoted by $det\,(A)\;\; or \;\; |A|$.

To learn the method of calculating the determinant, let's start with the simplest case, which is matrices of size 2×2 .

1) Determinant of a matrix order 2

when $n = 2$, the matrix A is structured as follows

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

In this case, the determinant of A is a cross product such that:

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{12}.a_{21}$$

Examples :

Let

$$A = \begin{pmatrix} -3 & 2 \\ -1 & 4 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}$$

Then,

- $det(A) = \begin{vmatrix} -3 & 2 \\ -1 & 4 \end{vmatrix} = (-3).4 - (-1).2 = -10$

- $det(B) = \begin{vmatrix} 2 & 3 \\ -4 & -6 \end{vmatrix} = (2).(-6) - (-4).(3) = 0.$

Now, let's see how to calculate the determinant of a matrix of dimension greater than 2.

2) Determinant matrix order n, $(n \geq 3)$

Definition : (the minor)

We call M_{ij} a minor of A is the determinant of the matrix formed by removing the $i-th$ row and the $j-th$ column from A. It is also referred to as the $(i,j)-th$ minor of A.

Exemple :

Let A be a matrix :

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 3 & 1 & 4 \\ -2 & -2 & 0 \end{pmatrix},$$

- The minor M_{12} is the determinant obtained by eliminating the 1st row and the 2nd column of matrix A, indeed:

$$M_{12} = \begin{vmatrix} 3 & 4 \\ -2 & 0 \end{vmatrix} = 8$$

- The minor M_{22} is the determinant obtained by eliminating the 2nd row and the 2nd column of the matrix :

$$M_{22} = \begin{vmatrix} 4 & 2 \\ -2 & 0 \end{vmatrix} = 4$$

- The minor M_{13} is the determinant obtained by eliminating the 1st row and the 3rd column of matrix A, it means:

$$M_{13} = \begin{vmatrix} 3 & 1 \\ -2 & -2 \end{vmatrix} = -4.$$

Remark :
A matrix has several minor determinants, as demonstrated in the following example.

Definition :

We call C_{ij} the cofactor of the element a_{ij}, given by the formula

$$\boldsymbol{C_{ij} = (-1)^{i+j}.M_{ij}}$$

The determinant of A (at the i-th row) is calculated using cofactor expansion as follows:

$$\det(A) = \sum_{j=1}^{n} a_{ij}.C_{ij}$$

Example

Let

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 3 & 1 & 4 \\ -2 & -2 & 0 \end{pmatrix},$$

We expand along the first row:

$$\det(A) = 4M_{11} - M_{12} + 2M_{13}$$

With

$$M_{11} = 8 \qquad M_{12} = 8 \quad et \quad M_{13} = -4$$

Therefore

$$\det(A) = 4 \times 8 - 8 + 2 \times (-4) = 16$$

Properties :

Let $A \, and \, B$ be tow matrices order n, satisfying:

- $\det(A.B) = \det(A).\det(B)$
- $\det(A^T) = \det(A)$
- $\det(k.A) = k.\det(A)$
- The determinant of a triangular or diagonal matrix is equal to the product of its diagonal elements.
- The determinant of a matrix containing a null row (or column) is 0.

IV. invertible matrix

Definition :

Let A be a square matrix of order n. The matrix A is invertible if and only if there exists a square matrix B of order n, such that

$$A \times B = I_n$$

And

$$B \times A = I_n$$

With I_n is the Identity matrix of order n.
Additionally, the inverse of A is denoted by A^{-1} and

$$A^{-1} = B.$$

Remark :

The concept of matrix inverses only applies to square matrices. This means that:

$$A \times A^{-1} = A^{-1} \times A = I_n$$

Example 01 : Let A be a square matrix of order 3 which satisfies the following relation

$$A^3 = 3A - 2I_3$$

Let's show that A is invertible. Indeed:

$$A^3 = 3A - 2I_3 \Leftrightarrow A^3 - 3A = -2I_3$$

$$\Leftrightarrow -\frac{1}{2}(A^3 - 3A) = I_3$$

$$\Leftrightarrow A\left[-\frac{1}{2}(A^2 - 3I_3)\right] = I_3$$

Thus, we have

$$\left[-\frac{1}{2}(A^2 - 3I_3)\right]A = I_3$$

Therefore, A is invertible and its inverse is

$$A^{-1} = -\frac{1}{2}(A^2 - 3I_3)$$

Example 02 :

Let A and B be the two matrices such that:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad and \quad B = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Since,

$$AB = BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then , , A is invertible, and

$$A^{-1} = B.$$

How to calculate the inverse of an n-order matrix?

It is recalled that the cofactor of the element a_{ij} is denoted by C_{ij} and defined by

$$\boldsymbol{C_{ij} = (-1)^{i+j}.M_{ij}}$$

Where $\boldsymbol{M_{ij}}$ is the minor determinant of the matrix A .

Definition :(cofactor matrix) :

We call the cofactor matrix (or adjoint matrix) of A, the square matrix of order n, denoted as $cof(A)$ and defined by:

$$cof(A) = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}$$

Proposition : if $det\,(A) \neq 0,\ then\ \ A\,is\ \ invertible$.

Définition :

The inverse matrix of A is also found using the following equation:

$$A^{-1} = \frac{1}{\det(A)}.\,cof(A)^t$$

Propriétés

Let $A, B\ and\ \ C$ three invertible matrices of order n

- $if\ A\ is$ invertible, then the matrix A^{-1} is unic.
- $det(A^{-1}) = \frac{1}{\det\,(A)}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^t)^{-1} = (A^{-1})^t$

Application : let the matrix A

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{pmatrix}$$

Show that A is invertible and calculate its inverseA^{-1}.

Chapter 03 :

Linear Equations Systems

A system of linear equations consists of several linear equations involving the same variables, which are called unknowns.

A system of n linear equations with p unknowns: $x_1, x_2, x_3, \dots, x_n$, is written in the following form:

$$(S) \quad \begin{cases} a_{11}x_1 + a_{12}.x_2 + a_{13} + \cdots + a_{1p}x_p = b_1 \\ a_{21}x_1 + a_{22}.x_2 + a_{23} + \cdots + a_{2p}x_p = b_2 \\ \vdots \\ a_{31}x_1 + a_{32}.x_2 + a_{33} + \cdots + a_{1p}x_p = b_n \end{cases}$$

Where :

- The a_{ij} are given real numbers, referred to as the coefficients of the system. (pour $1 \leq i \leq n \quad and \quad 1 \leq j \leq p$)
- The b_i are also real numbers representing the constants on the right-hand side of the system (S) ($for \ \ 1 \leq i \leq n$).
- The x_j are the unknowns of the system, where $1 \leq j \leq p$.

Solving the system (S) involves finding the values of x_j that satisfy all the equations of the system. The system (S) can be rewritten in matrix form as:

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{ip} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{np} \end{pmatrix}}_{A} \ . \ \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_p \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{pmatrix}}_{B}$$

$$A \qquad \times \qquad X \qquad = \qquad B$$

In other words, (S) is equivalent to

$$A \, . X = B$$

With $X = (x_1, x_2, \dots, x_j, \dots, x_p)$ represents the unknown to be determined.

Remark :

If all $b_i = 0$ (for $1 \leq i \leq n$), then the system is called homogeneous; otherwise (i.e., if at least one $b_i = 0$ is not equal to zero), it is called non-homogeneous.

I. Cramer's system :

Definition :

A system (S) is said to be a Cramer's system if it satisfies these three conditions:

- A is a square matrix, meaning it contains the same number of equations as unknowns. In other words, the number of unknowns equals the number of equations.
- A is invertible, meaning $det(A) \neq 0$

II. Methods of Solution :

Cramer's systems of linear equations are solved using one of the following methods:

- Method of matrix inversion (or Matrix inversion)
- Cramer's method
- Gauss method (Elimination Method) .

For this purpose, consider the following system, which is non-homogeneous and Cramer's system written in its matrix form.

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{ip} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{np} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_p \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{pmatrix}}_{B}$$

$$A \times X = B$$

1) Method of matrix inversion :

It is evident that in Cramer's systems, the matrix A^{-1} exists (because $det(A) \neq 0$) . Therefore, to determine the vector X it suffices to multiply both sides of the system by the inverse matrix A^{-1}. Indeed:

$$A.\boldsymbol{X} = B \iff A^{-1}.A.\boldsymbol{X} = A^{-1}.B$$

$$\iff I_n.\boldsymbol{X} = A^{-1}.B \qquad (because \quad A^{-1}.A = I_n \ )$$

$$\iff \boldsymbol{X} = A^{-1}.B \qquad (because \quad I_n.X = X \ )$$

This means that the values of the unknown vector �X are calculated from the matrix product $A^{-1}.B$.

Example :

Let the system (S_1) :

$$(s_1) \quad \begin{cases} 3x_1 + 2x_2 + x_3 = 4 \\ x_1 + x_2 + x_3 = 1 \\ x_1 - 2x_3 = -1 \end{cases}$$

To solve(S_1) , we first need to write (S_1) in matrix form:

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$$

It is in the form:

$$A.X = B$$

With

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \quad , \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$$

Consequently, the solution X of (S_1) is given by:

$$X = A^{-1}.B.$$

Now, let us calculate A^{-1}, the inverse of A , using the formula:

$$A^{-1} = \frac{1}{\det(A)} . com(A)^t$$

We have

$$det(A) = -1$$

Furthermore, after calculation, the matrix of cofactors of A is given by:

$$Com(A) = \begin{pmatrix} -2 & 3 & -1 \\ 4 & -7 & 2 \\ 1 & -2 & 1 \end{pmatrix}$$

It follows that:

$$A^{-1} = \frac{1}{-1} \begin{pmatrix} -2 & 4 & 1 \\ 3 & -7 & -2 \\ -1 & 2 & 1 \end{pmatrix}$$

It is deduced that:

$$X = A^{-1}.B = \begin{pmatrix} 2 & -4 & -1 \\ -3 & 7 & 2 \\ 1 & -2 & -1 \end{pmatrix}.\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \Longrightarrow X = \begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix}$$

2) Cramer Method's :

It is also called the method of determinants. Indeed,

Let (S) be the matrix system to solve.

$$A.X = B$$

And let Δ denote the determinant of the matrix A, such that:

$$\Delta = \det(A)$$

Thus, Δi is the determinant of the matrix obtained by replacing the i-th column of matrix A with the vector B (column of constants).

Therefore, the unknown $\boldsymbol{x_i}$ is obtained by calculating the following ratio:

$$\boldsymbol{x_i} = \frac{\Delta_i}{\Delta} \qquad pour\ 1 \leq i \leq n$$

Example (Application) :

Let the system :

$$A.X = B$$

Where

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \ , \ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad and \quad B = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$$

The determinant of this systeme is :

$$\Delta = |A| = -1$$

Let's calculate the value of x, indeed:

$$\Delta_x = \begin{vmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & -2 \end{vmatrix} = -5$$

So,

$$\boldsymbol{x} = \frac{\Delta_x}{\Delta}$$

$$\Longrightarrow \boldsymbol{x} = \frac{-5}{-\mathbf{1}}$$

$$\Longrightarrow \quad \boldsymbol{x} = \mathbf{5}$$

- Let's calculate the value of y:

$$\Delta_y = \begin{vmatrix} 3 & 4 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -2 \end{vmatrix} = 7$$

Then

$$\boldsymbol{y} = \frac{\Delta_y}{\Delta} = \frac{\mathbf{7}}{-\mathbf{1}}$$

$$\Longrightarrow \boldsymbol{y} = -\mathbf{7}$$

- Similarly, to find z, we have

$$\Delta_z = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = -3$$

Then,

$$\boldsymbol{z} = \frac{\Delta_z}{\Delta} = -\frac{\mathbf{3}}{-\mathbf{1}}$$

$$\Longrightarrow \boldsymbol{y} = \mathbf{3}$$

So, the solution of$(S_1)est$:

$$X = (x, y, z,) = (5, -7, 3).$$

3) Gauss Method's:

Given the following matrix system:

$$A.X = B \qquad\qquad (S)$$

To solve (S) using the Gauss method involves transforming the system matrix (S) into an upper triangular matrix using elementary row operations, and then solving the resulting system using the back-substitution method.

To successfully achieve this transformation, we first need to define the Gauss table, written as follows:

$$[A \;\; | \;\; B]$$

Definition : Elementary row operations on a matrix :

The Elementary row operations (on the rows of a linear system) include the following:

- Swapping two equations, which means interchanging two rows.
- Multiplying a row by a constant.
- Replacing an equation with a linear combination of two rows (or two equations).

Definition : (pivot)

A pivot is a value by which we must divide to solve the linear system. These are the diagonal elements of the square matrix (i.e., $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$). It is necessary for these pivots to be non-zero in order to determine the solution of the system.
Note: This method is also called the Gauss elimination method, or the Gauss pivot method.

Example (Application) :

Let's solve the system using the Gauss method:

$$A.X = B$$

Where

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \quad , \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad et \qquad B = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$$

The Gauss table is defined by:

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & -1 \end{array}\right] \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

We denote L_1, L_2 and L_3 as the rows defined in the Gauss table.

We also denote L'_1, L'_2 and L'_3 as the new rows calculated from L_1, L_2 and L_3.To transform the system into an upper triangular system, we first fix the first row $L_1$and apply the following operations:

$$L'_2 = L_1 + (-3)L_2$$

And

$$L'_3 = L_1 + (-3)L_3$$

We obtain the new table:

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 0 & -1 & -2 & -2 \\ 0 & 2 & -5 & 7 \end{array}\right] \begin{array}{l} L_1 \\ L'_2 \\ L'_3 \end{array}$$

To obtain the upper triangular matrix in the first part of the table, we use this operation while fixing the row L'_2

$$L''_2 = (2)L'_2 + L'_3$$

This results in

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -9 - & 9 \end{array}\right] \begin{array}{l} L_1 \\ L'_2 \\ L'_3 \end{array}$$

The new system obtained is:

$$\begin{pmatrix} 3 & 2 & -1 \\ 0 & -1 & -4 \\ 0 & 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix}$$

The system is rewritten as linear equations, starting from the last row and moving upwards to the first. Indeed,

$$\begin{cases} -3z = 9 \\ -y - 4z = 1 \\ 3x + 2y - z = 4 \end{cases}$$

By substitution, we find:

$$\begin{cases} z = -3 \\ y = 11 \\ x = -7 \end{cases}$$

I. Non-cramerian systems :

Let (S) be a linear system with n equations and p unknowns. (S) is non-Cramerian if:

1) $\boldsymbol{n > p}$, , meaning there are more equations than unknowns. In this case, the system is termed “ *overdetermined* ”.

2) $\boldsymbol{n < p}$, , indicating fewer equations than unknowns. In this case, the system is termed “ *underdetermined* ”

3) If $\boldsymbol{n = p}$ et $\boldsymbol{det\,(A) = 0}$, the system is square but non-invertible.

Solving overdetermined systems ($n > p$):

It suffices to follow these steps

a) Extract a subsystem with p equations and p unknowns such that the determinant associated with the subsystem is non-zero.

b) Solve the subsystem.
c) Check the solution obtained for the $n - p$ equations. There are two cases:
   - If the solution satisfies all the equations, we conclude that the global system has a unique solution.
   - If the solution does not satisfy all the equations, then it is clear that the global system has no solution.

Example: Consider the system:

$$(s)\quad \begin{cases} x + 2y = 1 \\ 3x - y = 2 \\ 5x - 4y = -2 \end{cases}$$

The system(s) contains 3 equations with 2 unknowns. Here, $n = 3\ and\ \ p = 2$. To solve (S), we choose the subsystem (S') with 2 equations:

$$(s')\quad \begin{cases} x + 2y = 1 \\ 3x - y = 2 \end{cases}$$

The resolution of (S') is very simple and yields $x = \frac{5}{7}\ \ et\ y = \frac{1}{7}$. In conclusion, we need to verify if this solution:

$$x = \frac{5}{7}\ \ et\ y = \frac{1}{7}$$

satisfies the last equation (the one that was not chosen).

$$5x - 4y = -2$$

We have

$$5x - y = 5\left(\frac{5}{7}\right) - 4\left(\frac{1}{7}\right) = \frac{21}{7} = 3\ \neq -2.$$

We deduce that the solution obtained from the subsystem (S') does not satisfy all the equations. Consequently, the system (S) has no solutions.

Solving underdetermined systems $(\boldsymbol{n} < \boldsymbol{p})$:

To determine the solution of this type of system, we need to:

1. Consider a subsystem that contains $\boldsymbol{n}$ equations with $\boldsymbol{n}$ unknowns and assume the remaining unknowns $\boldsymbol{n} - \boldsymbol{p}$, as constants.
2. The solution obtained demonstrates that underdetermined systems have infinitely many solutions.

Example: Consider the system:

$$(S)\quad \begin{cases} x + 2y - z = 1 \\ 3x - y + z = 2 \end{cases}$$

The system (S) contains 2 equations with 3 unknowns:: x, y *and* z (here, $n = 2$ *and* $p = 3$). To solve(S), we choose the subsystem (S') with 2 equations and 2 unknowns while assuming the third unknown as a constant:

Let's set:

$$z = \alpha \quad \text{where} \quad \alpha \in \mathbb{R}$$

The subsystem of (S) is:

$$\begin{cases} x + 2y - \alpha = 1 \\ 3x - y + \alpha = 2 \end{cases} \quad \Rightarrow \quad \begin{cases} x + 2y = 1 + \alpha \\ 3x - y = 2 - \alpha \end{cases}$$

Let's find the value of x and y in terms of α which gives:

$$x = \frac{5 - \alpha}{7} \quad and \quad y = \frac{4\alpha + 1}{7} \qquad where\ \ \alpha \in \mathbb{R}$$

Hence, (S) has infinitely many solutions of the form:

$$X = (x; y; z) = \left(\frac{5 - \alpha}{7};\quad \frac{4\alpha + 1}{7};\ \alpha\right) \qquad avec\ \alpha \in \mathbb{R}$$

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